

Geometric constructions in the theory of analytic complexity

V. K. Beloshapka

Abstract. Two geometric constructions are considered in the context of analytic complexity. Using the first construction, on the set of analytic functions, we build a metric invariant under the action of the gauge group. With the help of the second construction, we obtain a necessary differential algebraic condition for membership of a function in the tangent space to the class of bivariate functions of analytic complexity ≤ 2 at the point $z_0 = x^3y^2 + xy$. From this result we show that the polynomial $z = x^3y^2 + xy + \pi x^2y^3$ of degree 5 has analytic complexity 3.

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§ 1. Introduction

With the help of the superposition operation one can obtain functions of larger number of variables from those of smaller number of variables. In this field, plenty of excellent results are known, but still there are many open problems (see [1]–[5]). To be more precise, we need to define the class of functions and the number of variables. In the problem of representation of bivariate functions via univariate functions (the “2 \rightarrow 1” problem), as well as in many other similar problems, there are many interesting unsolved problems. The present paper follows the studies (see [6], [7], etc.), in which this problem was discussed in the context of analytic functions.

The hierarchy of the classes Cl^n , $n = 0, 1, \dots$, is defined inductively. The class Cl^0 consists of the analytic univariate functions (x or y), and the class Cl^{n+1} consists of the analytic functions locally representable in the form $z = c(A_n(x, y) + B_n(x, y))$, where A_n and B_n are Cl^n -functions, and $c(t)$ is an analytic univariate function. One of the main characteristics $N(z)$ of an analytic bivariate function is its *complexity* $N(z)$. Here, we say that $N(z) = n$ if $z \in Cl^n \setminus Cl^{n-1}$; if z is not contained in any of these classes, we put $N(z) = \infty$. Note that the complexity of an element of an analytic function evaluated at a single point coincides with that at any other non-singular point. The set of singular points for representation of a function holomorphic in a domain of a two-dimensional space by a superposition with minimum complexity forms a proper analytic subset of this domain.

When constructing the hierarchy, one can replace the base function $(x + y)$ with an arbitrary analytic bivariate function $\varphi(x, y)$, thereby obtaining the hierarchy

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Cl_φ^n , and, correspondingly, the relative complexity $N_\varphi(z)$. Fixing some germ of a bivariate function $\varphi(x, y)$ such that $\varphi'_x \varphi'_y \neq 0$, we can define an increasing sequence of classes of functions. We set $Cl_\varphi^1 = Cl^0$, and then, by induction, define Cl_φ^{n+1} as the set of functions featuring germs equivalent under the action of the gauge group (see below) to germs of functions of the form $c(\varphi(A_n(x, y), B_n(x, y)))$, where $A_n(x, y), B_n(x, y) \in Cl_\varphi^n$.

The characteristics $N(z)$ and $N_\varphi(z)$ of the complexity are invariant under some action. Let f be a germ of an analytic bivariate function at the origin such that $f(0, 0) = 0$ (a standard germ). We let G denote the group of germs of holomorphic mappings of the form $\{u \rightarrow \lambda u + o(u)\}$, $\lambda \neq 0$. We also set $\mathcal{G} = G \times G \times G$. Next, given $g = (a(u), b(u), c(u)) \in \mathcal{G}$, we denote $\varphi(x, y) = c(f(a(x), b(y)))$ by $g \circ f$. In this case, we say that φ and f are equivalent.

If f is a germ at (p, q) , then $(f(x - p, y - q) - f(p, q))$ is a *standard* germ. The equivalence relation of standard germs generates that of arbitrary germs and of complete analytic functions. Assume that in a neighbourhood D of a point (x_0, y_0) there are two holomorphic functions $F_1(x, y)$ and $F_2(x, y)$ (elements of the class of analytic functions). Assume that the germ $\tilde{F}_1(x, y, x_0, y_0)$ is equivalent to the germ $\tilde{F}_2(x, y, x_0, y_0)$. Then, for each path $\gamma(t) = (x(t), y(t))$ emanating from (x_0, y_0) and lying in $D \setminus \sigma$ (σ is a proper analytic subset), there exists a t -continuous family of germs $g_t = (a_t(s), b_t(s), c_t(s)) \in \mathcal{G}$ such that $\tilde{F}_2(x, y, x(t), y(t)) = g_t \circ \tilde{F}_1(x, y, x(t), y(t))$, that is, the germs F_1 and F_2 are equivalent at each point of their general domain of definition except a proper analytic subset (the discriminant subset). In this sense, local equivalence of germs ceases to be local. If f is a germ, then the class of complete analytic functions equivalent to the function generated by this germ will be denoted by $[f]$.

