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Associativity and Distributivity of Analytic Functions

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Abstract—It is proved that, up to natural equivalence, there exist exactly five locally associative analytic functions of two variables: 0, x, y, $(x+y)$, and xy. A description of all associative (nonlocally) polynomials is given. All distributive pairs where at least one of the functions is locally associative are also described.

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1. INTRODUCTION

The focus of complex analysis is holomorphic or, more broadly, multivalued analytic functions of one or more complex variables. A complex-valued function of a complex variable is an object that can be viewed from two points of view. The value of a function is a complex number and, therefore, arithmetic operations can be performed on functions. On the other hand, functions are mappings and, therefore, the operation of superposition (substitution) can be applied to them. Thus, having a certain basic stock of functions, as well as the arithmetic operations and the operation of superposition mentioned above, one can successively expand the basic stock of functions, constructing new classes. For example, one can take as a basic stock all analytic functions that depend only on one variable out of two possible ones: x or y. The set of such analytic functions will be denoted by Cl^0 (functions of analytical complexity zero). And then, using an arbitrary arithmetic operation $a \diamondsuit b$ and the substitution into an analytic function of one variable, we can form the class $\mathrm{Cl}^1 = \{z(x,y) = c(a(x) \diamondsuit b(y))\}$ (the functions of analytical complexity not higher than one). The resulting class does not depend on the choice of one of the four arithmetic operations, and we can assume that the operation is the addition. Based on Cl^1, we can construct Cl^2, and so on. Thus, a growing hierarchy of classes arises, with which many interesting questions are associated and to which a number of the author's works are devoted (see [1] and further). In the context of this study, it was shown that Cl^1 occupies a special place among functions of two variables.

The holomorphic function of two variables $F(x,y)$ can be subject to the same requirements that are usually imposed on binary algebraic operations: commutativity and associativity. And if we have a pair of such functions $F(x,y)$ and $G(x,y)$, then we can formulate the condition of distributivity (right and left). What commutative functions are, i.e., functions with the condition $F(y,x) = F(x,y)$, is quite clear. In this paper, the question of associative functions is considered. It will be refined and it will be shown (Corollary 8 to the main theorem, Theorem 3) that any locally associative function is contained in Cl^1 and is commutative.

On the other hand, an interest in associative analytic operations has taken shape as the theory of formal groups ([2], [3]). This theory arose as an analytic apparatus for constructing the Lie algebra of a Lie group. This theory is effectively used in class field theory. It should be noted that the theory of formal groups works with binary operations on multidimensional variables, while we consider operations

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on one-dimensional quantities. However, a binary operation $F(x,y)$, which defines a formal group, must satisfy, along with the associativity condition $F(F(x,y),z)=F(x,F(y,z))$, the conditions $F(x,0)=x$ and $F(0,y)=y$ as well (this means that zero is the neutral element). In our considerations presented in this paper, we do not impose any additional conditions on F other than associativity. Our object is an analytic function F, which a priori does not claim to realize any algebraic structure. As a result, the description of associative functions we obtain goes beyond the scope of this theory. Note also that the transformation M_g (see (3)), which is important for us, is a homomorphism of a formal group. In the theory of formal groups, it is customary to initially understand the basic operation F as a formal power series. Note that all the results obtained in the paper are transferred without change to the category of formal power series.

2. LOCAL ASSOCIATIVITY

A binary operation $x\diamondsuit y$ is said to be *associative* if $(x\diamondsuit y)\diamondsuit z=x\diamondsuit(y\diamondsuit z)$. Let us use the functional notation. Instead of $x\diamondsuit y$ we write $F(x,y)$, keeping in mind that F is a holomorphic function in a domain of the 2-dimensional complex space. The associativity condition takes the form $F(F(x,y),z)=F(x,F(y,z))$. In order for the expressions involved in this relation to be defined somewhere, there must be a point (a,b,c) in a 3-dimensional space such that (a,b) and (b,c) fall into the domain of F and if $p=F(a,b)$ and $q=F(b,c)$, then (p,c) and (a,q) also fall into the domain of F. The associativity property understood in this way is not local. Let us first consider the local version of this question. Namely, let $F(u,v)$ be a function of two variables, holomorphic in a neighborhood of the origin, and let $F(0,0)=0$. Then, as is easy to see, we can set $(a,b,c)=(0,0,0)$ and regard the associativity condition as a relation that is identically satisfied in a neighborhood of the origin in $\mathbf{C}^3$.

