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Complexity Classes of Analytic Functions and Their Tangent Spaces

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Abstract—A number of results on the nonrepresentability of analytic functions of two variables as superpositions of functions of one variable are obtained. The method of proof is based on a description of tangent spaces to complexity classes and properties of linear superpositions.

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1. INTRODUCTION

Given a set of objects and operations on them, we can construct new objects. There arises a hierarchy of constructed objects: those given initially, those constructed in the first iteration, etc. This inductive construction, which is based on the intuition of positive integers and counting, is omnipresent in mathematics [1]. It is understood as constructing the complex from the simple. This view of things suggests certain questions. Given a new object, we ask: Has this object entered our hierarchy, and if so, at which step? Questions of this type include classical questions concerning the solvability of algebraic equations in radicals, of differential equations in elementary functions, etc. A well-known such question is Hilbert's 13th problem (see [2]).

As is known, generally, the dependence of a root of a fifth-degree equation on the coefficients cannot be expressed in radicals. However, this will become possible if we add a certain algebraic function of one variable to the number of admitted arithmetic operations and radicals. Similarly, to make a sixth-degree equation solvable in radicals, it suffices to add an algebraic function of two variables, and in the case of a seventh-degree equation, a function of three variables. Hilbert, skipping the case of the sixth degree, asked a question about a function solving a seventh-degree equation: Can it be obtained by using only functions of two variables and their superpositions (substitutions)? It is already unnecessary to separately stipulate for the possibility of arithmetic operations and radicals. Of course, we must specify what functions of two variables are allowed. Hilbert's setting was unexpectedly broad: all continuous functions. As shown by Arnold [3] and Kolmogorov [4], any continuous function of any number n of variables $x_1, \dots, x_n$ on the n-cube can be represented in the form of the sum of $2n+1$ terms

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} c_q(a_{1q}(x_1) + \cdots + a_{nq}(x_n))$$

where $a_{1q}(t), \dots, a_{nq}(t)$ and $c_q(t)$, $q = 1, 2, \dots, 2n+1$, are continuous functions of one variable. Moreover, all inner functions $a_{jq}(t)$ are fixed and do not depend on f. Only the $2n+1$ outer functions $c_q(t)$ depend on f.

In the case of two variables, Kolmogorov's theorem says that any continuous function $f(x, y)$ on the square can be represented as the sum of five terms of the form $c_j(a_j(x) + b_j(y))$, $j = 1, \dots, 5$. In the context of this result, consider the nondecreasing sequence of function classes Kl^j_c, $j = 1, 2, \dots$, where

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each Kl_c^j is the set of all continuous functions on the square that can be represented as the sum of j arbitrary terms of the form $c(a(x)+b(y))$, where a, b, and c are continuous functions. In [5], it was shown that Kl_c^1 does not contain all continuous functions on the square, and in [6], this was proved for Kl_c^4. Thus,

$$\mathrm{Kl}_c^1 \subseteq \mathrm{Kl}_c^2 \subseteq \mathrm{Kl}_c^3 \subseteq \mathrm{Kl}_c^4 \subseteq \mathrm{Kl}_c^5 = \mathrm{Kl}_c^6 = \cdots,$$

i.e., the sequence stabilizes at the fifth term.

Consider the analytic analog $\{\mathrm{Kl}^j\}$ of the sequence $\{\mathrm{Kl}_c^j\}$. Let Kl^1 be the class of analytic functions of two variables of the form $z(x,y)=C(A(x)+B(y))$, where A, B, and C are analytic functions of one variable. Accordingly, Kl^n is the class of functions that can be locally represented as

$$z(x,y)=C_1\big(A_1(x)+B_1(y)\big)+\cdots+C_n\big(A_n(x)+B_n(y)\big). \tag{1}$$

Clearly, this sequence of classes does not decrease with increasing n.

It is appropriate to place the consideration of this sequence in the general context of *composition schemata, class hierarchies, differential ideals* and their generators, and the *complexity* of functions (see [7], [1]).