The set of all analytic functions of two complex variables (x, y) has a natural structure of a bundle of germs. From this point of view, a complete analytic function generated by some concrete germ is a connected component of this bundle. The above equivalence relation is an equivalence relation on the bundle of germs.

It is also worth pointing out that, up to this equivalence, all analytic univariate functions break up into the following four classes: $z = 0$, $z = 1$, $z = x$, $z = y$.

§ 2. The metric space of analytic functions

Our next aim is to equip the set of analytic bivariate functions \mathcal{A} with a metric. Here, it is assumed that the constants and univariate functions are deleted from this set. So, \mathcal{A} is the set of analytic functions such that $F'_x F'_y$ is not identically zero.

Given two germs of analytic bivariate functions $\varphi(x, y)$ and $\psi(x, y)$, we introduce the following characteristic:

$$\rho([\varphi], [\psi]) = \log(\max\{N_{[\varphi]}([\psi]), N_{[\psi]}([\varphi])\}).$$

Theorem 1. *The function ρ defines a metric on \mathcal{A} .*

Proof. Since $N_{[\varphi]}([\psi]) \geq 1$, we have $\rho([\varphi], [\psi]) \geq 0$. The equality $\rho([\varphi], [\psi]) = 0$ is equivalent to saying that $[\varphi] = [\psi]$. That the function ρ is symmetric is clear from the definition. It remains to verify the triangle inequality.

Let us employ the following property of the relative complexity: given three analytic functions (φ, ψ, χ) with $N_\varphi(\psi) \leq l$, $N_\psi(\chi) \leq m$, we have $N_\varphi(\chi) \leq lm$. This implies the claim. Indeed, for all three classes $[\varphi]$, $[\psi]$, $[\chi]$, we have

$$\rho([\varphi], [\chi]) \leq \rho([\varphi], [\psi]) + \rho([\psi], [\chi]),$$

proving the theorem.

In the actual fact, the metric ρ is a semi-metric, since it may assume the value ∞ . Nevertheless, ρ can be easily converted to a metric which assumes only finite values. Indeed, if $h(s) = s/(1 + s)$, then

$$\tilde{\rho}([\varphi], [\chi]) = h(\rho([\varphi], [\chi]))$$

is also a metric on \mathcal{A} such that $\tilde{\rho}([\varphi], [\chi]) \leq 1$ for any pair of functions.

Since ρ may assume finite and infinite values, one may equip \mathcal{A} with another equivalence relation, *viz.*, “to be at a finite distance”. In this case, \mathcal{A} splits into the disjoint union with respect to “types”. The distance between any two functions (classes) from the same type is finite; the distance between functions from different types is infinite.

Note that the set of types is not countable.

The above metric is suitable for dealing with the complexity of type $(2 \rightarrow 1)$, that is, with its help one can study problems of representability of bivariate functions by superpositions of univariate functions. This metric can also be extended to the general case. For example, for functions of three variables, the following two approaches are possible: $(3 \rightarrow 1)$ and $(3 \rightarrow 2)$. In both these cases, a suitable metric can be introduced. Indeed, the only property of the metric that requires justification is the triangle inequality. But this inequality follows from the multiplicative estimate for the relative complexity, which holds in very broad setting.

§ 3. Complexity classes as submanifolds in the space of functions

With the above complexity classes Cl^n one may associate the radical differential ideal I^n consisting of differential polynomials with rational coefficients which annihilate the functions from $z \in Cl^n$ (see [7]). By the Ritt–Raudenbush basis theorem [8], each such an ideal has a finite basis $D_n = (d_1, \dots, d_l)$. Since each class Cl^n is invariant under the action of the gauge group \mathcal{G} , this basis contains only polynomials not depending explicitly on the variables (x, y, z) (this polynomials depend only of the derivatives of the function z satisfying some homogeneity conditions).

The differential ideal I^1 has the single generator

$$d(z) = z'_y{}^2 z'_x z'''_{xxy} - z'_y{}^2 z''_{xy} z''_{xx} - z'_y z'_x{}^2 z'''_{xyy} + z''_{yy} z'_x{}^2 z''_{xy}.$$

Since $Cl^1 = \{z : d(z) = 0\}$, one can easily verify that the complexity of $z = x^2 + xy$ relative to $x + y$ is 2.