Definition 1. We say that F is *locally associative* if

$$\begin{gathered}F \text{ is holomorphic in a neighborhood of } (0,0),\quad F(0,0)=0,\quad \text{and}\\ F(F(x,y),z)=F(x,F(y,z)) \text{ in a neighborhood of the point } (0,0,0).\end{gathered}\tag{1}$$

Consider the group $\mathcal{G}$ of holomorphic transformations of the form

$$\mathcal{G}=\big\{u\to g(u)=\lambda u+o(u),\ \lambda\neq 0\big\}.\tag{2}$$

This group acts on holomorphic functions of two variables in a neighborhood of the origin as follows:

$$F(x,y)\to M_g\circ F(x,y)=F_g(x,y)=g\circ F(x,y)=g^{-1}\circ F\big(g(x),g(y)\big).\tag{3}$$

Let $F(x,y)$ be holomorphic in a neighborhood of the origin and $F(0,0)=0$. It is easy to verify that the assertion holds.

Proposition 2. *The action* (3) *preserves the local associativity*; *i.e., F is locally associative if and only if $M_g\circ F$ is locally associative for any $g\in\mathcal{G}$.*

Proof. Let us write out the associativity condition for

$$w(x,y)=M_g\circ F(x,y)=g^{-1}\circ F(g(x),g(y)),$$

i.e., $w(w(x,y),z)=w(x,w(y,z))$; we obtain

$$g^{-1}\circ F\big(g\circ g^{-1}\circ F(g(x),g(y)),g(z)\big)=g^{-1}\circ F\big(g(x),g\circ g^{-1}\circ F(g(y),g(z))\big),$$

i.e.,

$$F\big(F(g(x),g(y)),g(z)\big)=F\big(g(x),F(g(y),g(z))\big).$$

Assuming that $X=g(x)$, $Y=g(y)$, $Z=g(z)$, we see that $F(F(X,Y),Z)=F(X,F(Y,Z))$, i.e., the associativity condition of F. The statement is proved. □

The action (3) introduces an equivalence relation on the germs of functions $F(x,y)$ at the origin with the condition $F(0,0)=0$. The following main theorem is proved below.

Theorem 3. *Let $F(x,y)$ be locally associative. Then either it is equal to one of the three functions $F_1=0$, $F_2=x$, and $F_3=y$, or F is equivalent (in the sense of the $\mathcal{G}$-action) to one and only one of the two associative functions (in the canonical form) $F_4=x+y$ and $F_5=xy$.*

It should be noted that F_1, F_2, and F_3 are not changed by the action of $\mathcal{G}$. Using this action, new locally associative functions can be obtained only from F_4 and F_5. If in (3) we set $g(t)=\ln(t)$, then $g\circ(x+y)=xy$. But $\ln(t)$ is not holomorphic at zero, so this transformation does not belong to $\mathcal{G}$. Such transformations do not preserve the holomorphy property of F in a neighborhood of $(0,0)$ and the condition $F(0,0)=0$. With respect to the action of $\mathcal{G}$, the functions $x+y$ and xy are nonequivalent. They have different orders at $(0,0)$ (the minimum degree of a nonzero monomial in a power expansion), for the sum this is one, and for the product this is two. Note also that, among the five canonical locally associative functions, only $F_4=x+y$ fits into the context of formal groups.

Let us proceed to the proof of Theorem 3. In a neighborhood of the origin, the function F can be represented in the form $F(x,y)=p(x)+q(y)+xyf(x,y)$, where $p(x)=F(x,0)$, $q(y)=F(0,y)$, $f(x,y)=(F(x,y)-p(x)-q(y))/(xy)$, and $p(0)=q(0)=0$.