The classes introduced above are a special case of composition schemata and class hierarchies. Schemata of the form (1) are of depth (of composition) 1. We can define schemata S of any form and depth depending on any number $l\geq 1$ of independent functional parameters $f=(f_1(u),\dots,f_l(u))$, where the f_j are analytic functions of one variable. Each schema S determines the class $\mathrm{Cl}(S)$ of analytic functions which have local representations in the framework of this schema, i.e., $Z(x,y)=S(f)(x,y)$.

Also, to each schema there corresponds a radical differential ideal $I(S)$ (that is, a differential ideal coinciding with its radical) in the ring of differential polynomials. According to the Ritt–Raudenbush theorem (see [8] and [9]), this radical is finitely generated as a radical differential ideal. In particular, all classes consist of differential-algebraic functions (that is, analytic solutions of differential polynomials with rational coefficients).

Let $\mathcal{M}$ be the set of convergent power series in one complex variable without constant term, and let $\mathcal{M}^3=\mathcal{M}\oplus\mathcal{M}\oplus\mathcal{M}$ be a direct sum. An analytic function $Z_0(x,y)$ belongs to Kl^1 if it has a local representation of the form

$$Z_0(x,y)=z_0+C_0\big(A_0(x-x_0)+B_0(y-y_0)\big),\qquad m_0=(A_0,B_0,C_0)\in\mathcal{M}^3,$$

in a neighborhood of some point (x_0,y_0). Note that the existence of a representation of this form in a neighborhood of some point implies the existence of such a representation in a neighborhood of *almost every* point (namely, of any point in the complement of a proper analytic set).

Let us add an increment depending on a small one-dimensional parameter t to the functional parameter m_0, i.e., consider $m=m_0+t\delta m+o(t)$, where $\delta m\in\mathcal{M}^3$. The tangent space to Kl^1 at the point $Z_0=Z(m_0)$ is

$$T\,\mathrm{Kl}^1_{Z(m_0)}=\left\{\delta z=\frac{d}{dt}(Z(m_0+t\delta m))\bigg|_{t=0}\right\},$$
$$Z(m)(x,y)=Z(m_0)(x,y)+t\delta z(x,y)+o(t).$$

Thus, the tangent space is the set of linear increments δz. Clearly, the tangent space at any point is a vector space. We emphasize that we vary an analytic function $Z_0(x,y)$ which can be locally represented in the form of a power series, so that the perturbed function $Z(x,y)$ is a function of the same form, as well as the linear part $\delta z(x,y)$ of the increment.

On the plane of one complex variable u the group $G=\{u\to\lambda u+o(u),\ \lambda\neq 0\}$ of invertible holomorphic transformations preserving the origin acts. The direct product $G\times G\times G=G^3$ of three copies of this group acts on $\mathcal{M}^3$. If $(g_1,g_2,g_3)\in G^3$, then

$$(A(u),B(u),C(u))\to(A^*(u)=g_1\circ A(u),$$
$$B^*(u)=g_2\circ B(u),\qquad C^*(u)=g_3\circ C(u)).$$

Accordingly, G^3 acts on the elements of Kl^1 at a fixed (x_0, y_0, z_0) as a map

$$S(A, B, C)(x, y) \to S(A^*, B^*, C^*)(x, y).$$

If a schema S depends on l, rather than three, functional parameters, then we have an action of G^l on $\mathcal{M}^l$, which induces an action of G^l on $\mathrm{Cl}(S)$.

It is easy to describe the orbits of the action of G on $\mathcal{M}$. The germs having a simple zero at the origin form a single open dense orbit under postcomposition. Thus, $\mathcal{M}^l$ is an almost homogeneous space. Any schema inherits this almost homogeneity. Singular parameter values are those for which at least one of the corresponding germs has multiplicity different from 1.

There are natural operations on the set of schemata. In particular, consider the addition operation. Suppose given two schemata $S_1(m_1)$ and $S_2(m_2)$, where $m_1 \in \mathcal{M}^{l_1}$ and $m_2 \in \mathcal{M}^{l_2}$, i.e., m_1 and m_2 are independent parameters. Then we set $S(m) = S_1(m_1) + S_2(m_2)$, where $m = (m_1, m_2) \in \mathcal{M}^{(l_1+l_2)}$. The following proposition is quite obvious.

