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Model CR Surfaces: Weighted Approach

V. K. Beloshapka[*,**,1]

[*] *Department of Mechanics and Mathematics, Moscow State University, Moscow, 119991, Russia*
[**] *Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991, Russia*
E-mail: [1]vkb@strogino.ru

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Abstract. In the paper, a systematic construction of the theory of "weighted" model surfaces using the Bloom-Graham-Stepanova concept of the type of a CR-manifold is given. The construction is based on the Poincaré construction. It is shown how the use of weighted model surfaces expands the abilities of the method. New questions are being posed.

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1. INTRODUCTION

Among the approaches to the local analysis of CR-submanifolds of a complex space, the method of the model surface should be noted. This is an efficient analytical approach with more than a century of history. The basis of this method is a certain version of the implicit mapping theorem in the class of formal power series. The author of this technique is H. Poincaré, who used it both in the problems of celestial mechanics and in the area that is now called CR-geometry [1].

Recently, the scope of this approach has been extended to the class of arbitrary germs of finite Bloom–Graham type [2]. However, here the standard version of this method was used. This means that the variables in the complex tangent of the germ have the same weight. In a number of special cases, a more flexible technique of free assigning weights to complex tangent variables ([3, 4, 5]). Let us explain what was said above by an example in the paper [6].

Consider a hypersurface in the space $\mathbf{C}^{n+2}$ with coordinates $(z_1, \dots, z_n,\, \zeta,\, w = u + i\,v)$ given by the equation

$$v = 2\,\mathrm{Re}(z_1\bar{\zeta} + \dots + z_n\bar{\zeta}^n). \tag{1}$$

This hypersurface has the Bloom–Graham type $m = (2)$, the model surface $Q = \{v = 2\ \mathrm{Re}(z_1\bar{\zeta})\}$ is holomorphically degenerate, and the dimension of the algebra of its automorphisms is infinite, while the algebra of automorphisms of the hypersurface itself is finite-dimensional. This is what we obtain when using the standard approach, when the weights of all coordinates in the complex tangent (both z and ζ) are the same and equal to 1. However, if we arrange the weights differently, namely: ($[\zeta] = 1,\ [z_j] = 1 + n - j,\ [w] = n + 1$), then the surface becomes weighted homogeneous (of weight $n + 1$). Moreover, this hypersurface is holomorphically homogeneous. Using the "weighted" version of Poincaré's construction, one can obtain an estimate for the dimension of the automorphism algebra for the germ of the perturbed hypersurface

$$v = 2\mathrm{Re}\,(z_1\bar{\zeta} + \dots + z_n\bar{\zeta}^n) + o(n + 1).$$

It is clear that the correct viewpoint concerning this example is to consider this hypersurface as a weighted homogeneous one and that a "weighted" theory of model surfaces of finite type stands behind this example. Here the first step in the construction of such a theory, namely, the proof of the "weighted" analog of the Bloom–Graham theorem, has already been done by M. Stepanova in [10].

The objective of this paper is to systematically construct such a "weighted" theory of model surfaces, which is based on the weighted Bloom–Graham–Stepanova type of the germ in the same way in which the theory constructed in the paper [2] is based on the Bloom–Graham type [8].

2. Weights and the weighted type of the germ of a CR-manifold.

Poincaré's construction, which underlies the method, is mainly analytical. Therefore, for the two equivalent definitions of the type of a germ, the analytical definition using the form of local equations of the germ of a manifold is the main one for us.

The difference between the weighted version and the standard version lies in the possibility of choosing *different* weights for the coordinates of the complex tangent.

Let the coordinates of the ambient space $\mathbf{C}^N$ be divided into two groups, $z \in \mathbf{C}^n$ and $w \in \mathbf{C}^K$. Accordingly, each of the two coordinate subspaces is decomposed into a direct sum

$$\mathbf{C}^n = \mathbf{C}^{n_1} + \cdots + \mathbf{C}^{n_p}, \quad \mathbf{C}^K = \mathbf{C}^{k_1} + \cdots + \mathbf{C}^{k_q}.$$

Let two sets of positive integers be given,

$$\mu_1 < \mu_2 < \cdots < \mu_p, \quad m_1 < m_2 < \cdots < m_q.$$

Assign weights to the coordinates as follows:

$$[z_\alpha] = \mu_\alpha, \text{ where } z_\alpha \in \mathbf{C}^{n_\alpha}, \quad [w_\beta] = m_\beta, \text{ where } w_\beta \in \mathbf{C}^{k_\beta}.$$

It is implied that

$$\bar{z}_\alpha,\ \bar{w}_\beta,\ \operatorname{Re} z_\alpha,\ \operatorname{Im} z_\alpha,\ \operatorname{Re} w_\beta = u_\beta,\ \operatorname{Im} w_\beta = v_\beta$$

have the corresponding weights. Moreover, the differentiation with respect to the coordinate of weight ν obtains the weight $-\nu$.