Lemma 4. *If F is locally associative, then $p(x)$ is either x or 0. Similarly, $q(y)$ is either y or 0.*

Proof. Substitute $F(x,y)=p(x)+q(y)+xyf(x,y)$ into (1) and set $y=z=0$; then we obtain $p(x)=p(p(x))$. Let p vanish at zero with finite multiplicity $m>1$; then the multiplicity for $p(p(x))$ is $2m$. A contradiction, i.e., either p is identically zero, or $m=1$. Let $m=1$, i.e., $p(x)=\lambda(x+\mu x^n)+o(x^n)$, $\lambda\neq 0$; then $p(p(x))=\lambda^2(x+2\mu x^n)+o(x^n)$, whence we see that $\lambda=1$, $\mu=0$. Whence the assertion about $p(x)$ follows. The assertion about $q(y)$ is proved similarly. The lemma is proved. □

Thus, a locally associative function F has one of the four representations:

$$\begin{aligned}&(\text{case } 1)\quad F=xyf(x,y),\\ &(\text{case } 2)\quad F=x+xyf(x,y),\\ &(\text{case } 3)\quad F=y+xyf(x,y),\\ &(\text{case } 4)\quad F=x+y+xyf(x,y).\end{aligned}$$

Let us consider these cases.

Lemma 5. *If F is a locally associative function with case 2 representation, then $F(x,y)=x$, and if case 3 holds, then $F(x,y)=y$.*

Proof. Substituting $F(x,y)=x+xyf(x,y)$ into (1) and setting $y=0$, we obtain $xzf(x,z)=0$. Case 3 is similar. The proof of the statement is complete. □

Thus, the remaining two types of locally associative functions need further study:

$$\begin{aligned}F(x,y)&=xyf(x,y)\qquad(\text{case } 1),\\ F(x,y)&=x+y+xyf(x,y),\qquad(\text{case } 4).\end{aligned}$$

Case 1. Let $F(x,y)=xyq(y)+x^2y\varphi(x,y)$. Clearly, $yq(y)=F'_x(0,y)$. Let $(t\to g(t),\ g(0)=0,\ g'(0)\neq 0)$ be a holomorphic change in a neighborhood of zero and

$$\widetilde{F}(x,y)=g^{-1}\circ F\big(g(x),g(y)r\big)=xy\widetilde{q}(y)+x^2y\widetilde{\varphi}(x,y).$$

Then

$$y\widetilde{q}(y)=\widetilde{F}'_x(0,y)=g(y)q(g(y)).$$

That is, the function $Q(y)=yq(y)$ is transformed using the change $y\to g(y)$.

If $q(y)$ is not identically zero (case 1.1), then we can choose g so that, after the substitution, we obtain $\widetilde{q}(y)=y^k$, $k\geq 0$. Indeed, if q vanishes at zero with finite order k, then Q has the order $k+1$ and $Q(y)$ can be written out in the form $(\varphi(y))^{k+1}$, where φ has the order equal to one at zero and is

locally invertible. Setting $g=\varphi^{-1}$, we see after the substitution $Q(y)=y^{k+1}$ that $q(y)=y^k$. Thus, now $F(x,y)=xy^{k+1}+x^2yf(x,y)$. If $f=0$ in this case, then, as a direct verification shows, for the associativity of F it is necessary and sufficient that $k=0$. In this case, $F=xy$ (case 1.1.1.).

Let $f\neq 0$ (case 1.1.2). Let us substitute F into (1), reduce by xyz, and put $x=0$; we obtain

$$f(y,z)\big(yz(z^k+yf(y,z))\big)^k y-z^k y^k+z^k\big(yz(z^k+yf(y,z))\big)^k=0. \tag{4}$$

Let $m=\operatorname{ord}(f)$ be the order of f in $(0,0)$, i.e., m is the minimum among the degrees of nonzero monomials occurring in the expansion of f in a power series at the origin. If f_j is the sum of monomials from the expansion of f of degree j, then $f_m\neq 0$, and, for $j<m$, $f_j=0$. Let us consider the orders at zero of all three terms occurring in (4). Every term is nonzero, and thus the equality of all three orders is necessary for (4) to hold.