Proposition 1. *The tangent space to a sum is the sum of tangent spaces*:

$$T_{S(m)}\,\mathrm{Cl}(S) = T_{S_1(m_1)}\,\mathrm{Cl}(S_1) + T_{S_2(m_2)}\,\mathrm{Cl}(S_2).$$

Note that this sum of vector spaces is not necessarily direct: the summands may have a nontrivial intersection.

2. THE ABSENCE OF STABILIZATION IN THE Kl HIERARCHY

Let us fix the value of the functional parameter $m_0 = (A(x), B(y), C(u))$ and consider the tangent space $T_{Z(m_0)}\,\mathrm{Kl}^1$.

Proposition 2. *A function δz belongs to $T_{Z(m_0)}\,\mathrm{Kl}^1$ if and only if*

$$\delta z = C'\big(A(x) + B(y)\big)\big(a(x) + b(y)\big) + c\big(A(x) + B(y)\big) \tag{2}$$

for some $(a(x), b(y), c(u))$.

Proof. Consider a perturbation Dm of m in the direction (a, b, c), i.e.,

$$m_t = \big(A(x) + ta(x),\ B(y) + tb(y),\ C(u) + tc(u); x_0, y_0, z_0\big).$$

We have $Z(m_t)(x, y) = Z_0(x, y) + t\delta z(x, y) + o(t)$. Evaluating $\delta z(x, y)$, we obtain the required result. □

Let $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$. Setting $A(x) = \alpha \ln(x)$, $B(y) = \beta \ln(y)$, and $C(u) = \gamma \exp(u)$, we obtain $Z_0 = \gamma x^\alpha y^\beta$. Proposition 2 gives

$$\delta z = \gamma\big(a(x) + b(y)\big)x^\alpha y^\beta + c(x^\alpha y^\beta). \tag{3}$$

We set

$$X_{(\alpha,\beta)} = \frac{1}{\alpha}\frac{\partial}{\partial x} - \frac{1}{\beta}\frac{\partial}{\partial y}, \qquad D_x = x\frac{\partial}{\partial x}, \qquad D_y = y\frac{\partial}{\partial y}.$$

Clearly, the kernel of the operator $X_{(\alpha,\beta)}$ is formed by functions of the form $c(x^\alpha y^\beta)$. Let us give a differential criterion for the function $\delta z(x, y)$ to be representable in the form (3).

In what follows, we use the index notation for derivatives of functions of two variables, i.e., write z_{ij} instead of $z_{x^i y^j}$.

Proposition 3. *The following assertions hold*:

(a) *The function $\delta z(x,y)$ can be represented in the form* (3) *if and only if*

$$L_{(\alpha,\beta)}(\delta z) = -\alpha^2\beta y\delta z_{01} + \alpha^2 y^2\delta z_{02} + \alpha\beta^2 x\delta z_{10} - \alpha x y^2\delta z_{12} - \beta^2 x^2\delta z_{20} + \beta x^2 y\delta z_{21} + \delta z_{01}\alpha^2 y - \delta z_{11}\alpha y x - \beta^2 x\delta z_{10} + \beta x y\delta z_{11} = 0.$$

(b) $L_{(\alpha,\beta)} = (D_x - \alpha)(D_y - \beta)(\beta D_x - \alpha D_y)$.

(c) *Any two operators of the form $L_{(\alpha,\beta)}$ commute.*

(d) $L_{(\alpha,\beta)}(x^\lambda y^\mu) = (\lambda - \alpha)(\mu - \beta)(\beta\lambda - \alpha\mu)x^\lambda y^\mu$.

Proof. Let us successively eliminate the functions c, b, and a from relation (3). Applying $X_{(\alpha,\beta)}$ to this relation, we eliminate $c(x^\alpha y^\beta)$. Expressing $b'(y)$ in the resulting relation and differentiating with respect to x, we eliminate $b(y)$. Expressing $a''(x)$ and differentiating with respect to y, we eliminate $a(x)$ and, thereby, prove (a). Expressing $L_{(\alpha,\beta)}$ in terms of (D_x, D_y), we obtain (b). Assertion (c) holds because D_x and D_y commute and L is a polynomial with constant coefficients in the commuting operators D_x and D_y. Assertion (d) is verified by direct substitution. This completes the proof of the proposition. □

Propositions 2 and 3 imply the following assertion.