Let the local equations of the germ of a CR-manifold M_0 at the origin have the form

$$v_\beta = \Phi_\beta(z, \bar{z}, u) + o(m_\beta), \quad \beta = 1, \ldots, q, \tag{2}$$

where Φ_β is a quasihomogeneous polynomial vector of weight m_β, and the symbol $o(\nu)$ is treated as a vector function whose Taylor expansion at the origin contains terms whose weights are strictly greater than ν.

It is shown in [7] that, by simple invertible polynomial changes of coordinates in a neighborhood of the origin, one can always subject local equations of the form (2) to an additional reduction and reduce to the same form in which the weighted homogeneous forms Φ_j satisfy additional conditions. These conditions are related to the existence of invertible holomorphic transformations that preserve the form (2) but change the lower coordinate forms of the equation. Every holomorphic polynomial in the space $\mathbf{C}^N$ is a linear combination of monomials of the form (we use the multi-notation) $z_1^{\gamma_1} \ldots z_p^{\gamma_p} w_1^{\delta_1} \ldots w_q^{\delta_q}$. Every monomial of this kind is the ability to change some lower coordinate forms in the germ equation. The coordinate forms Φ themselves are real polynomials each of which is a linear combination of real monomials of the form

$$z_1^{\gamma_1} \ldots z_p^{\gamma_p} \bar{z}_1^{\bar{\gamma}_1} \ldots \bar{z}_p^{\bar{\gamma}_p} u_1^{\delta_1} \ldots u_q^{\delta_q}.$$

Using holomorphically polynomial transformations mentioned above, one can achieve the validity of the following two conditions:

(I) the coordinates of all forms do not contain monomials of the form
$z_1^{\gamma_1} \ldots z_p^{\gamma_p} u_1^{\delta_1} \ldots u_q^{\delta_q}$ and of the form $\bar{z}_1^{\gamma_1} \ldots \bar{z}_p^{\gamma_p} u_1^{\delta_1} \ldots u_q^{\delta_q}$ for any γ and δ,

(II) for every J, $1 \leqslant J \leqslant p$, none of the coordinates of the form Φ_J
contains terms of the form $c\, \phi(z, \bar{z}, u)\, u_1^{\delta_1} \ldots u_p^{\delta_p}$ such that
$\phi(z, \bar{z}, u)$ is a scalar coordinate of some form Φ_j for $j < J$, and
c is a nonzero constant.

Note that the first condition is the condition that pluriharmonic summands are absent. The other condition, unlike the first one, is of recurrent nature. In this case, the formulation “does not contain summands” should be treated as follows: it is impossible to compose an expression of the specified type from the monomials entering the coordinate form.

For every chosen weight ν, denote by $\mathcal{N}_\nu$ the real linear space of polynomials of the weight ν that satisfy conditions (I) and (II).

Definition (analytical): Choose some weight $\mu = ((\mu_1, n_1), \ldots, (\mu_p, n_p))$ on the variables of the group z. Let one can introduce coordinates in a neighborhood of a point $\xi \in M \subset \mathbf{C}^N$ in such a way that the local equation of M has the form (2), where the coordinate forms Φ satisfy conditions (I) and (II) and are also linearly independent. Then we say that the generating manifold M has a finite μ-type equal to $m = ((m_1, k_1), \ldots, (m_q, k_q))$.

Note that, since the minimal weight of the variables of the group z is μ_1, and the forms Φ do not contain pluriharmonic summands, it follows that the minimal admissible weight is $2\,\mu_1$. Thus,

$$2\,\mu_1 \leqslant m_1 < \cdots < m_q.$$

If we choose $\mu = ((1, n))$ as the weight, then we obtain a modified Bloom–Graham's analytical definition.

Before proceeding to a geometric definition, we recall the ordinary geometric definition of the *finiteness* of the type according to Bloom–Graham. This definition is not tied to the choice of any weights and remains without changes for us.