Let $k\leq m+1$; then

$$\operatorname{ord}\big(f(y,z)(yz(z^k+yf(y,z)))^k yr\big)=(m+1)+(k+2)k,$$
$$\operatorname{ord}(z^k y^k)=2k,\qquad \operatorname{ord}\big(z^k(yz(z^k+yf(y,z)))^k\big)=(k+2)k+k.$$

If $k>m+1$, then the orders are of the form

$$\operatorname{ord}\big(f(y,z)(yz(z^k+yf(y,z)))^k y\big)=(m+1)k+2k+m+1,$$
$$\operatorname{ord}(z^k y^k)=2k,\quad \operatorname{ord}\big(z^k(yz(z^k+yf(y,z)))^k\big)=(m+3)k+k.$$

In both cases, the order of the second term is minimal, which is $2k$. A contradiction, i.e., case 1.1.2 is impossible.

Now let $q(y)=0$ be an identity (case 1.2), i.e., $F(x,y)=x^k yf(x,y)$, $k\geq 2$.

If F is identically zero, then F is associative (1.2.1).

Otherwise, (1.2.2), we can assume that f is not divisible by x, i.e., $r(y)=f(0,y)$ is not identically zero. Let us write out the associativity condition. After reducing by $(xy)^k$, we obtain

$$f(y,z)f(x,y^k zf(y,z))-\big(x^{k-1}f(x,y)\big)^k f(x^k yf(x,y),z)=0. \tag{5}$$

Substituting $x=0$, we have $f(y,z)r(y^k zf(y,z))=0$. Whence it follows that $f(y,z)=0$ and, accordingly, $F(x,y)=0$.

Thus, we have proved the following statement.

Lemma 6. *If a function of the form $F=xyf(x,y)$ is associative, then it is either equal to 0 or equivalent to xy.*

Case 4. If $F(x,y)=x+y+xyf(x,y)$, then

$$A(x)=F'_y(x,0)=1+xf(x,0).$$

If $\widetilde{F}(x,y)=g^{-1}\circ F(g(x),g(y))$, $g\in\mathcal{G}$, then

$$\widetilde{A}(x)=\widetilde{F}'_y(x,0)=\frac{g'(0)}{g'(g(x))}A(g(x)).$$

We choose for g the solution to the differential equation $g'(u)=A(g(u))$, $g(0)=0$. In this case, $g'(0)=A(0)=1$. Then $A(g(x))=g'(g(x))$, and we obtain $\widetilde{A}(x)=1$.

Therefore, in what follows we assume that $F'_y(x,0)=1$ and $f(x,y)$ is divisible by y. If this is not identically zero, then (after changing the notation) we write $F(x,y)=x+y+xy^m f(x,y)$, where $m\geq 2$ and $f(x,0)$ is not identically zero. Substituting this expression into (1), we obtain $E(x,y,z)=0$.

Substituting $y=0$ into E'_y, reducing this expression by z^{m-1}, and substituting $z=0$, we obtain $mf(x,0)=0$. This contradiction shows that f is identically zero and $F=x+y$.

Thus, we have proved the following assertion.

Lemma 7. *If a function of the form $F = x + y + xyf(x, y)$ is associative, then it is equivalent to $x + y$.*

The proof of Lemma 7 completes the proof of Theorem 3.

Corollary 8 (to Theorem 3)**.** (a) *If $F(x, y)$ is associative, then $F \in \mathrm{Cl}^1$ (for the class of functions of analytic complexity at most one, see* [1]).

(b) *Let $F(x, y)$ be associative and depend on both variables* (*i.e., $F'_x F'_y$ is not identically zero*). *Then F is commutative* (*i.e., $F(y, x) = F(x, y)$*).

Proof. All functions in Theorem 3 are contained in Cl^1. The transformations (3) do not change the complexity. This proves (a). If $F'_x F'_y \neq 0$, then F is equivalent to either $(x + y)$ or xy, but these functions are commutative, and the transformations (3) do not violate the commutativity. This proves (b). □

3. NONLOCAL ASSOCIATIVITY

Let a function $F(x, y)$ be nonconstant and meromorphic in $\mathbf{C}^2$, i.e., $F(x, y)$ Is $P(x, y)/Q(x, y)$, where P and Q are coprime entire functions. Then, in order to write out the associativity condition, it is not necessary to impose the condition $F(0, 0) = 0$. Indeed, the common domain of both parts of (1) is the open dense set

$$\big\{(x, y, z)\colon Q(F(x, y), z) \neq 0\big\} \cap \big\{(x, y, z)\colon Q(x, F(y, z)) \neq 0\big\} \cap \big\{(x, y, z)\colon Q(x, y) \neq 0\big\};$$

therefore, the following definition can be given.