Corollary 4. *Let $\delta z(x,y)$ be an element of the tangent space to* Kl^n *at the point*

$$Z_0(x,y) = \sum_{j=1}^{n} \gamma_j x^{\alpha_j} y^{\beta_j},$$

and let L be the operator obtained by successively applying the commuting operators $L_{(\alpha_j,\beta_j)}$, $1 \le j \le n$. Then $L(\delta z) = 0$.

Expressions for a general element of the tangent space to a class S at a fixed point $Z_0 = S(m_0)(x,y)$ are composition schemata, too, but of somewhat more general form. Initially, we constructed schemata from free functional variables and arithmetic actions. Now we allow schemata in which certain functional variables are assigned fixed values, which are fixed analytic functions of one variable. Let us introduce appropriate notation. We use $z = S(m,M)(x,y)$ to denote a schema in which m is the group of free parameters and M is the group of fixed parameters. For example, in the schema of Proposition 2, the set m of free parameters is (a,b,c) and the set M of fixed parameters is (A,B,C). When we pass to this larger class of schemata, the assertion that any schema is defined by equations in the form of differential polynomials in a differential ring with independent variables x and y, a dependent variable z, and differentiations with respect to x and y over $\mathbb{Q}$ needs the following correction, which does not affect the proofs. The defining equations are now differential polynomials with coefficients in the differential extension obtained by adjoining the elements of M to $\mathbb{Q}$. For the example in Proposition 3, this is the extension of $\mathbb{Q}$ generated by adjoining the coordinates of the vectors α, β, and γ.

The results presented in what follows are obtained by combining two considerations.

The first consideration [10]**.** Given a schema S, let $Z_0 = S(m_0)(x,y)$ for a nonsingular value m_0. Suppose that $\delta z(x,y)$ does not belong to the tangent space to $\mathrm{Cl}(S)$ at $Z_0 = S(m_0)(x,y)$. Then the 1-dimensional line $Z_t = Z_0 + t\delta z$ cannot be entirely contained in $\mathrm{Cl}(S)$.

This consideration has a variation. Suppose given two composition schemata S' and S'' such that the intersection of the corresponding classes is nonempty and $W(x,y) \in \mathrm{Cl}(S') \cap \mathrm{Cl}(S'')$. If $\mathrm{Cl}(S') \subseteq \mathrm{Cl}(S'')$, then the tangent space to $\mathrm{Cl}(S')$ at the point W is contained in the tangent space to $\mathrm{Cl}(S'')$ at the same point. Therefore, if there exists a vector in the former space not contained in the latter, then $\mathrm{Cl}(S')$ is not contained in $\mathrm{Cl}(S'')$.

The second consideration [11]**.** The existence of a defining system of differential polynomials for a class, in particular, for a class determining a tangent space, has the following consequence. Since the line $Z_t = Z_0 + t\delta z$ is not entirely contained in $\mathrm{Cl}(S)$, it follows that the values of t for which $Z_t = Z_0 + t\delta z \in \mathrm{Cl}(S)$ are algebraic over the relevant coefficient field. Each value $t = t_0$ not satisfying this condition gives an example of a function not contained in $\mathrm{Cl}(S)$.

This opens the possibility of constructing such examples based on the Lindemann–Weierstrass theorem. This theorem says that if $\alpha_1, \alpha_2, \dots, \alpha_m$ are algebraic numbers *linearly* independent over $\mathbb{Q}$, then the exponentials $\exp(\alpha_1), \exp(\alpha_2), \dots, \exp(\alpha_m)$ are *algebraically* independent over $\mathbb{Q}$. The simplest example of such a set is the sequence of square roots of primes.

Let p_n be the nth prime. We set

$$Z_n(x,y) = \sum_{j=1}^{n} \exp(-\sqrt{p_j}) x^j y^{j^2}.$$

Clearly, $Z_n(x,y) \in \mathrm{Kl}^n$.