The class Cl^2 consists of the functions of the form

$$z = S(C(A(x) + B(y)) + R(P(x) + Q(y))). \tag{3.1}$$

The ideals forming I^n can be found by certain algorithms. However, this requires immense amount of computation, which makes even the calculation of D_2 unfeasible (see [9]).

It can be easily shown that all polynomials of degree 2 lie in the class Cl^2 . It is also clear that a generic polynomial of degree 11 has complexity > 2 . The problem of constructing a concrete example of a polynomial of complexity 3 is much more involved. Examples of functions (in particular, polynomials) of any prescribed complexity were given in [10]. However, in these examples, even the polynomial of complexity 3 is of astronomically large degree.

In the present paper, we propose to consider the complexity classes as infinite-dimensional submanifolds of some function space. On this way, it will prove possible to constructively describe the tangent space to Cl^2 at some point. In turn, this will enable us to construct explicitly a polynomial of degree 5 of analytic complexity 3.

Let us proceed with this construction.

Let the function $Z_0 = x^3y^2 + xy$ be considered as a point in Cl^2 . The function Z_0 can be written as (3.1) with

$$\begin{aligned} S_0(u) &= u, & C_0(u) &= \exp(u), & R_0(u) &= \exp(u), \\ A_0(u) &= 3 \ln(u), & B_0(u) &= 2 \ln(u), & P_0(u) &= \ln(u), & Q_0(u) &= \ln(u). \end{aligned}$$

Let us perturb this family of seven functions as follows:

$$\begin{aligned} S(u) &= u + ts(u), & C(u) &= \exp(u) + tc(u), & R(u) &= \exp(u) + tr(u), \\ A(u) &= 3 \ln(u) + ta(u), & B(u) &= 2 \ln(u) + tb(u), \\ P(u) &= \ln(u) + tp(u), & Q(u) &= \ln(u) + tq(u), \end{aligned}$$

here, each of the functions is augmented with a perturbation linear with respect to the parameter t .

So, we draw the straight line with direction $(s(u), c(u), r(u), a(u), b(u), p(u), q(u))$ through the point $(u, \exp(u), \exp(u), 3 \ln(u), 2 \ln(u), \ln(u), \ln(u))$ in the space of seven function parameters

$$(S(u), C(u), R(u), A(u), B(u), P(u), Q(u)).$$

Here, the functions (s, c, r, a, b, p, q) are analytic, and the domain of definition of the composition is non-empty. In the perturbed expression $Z(x, y, t) = (x^3y^2 + xy) + tz(x, y) + o(t)$, we single out the term linear with respect to t . As a result, we get the parametric description of $T Cl^2_{(x^3y^2+xy)}$, which is the tangent space to Cl^2 at the point $Z_0 = (x^3y^2 + xy)$:

$$z(x, y) = s(x^3y^2 + xy) + c(x^3y^2) + x^3y^2(a(x) + b(y)) + r(xy) + (p(x) + q(y))xy. \quad (3.2)$$

Our next aim is to change from the parametric description of the tangent space to the relations that define this space. To this end, we exclude the function parameters (s, c, a, b, p, q, r) from relation (3.2). In what follows, derivatives will be indicated by appending a lower index; that is, $a_p, b_q, z_{m,n}$, etc. The set of the derivatives of the function z of orders $\leq n$ inclusively will be denoted by $z^{(n)}$ ($a^{(n)}$, etc., have the same meaning).

We set

$$\Delta_s = \frac{\partial/\partial x}{3x^2y^2 + y} - \frac{\partial/\partial y}{2x^3y + x}, \quad \Delta_c = \frac{\partial/\partial x}{3x^2y^2} - \frac{\partial/\partial y}{2x^3y}, \quad \Delta_r = \frac{\partial/\partial x}{y} - \frac{\partial/\partial y}{x}.$$

The kernel of Δ_s consists of the functions of the form $s(x^3y^2 + xy)$, the kernel of Δ_c consists of functions of the form $c(x^3y^2)$, and the kernel of Δ_r consists of functions of the form $r(xy)$. Applying the differential operator Δ_s to (3.2), we get

$$\begin{aligned} e_1(a^{(1)}, b^{(1)}, c_1, p_1, q^{(1)}, r^{(1)}, z^{(1)}) &= 2a_1x^6y^3 - 3b_1x^5y^4 + a_1x^4y^2 + 2p_1x^4y^2 \\ &\quad - b_1x^3y^3 - 3q_1x^3y^3 + a_0x^3y^2 + b_0x^3y^2 + c_1x^3y^2 - p_0x^3y^2 - q_0x^3y^2 \\ &\quad - r_1x^3y^2 - 2z_{1,0}x^3y + 3z_{0,1}x^2y^2 + p_1x^2y - q_1xy^2 - z_{1,0}x + z_{0,1}y = 0. \end{aligned} \quad (3.3)$$