Definition 9. We say that a meromorphic function $F(x, y)$ in $\mathbf{C}^2$ is *associative* if

$$F(F(x, y), z) = F(x, F(y, z))$$

at all points at which both sides of the equality are defined.

In particular, this definition is valid for rational, integer, or polynomial functions.

Theorem 10. *All associative polynomials $F(x, y)$ are contained in the following list* (p *and* q *are arbitrary constants*):

$$F_1 = 0, \qquad F_2 = x, \qquad F_3 = y, \qquad F_4 = x + y + pxy, \qquad F_5 = pxy,$$
$$F_6 = q + pq(x + y) + p(p - 1)xy.$$

Proof. Let

$$F(x, y) = a_0(x) + a_1(x)y + \cdots + a_m(x)y^m$$

be a nonzero associative polynomial of degree m with respect to y, i.e., $a_m(x) \neq 0$. The term of the polynomial $F(F(x, y), z)$ of the highest degree with respect to z is $a_m(F(x, y))z^m$, and the leading term with respect to z of the polynomial $F(x, F(y, z))$ is $(a_m(y)z^m)^m$, from which it follows that $m \leq 1$, i.e., $F(x, y) = a_0(x) + a_1(x)y$. Substituting into (1) and separating the powers with respect to z, we obtain

$$a_0(x) + a_1(x)a_0(y) - a_0(a_0(x) + a_1(x)y) = 0,$$
$$a_1(x)a_1(y) - a_1(a_0(x) + a_1(x)y) = 0. \tag{6}$$

Let m_0 be the power of $a_0(x)$ and m_1 be the power of $a_1(x)$; then it follows from (6) that $m_0 \leq 1$. Let $m_0 = 0$, i.e., a_0 is a constant. Then the first relation (6) takes the form $a_1(x)a_0 = 0$. And if $a_0 \neq 0$, then $a_1(x) = 0$ and $F = a_0$ is a constant. If $a_0 = 0$, then the second relation (6) takes the form

$$a_1(x)a_1(y) - a_1(a_1(x)y) = 0,$$

whence it follows that $m_1 \leq 1$. Thus, the degrees of F both in x and in y do not exceed 1. Substituting the general form of such a polynomial into (6), we finally find the complete list of solutions, which is presented in the formulation. The theorem is proved. □

Note that all the polynomials thus obtained have analytical complexity not exceeding one.

If we consider these polynomials up to the action of $\mathcal{G}$, then $(F_1, F_2, F_3, F_4, F_5)$ give all five canonical associative functions in Theorem 3. To reduce the family F_6 to the canonical form, we consider the action of M_h (see (3)) generated by the shift $h(t) = t + a$. The (nonlocal) associativity is preserved (although h is not contained in $\mathcal{G}$). Then

$$\begin{aligned}\widetilde{F}_6(x,y) &= F_6(x+a, y+a) - a \\ &= p^2q(x+a)(y+a) + pq(x+a) + pq(y+a) - (x+a)(y+a)p + q - a.\end{aligned}$$

Therefore,

$$\widetilde{F}_6(0,0) = F_6(a,a) - a = (ap+1)(pqa - a + q),$$

and if $a = q/(1-pq)$, then $\widetilde{F}_6(x,y) = xyp(pq-1)$, and this function, by a transformation in $\mathcal{G}$, either turns into xy or is 0.

This technique can also be used to describe more general situations. Indeed, if there exists a constant a such that $F(a,a) - a = 0$, then, for $h(u) = u + a$, the function

$$\widetilde{F} = M_h \circ F = F(a+x, a+y) - a$$

is locally associative, and Theorem 3 applies to this function.