Theorem 5. *The sequence of classes* Kl^n *is strictly increasing*; *i.e.,* $\mathrm{Kl}^{n+1} \neq \mathrm{Kl}^n$ *for all* $n \geq 1$. *Moreover,* $Z_{n+1}(x,y) \in \mathrm{Kl}^{n+1} \setminus \mathrm{Kl}^n$.

Proof. Clearly, $Z_{n+1}(x,y) \in \mathrm{Kl}_{n+1}$. Let us prove that $Z_{n+1}(x,y) \notin \mathrm{Kl}^n$. Consider the tangent space to Kl^n at the point corresponding to Z_n for the obvious set of functional parameters, i.e.,

$$A_j(x) = \alpha_j \ln(x), \qquad B_j(y) = \beta_j \ln(y), \qquad C_j(u) = \gamma_j \exp(u),$$
$$\text{where} \quad \alpha_j = j, \quad \beta_j = j^2, \quad \gamma_j = \exp(\sqrt{-p_j}).$$

According to Propositions 1 and 2, a general element of the tangent space is the sum

$$\delta z = \delta z_1 + \cdots + \delta z_n, \tag{4}$$

where the δz_j have the form described in Proposition 2. Let $L_j = L_{\alpha_j,\beta_j}$ be the same as in Proposition 3. Successively applying the operators $L_1, L_2, \dots, L_n$ to (4) and using their commutativity, we see that the condition $L(\delta z) = L_1 \circ L_2 \circ \cdots \circ L_n(\delta z) = 0$ is necessary for the function δz to belong to the tangent space. By Proposition 3, any monomial of the form $x^\lambda y^\mu$ is an eigenvector of the operator L, and the corresponding eigenvalue has the form

$$\prod_{j=1}^{n} (\beta_j - \mu)(\alpha_j - \lambda)(\beta_j \lambda - \mu \alpha_j) = \prod_{j=1}^{n} (j^2 - \mu)(j - \lambda)(j^2 \lambda - j\mu).$$

It follows that, setting $\lambda = (n+1)$ and $\mu = (n+1)^2$, we obtain a nonzero value. Therefore, the line $Z = Z_n + t x^{n+1} y^{(n+1)^2}$ cannot be entirely contained in Kl^n. But Kl^n is the set of common zeros of a system of differential polynomials with coefficients in the extension of the field of rational numbers obtained by adjoining $\exp(-\sqrt{p_1}), \dots, \exp(-\sqrt{p_n})$. Therefore, those values of t for which $Z \in \mathrm{Kl}^n$ are algebraic over this extension. According to the Lindemann–Weierstrass theorem [12], the number $\gamma_{n+1} = \exp(-\sqrt{p_{(n+1)}})$ is not algebraic over this extension (in view of algebraic independence). Thus,

$$Z_{n+1} = Z_n + \exp(-\sqrt{p_{(n+1)}}) x^{(n+1)} y^{(n+1)^2}$$

does not belong to Kl^n. This completes the proof of the theorem. □

3. COMPARISON OF THE Kl AND Cl HIERARCHIES

Along with the system of classes Kl^n of analytic functions in two variables, we can consider the system of classes Cl^n, which are defined as follows. For $n = 1$, we set $\mathrm{Cl}^1 = \mathrm{Kl}^1$. Next, Cl^{n+1} is the set of analytic functions which can be locally represented in the form $Z(x,y) = C(A_n(x,y) + B_n(x,y))$, where $A_n, B_n \in \mathrm{Cl}^n$ and $C(u)$ is an analytic function of one variable. The system of classes Cl^n, as well as that of Kl^n, is infinitely expanding.

Using the same method as above, we prove that $\mathrm{Cl}^2 \not\subset \mathrm{Kl}^n$ for all n.