Let D_1 be the distribution of complex tangents defined on M in a neighborhood of ξ, i.e., $D_1 = T^c_M$. This distribution can be specified using a basis family of $2\,n$ smooth real vector fields. Further, define an infinite sequence of distributions D_ν given inductively:

$$D_{\nu+1} = [D_\nu, D_1] + D_\nu, \quad \nu = 1, 2, \dots .$$

Further, let $D_\nu(\xi)$ be the value of D_ν at the point ξ. Thus,

$$T^c\, M_\xi = D_1(\xi) \subset D_2(\xi) \subset \dots \subset D_\nu(\xi) \subset \dots$$

Since this nondecreasing sequence consists of subspaces of TM_ξ, it follows that it stabilizes at some step. If the last subspace coincides with TM_ξ, then we say that M, at the point ξ, is a manifold *of finite type* (if not, *of infinite type*).

We assume below that M, at the point ξ, is a manifold of finite type. Let us return to our "weighted" situation. It is clear that, for a manifold given by equations (2), the subspace $D_1(0)$ coincides with the space generated by the derivations on the space $\mathbf{C}^n$ of he coordinate z and obtains the decomposition corresponding to the weight μ. In accordance with our convention, the differentiations with respect to coordinates obtain weights equal to minus the weights of the coordinates. Due to the finiteness of the type at the point, the inductive process described above gives all the tangent space at the point in a certain number of steps. Thus, every vector of the tangent space obtains an expression in terms of differentiations with respect to coordinates spaces $\mathbf{C}^n$. As a result, we obtain

$$T^c_0 M = \mathcal{D}_1(0) \subset \mathcal{D}_2(0) \subset \dots \subset \mathcal{D}_{max}(0) = T_0\, M,$$

where the subspace $\mathcal{D}_\nu(0)$ is formed by the differentiations of weight $-\nu$. Let $2 \leqslant m_1 < m_2 < \dots < m_l$ be the values of the weights for which the dimension grows, i.e., $\dim \mathcal{D}_{m_j}(0) > \dim \mathcal{D}_{m_j - 1}(0)$, and denote by k_j the value of this growth, i.e., $\dim \mathcal{D}_{m_j}(0) - \dim \mathcal{D}_{m_j-1}(0) > 0$.

The family $m = ((m_1, k_1), \dots, (m_q, k_q))$ gives the second, **geometric**, **definition** of the μ-type. According to [7], these definitions are equivalent. Note that, if the type of the manifold at a point were infinite, then, as in the standard case, we could speak about the μ-type at the point, completing our family with the pair (∞, d), where $d = K - (k_1 + \dots + k_l)$ stands for the defect at the point (see also [10]). In our case of finite type, the defect is equal to zero.

Statement 1 (see [7]).
(a) The geometric and analytic definitions of the μ-type of a germ are equivalent.
(b) The μ-type of a germ is invariant under locally holomorphic transformations that preserve the weight expansion of the complex tangent at the center of the germ.
(c) If the germ is of finite type $m = ((m_1, k_1), \dots, (m_q, k_q))$, then, for all $\alpha = 1, \dots, p$,

$$k_\alpha \leqslant \dim \mathcal{N}_\alpha.$$

Note that there is a significant difference between the distribution of weights in the variable z (the value μ) and the distribution of weights in the variable w (the value m). If we assign μ as we like, then, for chosen M_ξ and μ, the value m, i.e., μ-type of the germ, is recovered uniquely. Therefore, we can write $m = m(M_\xi, \mu)$.

3. Lower components of the mapping

Let there be a locally invertible holomorphic mapping $\chi = (f, g)$ of the form

$$(z_\alpha \to f_\alpha(z, w), \;\; w_\beta \to g_\beta(z, w)), \;\alpha = 1, \dots, p, \;\; \beta = 1, \dots, q,$$

of the germ at the origin M_0 of finite type m given by equations in the reduced form,

$$v_\beta = \Phi_\beta(z, \bar z, u) + F_\beta(z, \bar z, u), \quad j = 1, \dots, q, \tag{3}$$

into another similar germ $\tilde{M}_0$,

$$v_\beta = \tilde{\Phi}_\beta(z,\bar{z},u) + \tilde{F}_\beta(z,\bar{z},u), \quad j = 1,\dots,q, \tag{4}$$

where F_β and $\tilde{F}_\beta$ are $o(m_\beta)$ preserving the origin and the weight expansion of the complex tangent at zero.

Consider the surfaces given by the equations

$$Q = \{v_\beta = \Phi_j(z,\bar{z},u)\}, \quad \tilde{Q} = \{v_\beta = \tilde{\Phi}_\beta(z,\bar{z},u)\}, \quad \beta = 1,\dots,q. \tag{5}$$

We shall say that Q and $\tilde{Q}$ are *μ-model surfaces* of the germs M_0 and $\tilde{M}_0$.