Proposition 11. *Let $F(x,y)$ be associative and meromorphic in $\mathbf{C}^2$, and let there exist an $a \in \mathbf{C}$ such that $F(a,a) = a$. Then*

$$\begin{gathered}\textit{either} \quad F(x,y) = \text{const}, \qquad \textit{or} \quad F(x,y) = x, \qquad \textit{or} \quad F(x,y) = y, \\ \textit{or} \quad F(x,y) = g^{-1} \circ \big(g(x-a) + g(y-a)\big) + a, \\ \textit{or} \quad F(x,y) = g^{-1} \circ \big(g(x-a)g(y-a)\big) + a, \quad \textit{where } g \in \mathcal{G}.\end{gathered}$$

Proof. Applying Theorem 3 to $\widetilde{F}(x,y) = (F(a+x, a+y) - a)$, we see that $M_{g^{-1}} \circ \widetilde{F}(x,y) = F_j(x,y)$, where F_j is one of the five canonical locally associative functions. Thus, $\widetilde{F}(x,y) = M_g \circ F_j(x,y)$ and $F(x,y) = M_g \circ [F_j(x-a, y-a) + a]$. Successively substituting the functions in Theorem 3 for F_j, we obtain the assertion of the theorem. □

What can be said about the function $g \in \mathcal{G}$ if it is known that F is rational and falls into the class of addition or multiplication?

Consider the additive case. Let a rational function F satisfy the relation

$$g(F(x,y)) = g(x-a) + g(y-a) + a.$$

We assume that the function $F'_x F'_y$ is not identically zero (i.e., F is a function of two variables). Differentiating the relation with respect to x and y, we see that, in a neighborhood of the point (a,a), the following relation holds:

$$g'(F(x,y))F''_{xy}(x,y) + g''(F(x,y))F'_x(x,y)F'_y(x,y) = 0. \tag{7}$$

Let us pass from the coordinates (x,y) to the coordinates (x,t), where $t = F(x,y)$; then this relation defines y as an algebraic function $y = Y(x,t)$. And relation (7) takes the form

$$\frac{g''(t)}{g'(t)} = -\frac{F''_{xy}(x, Y(x,t))}{F'_x(x, Y(x,t))F'_y(x, Y(x,t))} = A(t), \qquad \text{where} \quad A(t) \text{ is algebraic.} \tag{8}$$

Moreover, the fact that the right-hand side (8) does not depend on x is a differential condition on F (an implicit differentiation). If we denote the operation of taking an antiderivative by D^{-1}, then the solution of (8) can be written in the form

$$g(t) = D^{-1}\big(\exp(D^{-1}(A))\big)(t).$$

If $(a_1,\dots,a_s)$ is the set of singular points of the function $A(t)$ on the complex plane of the variable t, then the function $g(t)$ is analytic on the plane without these points. However, unlike $A(t)$, the function $g(t)$ can be infinite-valued and have nonalgebraic branch points. Similarly, we can analyze the case of $g(F(x,y)) = g(x-a)g(y-a)+a$.

Note also that if we use the transformations M_h generated by the fractional-linear transformations

$$h(u) = \frac{\alpha u+\beta}{\gamma u+\delta}, \qquad \det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \neq 0,$$

then, applying them to $x+y$ and xy, we obtain two 3-parameter families of associative rational functions of degree two:

$$M_h\circ(x+y) = \frac{-2\alpha\delta\gamma xy+\beta\gamma^2 xy-\alpha\delta^2 x-\alpha\delta^2 y-\beta\delta^2}{\alpha\gamma^2 xy+\beta\gamma^2 x+\beta\gamma^2 y-\alpha\delta^2+2\beta\delta\gamma}$$

$$M_h\circ(xy) = -\frac{-\alpha^2\delta xy+\beta\gamma^2 xy-\alpha\beta\delta x-\alpha\beta\delta y+\beta\delta\gamma x+\beta\delta\gamma y-\beta^2\delta+\beta\delta^2}{-\alpha^2\gamma xy+\alpha\gamma^2 xy-\alpha\beta\gamma x-\alpha\beta\gamma y+\alpha\delta\gamma x+\alpha\delta\gamma y+\alpha\delta^2-\beta^2\gamma}.$$