Next, expressing the function c_1 from (3.3) and applying Δ_c , we have

$$\begin{aligned} e_2(a^{(2)}, b^{(2)}, p^{(2)}, q^{(2)}, z^{(2)}) &= 2z_{1,0}x + 3z_{0,1}y - 2p_2x^3y - 3q_2xy^3 - 5z_{1,1}xy \\ &\quad - 4a_2x^7y^3 - 9b_2x^5y^5 - 4p_2x^5y^2 - 2a_2x^5y^2 - 9q_2x^3y^4 - 3b_2x^3y^4 - 12z_{1,1}x^3y^2 \\ &\quad + 4z_{2,0}x^4y + 9z_{0,2}x^2y^3 - r_2x^4y^3 + 2z_{2,0}x^2 + 2p_1x^4y^2 - 12q_1x^3y^3 + 3z_{0,2}y^2 \\ &\quad - 6a_1x^6y^3 - 6b_1x^5y^4 - 4a_1x^4y^2 - 6z_{1,0}x^3y + 6z_{0,1}x^2y^2 - p_1x^2y - 4q_1xy^2 = 0. \end{aligned} \quad (3.4)$$

Expressing the function r_2 from (3.4) and applying Δ_r to the resulting equality, we obtain

$$\begin{aligned} e_3(a^{(3)}, b^{(3)}, p^{(3)}, q^{(3)}, z^{(3)}) &= -6z_{0,1}y - 4a_3x^8y^3 - 2a_3x^6y^2 - 4p_3x^6y^2 + 4z_{3,0}x^5y \\ &\quad - 16z_{2,1}x^4y^2 + 21z_{1,2}x^3y^3 - 2p_3x^4y - 7z_{2,1}x^2y + 8xz_{1,2}y^2 + 9x^5y^6b_3 + 3x^3y^5b_3 \\ &\quad + 9x^3y^5q_3 - 9x^2y^4z_{0,3} + 3xy^4q_3 + 2, z_{3,0}x^3 - 3y^3z_{0,3} - 3p_2x^3y + 13q_2xy^3 \\ &\quad + 6z_{1,1}xy - 18a_2x^7y^3 + 15b_2x^5y^5 - 10p_2x^5y^2 - 8a_2x^5y^2 + 30q_2x^3y^4 + 6b_2x^3y^4 \\ &\quad + 14z_{2,0}x^4y - 24z_{0,2}x^2y^3 + 4z_{2,0}x^2 - 12z_{0,2}y^2 - 12a_1x^6y^3 - 4a_1x^4y^2 - 2p_1x^4y^2 \\ &\quad + 12q_1x^3y^3 + 6z_{1,0}x^3y - 6z_{0,1}x^2y^2 + 8q_1xy^2 = 0. \end{aligned} \quad (3.5)$$

So, we have constructed a third-order differential polynomial which does not depend on $(s(x^3y^2 + xy), c(x^3y^2), r(xy))$, but depends only on $(a(x), b(y), p(x), q(y))$ and the function $z(x, y)$. Our aim is to use relation (3.5) and some differential corollaries thereof to find some relation involving only the function z ; that is, we will exclude in succession the functions $(q(y), b(y), p(x), a(x))$ (in this order). The calculations were done using Maple; the code is available at <http://vkb.strogino.ru/> (section ‘‘Miscellaneous’’, item 3).

The sizes (the number of monomials) and the differential orders of intermediate differential corollaries will increase in the course of calculations. We will not give them explicitly, but instead we will follow the scheme of calculations, and note the differential orders of polynomials and the number of the involved monomials.

Expressing the function $q_3 = Q_3(a^{(3)}, b^{(3)}, p^{(3)}, q^{(2)}, z^{(3)})$ from (3.5) and using the condition that q_3 does not depend on x , we have $e_4(a^{(4)}, b^{(3)}, p^{(4)}, q^{(2)}, z^{(4)}) = 0$ (with 68 monomials).