Theorem 6. (a) *For any n, the class Cl^2 is not contained in Kl^n.*

(b) *Moreover,*

$$Z_n(x,y) = (x^2y + xy^2)^{(n-1)} + \pi \exp(x^2y + xy^2) \in \mathrm{Cl}^2 \setminus \mathrm{Kl}^n\,.$$

Proof. We can assume that $n \geq 2$. Let $Z_0 = (xy(x+y))^{n-1}$. This polynomial is the sum of n monomials and hence belongs to Kl^n. On the other hand, since the polynomial $(xy(x+y))$ belongs to Cl^2, it follows that any one-variable analytic function of this polynomial, in particular, any power of it, belongs to Cl^2. Thus, $Z_0 \in \mathrm{Kl}^n \cap \mathrm{Cl}^2$. If the inclusion $\mathrm{Cl}^2 \subseteq \mathrm{Kl}^n$ held for some n, then any vector in the tangent space to Cl^2 at Z_0 would belong to the tangent space to Kl^n.

Consider the linear operators

$$L_j = L_{\alpha,\beta}, \qquad \text{where} \quad \alpha = 2(n-1) - j, \quad \beta = (n-1) + j, \quad j = 0, 1, 2, \ldots, n-1.$$

Let L be their successive composition. Applying Corollary 4, we see that if δz belongs to the tangent space to Kl^n at Z_0, then $L(\delta z) = 0$.

The class Cl^2 depends on seven functional parameters, and the schema has the form

$$Z = K\big(C(A(x) + B(y)) + R(P(x) + Q(y))\big).$$

To obtain Z_0, it suffices to set

$$A_0(x) = 2\ln(x), \qquad B_0(y) = \ln(y), \qquad C_0(u) = \exp(u),$$
$$P_0(x) = \ln(x), \qquad Q_0(y) = 2\ln(y), \qquad R_0(u) = \exp(u), \qquad K_0(u) = u^{(n-1)}.$$

Perturbing the last coordinate $K(u) = K_0(u) + tk(u) + o(t)$, we obtain a vector tangent to Cl^2 at Z_0 of the form $\delta z = k(x^2y + xy^2)$. Suppose that $\mathrm{Cl}^2 \subseteq \mathrm{Kl}^n$. Taking the function $\exp(u)$ as $k(u)$, we see that $L(\exp(xy(x+y))) = 0$. All monomials x^py^q are eigenfunctions of L. The power expansion of $\exp(xy(x+y))$ contains homogeneous components of the form $(xy(x+y))^{2N-2}$, and these components contain monomials of the form $x^{(4N-5)}y^{(2N-1)}$, $N \geq 0$, with nonzero coefficients. Therefore, the relation $L(\exp(xy(x+y))) = 0$ implies $L(x^{4N-5}y^{2N-1}) = 0$ for all $N \geq 0$. However, the eigenvalue of $L_{\alpha\beta}$ corresponding to this monomial is

$$3(2Nj - 2j - n + 1)(2N - n - j)(4N - 3 - 2n + j).$$

Thus, the corresponding eigenvalue of L equals

$$3^n \prod_{j=0}^{n-1} (2Nj - 2j - n + 1)(2N - n - j)(4N - 3 - 2n + j)$$

and does not vanish at $N > n$.

This proves that $\mathrm{Cl}^2 \nsubseteq \mathrm{Kl}^n$ for any n. Next, as in the proof of Theorem 5, consider the line $Z_t = Z_0 + t\exp(x^2y + xy^2)$. It is not contained in the class Kl^n. This class is defined by equations being differential polynomials with rational coefficients. Therefore, those values of t for which $Z_t \in \mathrm{Kl}^n$ belong to the algebraic closure of the field of rational numbers (that is, to the field of algebraic numbers). Thus, if t is not an algebraic number, e.g., $t = \pi$, then $Z_t(x,y) \notin \mathrm{Kl}^n$. This completes the proof of the theorem. □

Theorem 5 suggests considering the function

$$f(x,y) = \sum_{n=1}^{\infty} \exp(-\sqrt{p_n}) x^n y^{n^2}.$$

This analytic function is a solution of the equation $D_y f = (D_x)^2 f$ (an analog of the heat equation in the Fuchsian derivatives D_x and D_y). Applying an argument of [11] to the function f, one can show that this function is not contained not only in $\mathrm{Kl} = \bigcup_1^\infty \mathrm{Kl}^n$ but also in the much larger class $\mathrm{Cl} = \bigcup_1^\infty \mathrm{Cl}^n$.

Let $E(s)$ denote the integer part of a real number s.