4. LOCALLY DISTRIBUTIVE PAIRS

Let there be two binary operations $x\oplus y$ and $x\diamondsuit y$. Consider two relations

$$\begin{aligned}&(R)\quad (x\oplus y)\diamondsuit z = (x\diamondsuit z)\oplus(y\diamondsuit z),\\ &(L)\quad z\diamondsuit(x\oplus y) = (z\diamondsuit x)\oplus(z\diamondsuit y).\end{aligned}\tag{9}$$

It is customary to say that the operation $\diamondsuit$ is distributive with respect to $\oplus$ from the right if the first relation is satisfied, and from the left if the other holds. Moreover, if the second operation is commutative, then the difference between right and left distributivity disappears.

Let us write out these operations below using the functional notation, i.e., now $a\oplus b = F(a,b)$ and $a\diamondsuit b = G(a,b)$. Then (9) takes the form

$$\begin{aligned}&(R)\quad G(F(x,y),z) = F(G(x,z),G(y,z)),\\ &(L)\quad G(z,F(x,y)) = F(G(z,x),G(z,y)).\end{aligned}$$

Let in a neighborhood of the point $(0,0)$ there be a pair of holomorphic functions $F(x,y)$ and $G(x,y)$, where $F(0,0)=G(0,0)=0$.

Definition 12. We say that G is *locally right distributive* with respect to F if (R) holds and *locally left distributive* with respect to F if (L) holds.

The group $\mathcal{G}$ (see (2)) acts on the pairs (F,G):

$$\begin{gathered}(F,G)\to(M_g\circ F, M_g\circ G) = g\circ(F,G),\\ g\circ F(x,y) = g^{-1}\big(F(g(x),g(y))\big), \qquad g\circ G(x,y) = g^{-1}\big(G(g(x),g(y))\big), \qquad g\in\mathcal{G}.\end{gathered}\tag{10}$$

Proposition 13. *The action of $\mathcal{G}$ on the pairs* (10) *preserves both the right and left local distributivity.*

This is verified directly.

Let one of the functions of the pair, say, F, be locally associative. Then, when studying locally distributive pairs (F,G), we can, using Theorem 3, assume that F is given in locally canonical form. Substituting all five locally associative functions for F into both distributivity conditions, we obtain ten conditions on $G(x,y)$ (five right and five left):

$$\begin{array}{lll}
F_1=0, & (R_1)\quad G(0,z)=0, & (L_1)\quad G(z,0)=0,\\
F_2=x, & (R_2)\quad 0=0, & (L_2)\quad 0=0,\\
F_3=y, & (R_3)\quad 0=0, & (L_3)\quad 0=0;\\
F_4=x+y, & (R_4)\quad G(x+y,z)=G(x,z)+G(y,z), & \\
 & (L_4)\quad G(z,x+y)=G(z,x)+G(z,y); & \\
F_5=xy, & (R_5)\quad G(xy,z)=G(x,z)G(y,z), & \\
 & (L_5)\quad G(z,xy)=G(z,x)G(z,y). &
\end{array}\tag{11}$$

When writing out solutions of distributivity equations, in addition to the conditions of right and left distributivity, we will also note the condition (RL) of two-sided distributivity.

Proposition 14. *The list of solutions of Eqs.* (11) *has the following form*:

$$F = 0,$$
$$(R)\quad G(x,y) = xr(x,y), \qquad (L)\quad G(x,y) = yl(x,y), \qquad (RL)\quad G(x,y) = kxy;$$

$$F = x \text{ or } y,$$
$$(RL)\quad G(x,y) \text{ is an arbitrary function};$$

$$F = x + y,$$
$$(R)\quad G(x,y) = xq(y), \qquad (L)\quad G(x,y) = yp(x), \qquad (RL)\quad G(x,y) = kxy$$

$$F = xy,$$
$$(R)\quad G(x,y) = 0 \text{ or } x^n, \qquad (L)\quad G(x,y) = 0 \text{ or } y^n, \qquad (RL)\quad G(x,y) = 0.$$