Expressing q_2 from $e_4 = 0$, we find that $q_2 = Q_2(a^{(4)}, b^{(3)}, p^{(4)}, q_1, z^{(4)})$. Since q_2 is independent of x , we have $e_5(a^{(5)}, b^{(3)}, p^{(5)}, z^{(5)}) = 0$ (with 52 monomials, not depending on q). Since $(Q_2)'_x = Q_3$, we have $e_6(a^{(4)}, b^{(4)}, p^{(4)}, z^{(5)}) = 0$ (with 75 monomials).

Let us now exclude b . Expressing the function b_3 from $e_5 = 0$, we obtain $b_3 = B_3(a^{(5)}, b_2, p^{(5)}, z^{(4)})$. Since b_3 is independent of x , we have $e_7(a^{(6)}, p^{(6)}, z^{(6)}) = 0$ (with 61 monomials, not depending on b). Expressing the function b_4 from $e_6 = 0$, we get $b_4 = B_4(a^{(5)}, b_2, p^{(5)}, z^{(6)})$. Since b_4 is independent of x , we have $e_7(a^{(6)}, p^{(6)}, z^{(6)}) = 0$ (with 61 monomials, not depending on b). Using the relation $(B_3)'_y = B_4$, we get $e_8(a^{(5)}, b_2, p^{(5)}, z^{(6)}) = 0$ (with 131 monomials). Expressing the function b_2 from $e_8 = 0$, we have $b_2 = B_4(a^{(5)}, p^{(5)}, z^{(6)})$. Since b_2 is independent of x , we find that $e_9(a^{(6)}, p^{(6)}, z^{(7)}) = 0$ (with 148 monomials). Using $(B_2)'_y = B_3$, this gives $e_{10}(a^{(5)}, p^{(5)}, z^{(7)}) = 0$ (with 134 monomials).

Let us exclude p . Expressing the function p_5 from $e_{10} = 0$, we get $p_5 = P_5(a^{(5)}, p^{(4)}, z^{(7)})$. Since p_5 is independent of y , we have $e_{11}(a^{(5)}, p^{(4)}, z^{(8)}) = 0$ (with 247 monomials). Substituting p_6 with the expression for $(P_5)'_x$, we get $ee_7(a^{(6)}, p^{(4)}, z^{(8)}) = 0$ (with 336 monomials). Similarly, substituting e_9 with the expressions for p_5 and p_6 , we get $ee_9(a^{(6)}, p^{(4)}, z^{(8)}) = 0$ (with 404 monomials). Expressing the function p_4 from $e_{11} = 0$ and substituting the resulting expression into $ee_7 = 0$ and into $ee_9 = 0$, we get $eee_7(a^{(6)}, z^{(8)}) = 0$ (with 354 monomials, not depending on p) and $eee_9(a^{(6)}, z^{(8)}) = 0$ (with 418 monomials, not depending on p).

Let us exclude a from the last two relations. Expressing a_6 from $eee_7 = 0$, we have $a_6 = A_{61}(a^{(5)}, z^{(8)})$, and expressing a_6 from $eee_9 = 0$, we find that $a_6 = A_{62}(a^{(5)}, z^{(8)})$. Since $A_{61} = A_{62}$, we have $e_{12}(a^{(5)}, z^{(8)}) = 0$ (with 266 monomials). Using $e_{12} = 0$ we get $a_5 = A_5(a^{(4)}, z^{(8)})$. Differentiating A_5 , we get another expression for a_6 , namely, $a_6 = A_{63}(a^{(4)}, z^{(9)})$. Since $A_{63} = A_{61}$, we have $e_{13}(a^{(4)}, z^{(9)}) = 0$ (with 533 monomials). From $e_{13} = 0$ we have $a_4 = A_4(a^{(3)}, z^{(9)})$. Differentiating A_4 , we get another expression for a_5 , namely, $a_5 = AA_5(a^{(3)}, z^{(10)})$. Since $A_5 = AA_5$, we have $e_{14}(z^{(10)}) = 0$ (705 monomials, not depending on a).

The polynomial e_{14} of order 10 is the result of our calculations. This expression is linear with respect to the derivatives if the function z is of order from 1 to 10.

So, we have the following result.