Proposition 7. *The following inclusions hold*:

(a) $\mathrm{Kl}^m \subset \mathrm{Cl}^n$ *for* $n = \lceil \log_2 m \rceil + 1$.

(b) $\mathrm{Kl} \subset \mathrm{Cl}$.

Proof. The inclusion $\mathrm{Kl}^2 \subset \mathrm{Cl}^2$ obviously follows from $\mathrm{Kl}^1 = \mathrm{Cl}^1$; hence $\mathrm{Kl}^3 \subset \mathrm{Kl}^4 \subset \mathrm{Cl}^3$ and so on. Accordingly, if $m \le 2^{n-1}$, then $\mathrm{Kl}^m \subset \mathrm{Cl}^n$. This gives (a), which immediately implies (b). This completes the proof of the proposition. □

Proposition 8. *Let* $\Omega \subset \mathbb{C}$ *be a connected open set, and let* $f(x, y, t)$ *be holomorphic on the product of the bidisk in* (x, y) *with* Ω. *Suppose that for every* n *there exists* $t_n \in \Omega$ *such that* $f(x, y, t_n) \notin \mathrm{Cl}(S_n)$. *Then there exists* $T \in \Omega$ *such that* $f(x, y, T) \notin \mathrm{Cl}(S_n)$ *for every* n.

Proof. Let P_n be a set of differential polynomials defining $\mathrm{Cl}(S_n)$. By assumption, P_n contains a polynomial p_n such that the function $p_n(f(x, y, t_n))$ is holomorphic in the bidisk $\{|x - a| < r_1,\ |y - b| < r_2\}$ and does not vanish identically on this bidisk. Therefore, the set N_n of points t in B for which $p_n(f(x, y, t))$ vanishes identically is discrete and at most countable. Hence $N = \bigcup N_n$ is at most countable as well. Thus, $B \setminus N$ is nonempty (this is the disk minus a countable set). If T is a point in this complement, then $f(x, y, T) \notin \mathrm{Cl}(S_n)$ for all n. This completes the proof of the proposition. □

Let us use the Lagrange interpolation formula [13] for the auxiliary complex variable t (the variable ζ is treated as a parameter). We set

$$g(\zeta, t) = \sum_{n=1}^{\infty} \frac{(\zeta^{n-1} + \pi \exp(\zeta))}{2^n} \frac{\sin(t - \pi n)}{(t - \pi n)}.$$

If $|\zeta| < 1$, then $g(\zeta, t)$ is holomorphic with respect to both variables in the product of the disk and the plane. Moreover, $g(\zeta, \pi n) = (\zeta^{n-1} + \pi \exp(\zeta))/2^n$. Set $f(x, y, t)$ to be $g(\zeta, t)$, where $\zeta = xy(x + y)$. If $|x| \le 1/4$ and $|y| \le 1/4$, then $|\zeta| \le 1/8$ and $|\zeta^{n-1} + \pi \exp(\zeta)| \le 5$ for all n. It follows that the $f(x, y, n)$ are functions proportional to those in Theorem 6.

Thus, the assumptions of Proposition 8 are satisfied (for $\Omega = \mathbb{C}$ and $t_n = \pi n$), and applying this proposition, we see that, for almost every complex number T (outside the countable set of singular values), the function $f(x, y, T)$ does not belong to Kl. Its membership in Cl^2 is obvious, because $\zeta(x, y)$ and hence $g(\zeta, t)$ with fixed t belong to Cl^2 and $f = g(\zeta, t)$. We have obtained the following result.

Theorem 9. *For all but countably many complex numbers* T, *the function*

$$f(x, y, T) = \sum_{n=1}^{\infty} \left(\frac{(xy(x + y))^{n-1} + \pi \exp(xy(x + y))}{2^n} \right) \frac{\sin(T - \pi n)}{(T - \pi n)}$$

belongs to $\mathrm{Cl}^2 \setminus \mathrm{Kl}$.

Both classes Kl and Cl are vector spaces over the field $\mathbb{C}$. The space of polynomials in (x, y) is a proper subspace of Kl, which, in turn, is a proper subspace of Cl by Theorem 9.