Proof. Let us consider the case $F = x + y$. The right distributivity condition is of the form

$$G(x + y, z) = G(x, z) + G(y, z).$$

In this relation, the second variable z occurs as a parameter. Let $f(t)$ be $G(t, z)$ for some chosen value z. The equation is of the form $f(x + y) = f(x) + f(y)$. Applying the operator $(\partial/\partial x - \partial/\partial y)$ to this relation, we obtain $f'(x) = f'(y)$. Therefore, f is linear and $G(x, y) = xq(y)$. The remaining cases are treated similarly. The statement is proved. □

Corollary 15. *The list of pairs of functions (F, G), where F is locally associative and G is locally two-way distributive with respect to F, has the form*

$$(F = 0,\ G = kxy), \qquad (F = x,\ G = f(xy)), \qquad (F = y,\ G = f(xy)),$$
$$(F = x + y,\ G = kxy), \qquad (F = xy,\ G = 0), \qquad k \text{ is an arbitrary constant}.$$

Note that in all cases except for $F = x$ or y, the corresponding function G is also locally associative.

The relation "to be distributive" is not symmetric; F and G enter into it unequally. Using Theorem 3, we can find out the functions F with respect to which the locally associative functions G are locally distributive. To do this, using a transformation M_g, we translate G into one of the five canonical forms and find the type of function F with respect to which the function G is distributive. Simple calculations allow us to give a complete answer. In all cases where the canonical form is commutative, the difference between right and left distributivity disappears.

Theorem 16. *All pairs (F, G) such that the function G is locally distributive with respect to F and G is locally associative and written in a canonical form* (*see Theorem* 3) *are contained in the following list*:

$$(RL)\quad (F \text{ is arbitrary},\ G = 0),$$
$$(R)\quad (F \text{ is arbitrary},\ G = x), \qquad (L)\quad (F = x + f(x,y)(x - y),\ G = x),$$
$$(L)\quad (F \text{ is arbitrary},\ G = y), \qquad (L)\quad (F = y + f(x,y)(x - y),\ G = y),$$
$$(RL)\quad (F = x + \varphi(x - y),\ G = x + y),$$
$$(RL)\quad (F = \lambda x + \mu y,\ G = xy).$$

Note that the functions $x + \varphi(x - y)$ (a pair for $(x + y)$) and $\lambda x + \mu y$ (a pair for xy) are associative in exceptional cases only. Thus, the function $F = x + \varphi(x - y)$ is locally associative in two cases only: $F = x$ for $\varphi(t) = 0$ and $F = y$ for $\varphi(t) = -t$. And the function $\lambda x + \mu y$ is locally associative in four cases only: 0, x, y, and $x + y$.

In conclusion we note that our description of nonlocally associative rational functions remains incomplete. It does not cover rational functions $F(x, y)$ such that $F(x, x) - x$ has no zeros. The question

of describing these rational functions is open. In particular, two questions remained outside the scope of our considerations: (a) Do associative rational functions of degree greater than two exist? (b) Do associative rational functions of analytic complexity greater than one exist?

Note also that the transformation M_g, which is used in this paper, can be regarded as a transformation of functions of one variable, i.e., $M_g(f(x)) = g^{-1}(f(g(x))$. This is the kind of equivalence of germs of functions holomorphic at the origin that is fundamental in questions of holomorphic dynamics ([4], [5]). It would be interesting to understand how the classification of germs of holomorphic functions of two variables is structured with respect to the action M_g of the group $\mathcal{G}$, which played a leading role in this paper.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

REFERENCES

1. V. K. Beloshapka, "Analytic complexity of functions of two variables," Russ. J. Math. Phys. **14** (3), 243–249 (2007).
2. J.-P. Serre, *Lie algebras and Lie groups. 1964 lectures, given at Harvard University, 2nd ed.* (Springer-Verlag, Berlin et al., 1992).
3. M. Hazewinkel, *Formal Groups and Applications*, in *Pure Appl. Math.* (Academic Press, New York–London, 1978), Vol. 78.
4. S. M. Voronin, "Analytic classification of germs of conformal mappings $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with identity linear part," Funct. Anal. Appl. **15** (1), 1–13 (1981).
5. J. Milnor, *Dynamics in One Complex Variable, 3rd ed.* (Princeton University Press, Princeton, 2006).

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