Proposition. a) *The tangent space $T Cl^2_{(x^3y^2+xy)}$ to Cl^2 at the point $Z_0 = (x^3y^2 + xy)$ consists of the analytic functions $z(x, y)$ of the form*

$$z(x, y) = s(x^3y^2 + xy) + c(x^3y^2) + x^3y^2(a(x) + b(y)) + r(x, y) + (p(x) + q(y))xy,$$

where (a, b, c, p, q, r, s) are analytic univariate functions such that the sum is defined in some non-empty domain.

b) *The relation $e_{14}(z^{(10)}) = 0$ is a necessary condition that $z(x, y)$ has a representation from assertion a).*

c) *The polynomial $e_{14}(z^{(10)})$ (with 705 monomials) is a linear expression of the derivatives of the function z of orders ≤ 10 (with 65 derivatives); the coefficients are polynomials of (x, y) with integer coefficients lying in $[-306021888, 306021888]$.*

Having at our disposal a necessary condition for membership of a function in the tangent space, we can easily give examples of simple expressions not lying in this

tangent space. For example, substituting $z = x^2y^3$ into e_{14} , we get

$$\begin{aligned} \frac{e_{14}(z)}{-120x^2y^3} &= 32256x^{28}y^{14} + 486432x^{26}y^{13} + 414672x^{24}y^{12} + 693120x^{22}y^{11} \\ &\quad - 142140x^{20}y^{10} - 107594x^{18}y^9 + 295122x^{16}y^8 + 127164x^{14}y^7 - 49287x^{12}y^6 \\ &\quad - 28929x^{10}y^5 + 21606x^8y^4 + 5541x^6y^3 + 501x^4y^2 - 54x^2y + 40 \neq 0. \end{aligned}$$

Let $a \in \mathbf{C}^{21}$ be the family of coefficients of a general polynomial of (x, y) of degree ≤ 5 , and $p(x, y, a)$ be a polynomial with coefficients a , and let $\sigma_2 = \{a \in \mathbf{C}^{21} : N(p(x, y, a)) \leq 2\}$.

Theorem 2. a) *The complexity of the polynomial*

$$W = x^3y^2 + xy + \pi x^2y^3$$

relative to $(x + y)$ is 3, that is, $N(W) = 3$.

b) *The set σ_2 is a proper algebraic subset of \mathbf{C}^{21} , and σ_2 is the set of common zeros of the family of polynomials of a with integer coefficients.*

Proof. That the complexity of W is ≤ 3 is clear. So, it suffices to show that $W \notin Cl^2$. Consider the straight line l passing through $(x^3y^2 + xy)$ in the direction $z = x^2y^3$, that is, $l = \{Z(x, y, t) = (x^3y^2 + xy) + tx^2y^3\}$. If l would wholly lie in Cl^2 , then its direction vector would be a tangent vector. But this is not so. Therefore, l does not lie in Cl^2 . Let $(d_1(Z(x, y)), \dots, d_s(Z(x, y)))$ be the differential polynomials defining Cl^2 ; we also consider their restriction to l . We thus have the family of polynomials $(P_1(t), \dots, P_s(t))$ of t with integer coefficients, not all these polynomials are identically zero. Let $\tilde{P}(t)$ be such a polynomial. In this case, if, for some t , the function $Z(x, y, t)$ lies in Cl^2 , then $\tilde{P}(t) = 0$. The zeros of this polynomial form a finite family of algebraic numbers (over the field \mathbf{Q}). The number π is non-algebraic, and hence $(x^3y^2 + xy + \pi x^2y^3) \notin Cl^2$. This proves assertion a).

Substituting the general polynomial $Z = p(x, y, a)$ of degree 5 into (d_1, \dots, d_s) , we get the family of polynomial relations of $a \in \mathbf{C}^{21}$ with integer coefficients. There is a polynomial $p(x, y, a_0)$ of degree 5 which does not lie in Cl^2 , and hence these relations define a proper algebraic subset of \mathbf{C}^{21} . This proves assertion b), and, therefore, the theorem.

Although the differential polynomials defining the second class are not given explicitly, there is an algorithm for their construction. By analyzing this algorithm we can obtain upper estimates for the differential order, the number of monomials, and the size of their coefficients (these quantities are integer). Using such estimates, one can easily find an upper bound for the absolute values of the roots of the polynomial $P(t)$ from the proof of Theorem 2. This enables us to replace the non-algebraic coefficient π in the polynomial W by some very large (or very small) rational number. This will give us an example of an integer polynomial of degree 5 and complexity 3.

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V. K. Beloshapka (Valerii K. Beloshapka)
 Lomonosov Moscow State University,
 Moscow Center for Fundamental and Applied
 Mathematics
E-mail: vkb@strogino.ru

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