Note also that δz, which is an element of the tangent space to $\mathrm{Cl}(S)$ for an arbitrary composition schema S at some point, belongs to a special class of schemata. An example of such a class is given by Proposition 2. The element δz of the tangent space to Cl^2 at the point $Z_0 = x^\alpha y^\beta + x^\lambda y^\mu$ has the form

$$\begin{aligned} \delta z = (a(x) + b(y))x^\alpha y^\beta + c(x^\alpha y^\beta) + (p(x) + q(y))x^\lambda y^\mu \\ + r(x^\lambda y^\mu) + s(x^\alpha y^\beta + x^\lambda y^\mu). \end{aligned}$$

Composition schemata of this form are said to be *linear*. They depend on each of the free functional parameters linearly. Therefore, the set of functions of the above form is a vector space. In [14], Vitushkin

and Henkin considered continuous and smooth superpositions of this form. In our context of functions of two variables, by linear superpositions they meant schemata of the form

$$z(x,y) = f_1(\varphi_1(x,y)) + \cdots + f_m(\varphi_m(x,y)),$$

where the φ_j are fixed functions of two variables and the f_j are arbitrary functions of one variable (in a given class). We refer to schemata of this type as *linear schemata*.

Let S be any schema, and let $\mathrm{Cl}(S)$ be the corresponding class. Consider a fixed function $z_0(x,y) \in \mathrm{Cl}(S)$ and the schema $T_{z_0}S$ representing the tangent space to $\mathrm{Cl}(S)$ at the point z_0.

Proposition 10. *The schema $T_{z_0}S$ is linear.*

Note that, in this linear schema, the inner functions φ_j are determined by the chosen representation (parameter value) of z_0 and remain fixed in the tangent schema, and the outer functions f_j are essentially coordinates in the tangent space.

4. CONCLUSIONS

Schemata are functions of functional arguments. In functional analysis, linear functions of functional arguments are called linear functionals. Thus, we can say that a schema is a nonlinear functional. In the theory of differential equations and mathematical physics, schemata are used when the form of a solution to an equation is fixed a priori; such situations are called ansatzes. Like functions, schemata have a dual nature: they can be treated as numbers or as maps. On schemata treated as numbers we can perform arithmetic operations and on schemata treated as maps, the superposition operation. Accordingly, the complexity of a schema can be defined in different ways, as the depth (the number of embedded compositions) or as the number of arithmetic operations.

We defined Kl^n based on Kl^1 as the sum $\mathrm{Kl}^1 + \mathrm{Kl}^1 + \cdots + \mathrm{Kl}^1$ of n terms. This is what any schema of superposition depth 1 constructed by using addition looks like. As n grows, the depth does not change and remains equal to 1, and only the number of additions increases. If we took Cl^2 as a basis and formed the sum $\mathrm{Cl}^2 + \mathrm{Cl}^2 + \cdots + \mathrm{Cl}^2$ of n terms, then we would obtain a general form of schemata of depth 2 based on addition. The composition depth of the class Cl^n equals n and increases without bound with n.

We have seen above that, when we pass from continuous functions to analytic ones, the situation for the Kl hierarchy changes drastically. Consider the continuous version Cl_c of the hierarchy Cl. It consists of nondecreasing sequences of classes Cl^n_c, which are defined in the same way as Cl^n but for continuous functions instead of analytic ones. It is easy to show that $\mathrm{Kl}^5_\mathrm{c} \subseteq \mathrm{Cl}^4_\mathrm{c}$. Moreover, by virtue of Kolmogorov's theorem, the sequence Kl^n_c stabilizes: each class, starting with Kl^5_c, absorbs all continuous functions. Therefore, the sequence Cl^n stabilizes starting with Cl^4_c or earlier. It is also known that the class $\mathrm{Cl}^1_\mathrm{c} = \mathrm{Kl}^1$ does not contain all continuous functions [3]; thus, $\mathrm{Cl}^1_\mathrm{c} \neq \mathrm{Cl}^4_\mathrm{c}$.

Question 11. At which step does the sequence $\{\mathrm{Cl}^n_\mathrm{c}\}$ stabilize?

There are three possible answers, $n = 2$, $n = 3$, and $n = 4$, but only one of them is correct.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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