In what follows, we use the expansions

$$f_\alpha = \sum f_{\alpha\gamma}, \ \ g_\beta = \sum g_{\beta\gamma}, \ \ F_\beta = \sum F_{\beta\gamma}, \ \ \tilde{F}_\beta = \sum \tilde{F}_{\beta\gamma},$$

where $f_{\alpha\gamma},\ g_{\beta\gamma},\ F_{\beta\gamma},\ \tilde{F}_{\beta\gamma}$ are the components of the weight γ.

In this notation, the fact that the mapping preserves the origin and the decomposition of the complex tangent into the weight components means that

$$f_\alpha = C_\alpha\, z_\alpha + \tau_\alpha(z,w) + o(\mu_\alpha),$$

where C_α is a nondegenerate transformation of the linear space $\mathbf{C}^{n_\alpha}$, and $\tau_\alpha(z,w)$ is a vector polynomial of the weight μ_α, which can depend on z_γ only for $\gamma < \alpha$ and from w_δ such that $m_\delta < \mu_\alpha$. Thus, $\tau_\alpha(z,w)$ does not contain linear terms. Write

$$\begin{gathered} C\,z = (C_1\,z_1,\dots,C_p\,z_p), \quad \tau(z,w) = (\tau_1(z,w),\dots,\tau_p(z,w)) \\ \rho\,w = (\rho_1\,w_1,\dots,\rho_q\,w_q), \quad \theta(z,w) = (\theta_1(z,w),\dots,\theta_q(z,w)). \end{gathered}$$

Writing out that the image of M_0 is contained in $\tilde{M}_0$, we obtain the identity

$$\operatorname{Im} g = \tilde{\Phi}(f,\bar{f},\operatorname{Re} g) + \tilde{F}(f,\bar{f},\operatorname{Re} g) \text{ for } w = u + i(\Phi + F). \tag{6}$$

Consider the lower components of identity (6).

Let us begin with the group of variables w_1. In the weights from 1 to $(m_1 - 1)$, we obtain $\operatorname{Im} g_{1\nu} = 0$, where $1 \leqslant \nu \leqslant m_1 - 1$. Taking into account that a uniform form of a weight $\nu < m_1$ is a holomorphic form of the variables of the group z, we conclude that $g_{11} = g_{12} = \dots = g_{1(m_1-1)} = 0$.

In the weight m_1, we have $g_{1m_1} = a(z) + \rho_1\,w_1$, $\quad f_\alpha = C_\alpha\,(z_\alpha + \tau_\alpha(z,w))$, where ρ_1 and C_α are linear and invertible and $a(z)$ is a holomorphic homogeneous form of the weight m_1. We obtain

$$\operatorname{Im}(a(z) + \rho_1\,(u_1 + i\,\Phi_1)) = \tilde{\Phi}_1(C\,(z + \tau(z,w)), \overline{C\,(z + \tau(z,w))}).$$

Since m_1 is the lowest weight in the weight group w, it follows that, the variables of the group w cannot enter the coordinates τ on which Φ_1 depends. That is, τ may depend on w, but not for the coordinates on which Φ_1 depends.

Separating in this relation the component that is holomorphic in z and taking into account that Φ_1 and $\tilde{\Phi}_1$ do not contain holomorphic summands, we see that $a(z) = 0$. From the linear component in u_1 we obtain $\operatorname{Im}\rho_1\,u_1 = 0$. Then we have the following relation:

$$\rho_1\Phi_1(z,\bar{z}) = \tilde{\Phi}_1(C\,(z + \tau(z)), \overline{C\,(z + \tau(z))})/ \tag{7}$$

Note that this relation is equivalent to the fact that the mapping $(z \to C\,(z+\tau(z)), \quad w_1 \to \rho_1\,w_1)$ translates the "truncated" model surface $Q(1) = \{v_1 = \Phi_1(z,\bar{z})\}$ of the space $\mathbf{C}^{n+k_1}$ of type (m_1,k_1) into another "truncated" surface $\tilde{Q}(1) = \{v_1 = \tilde{\Phi}_1(z,\bar{z})\}$ of the same type.

Let us pass to the coordinate w_2. The components $g_{2\nu}$, where $\nu < m_2$, are expressions of the form $\sum \psi_{\alpha\beta}\,(z,w_1)$, where $\psi_{\alpha\beta}\,(z,w_1)$ is a holomorphic multilinear form of the weight α in z and β in w_1, and $\alpha + m_1\,\beta = \nu$. The components of identity (6) of weights $\nu < m_2$ give

$$\operatorname{Im}\,(\sum \psi_{\alpha\beta}\,(z, u_1 + i\,\Phi_1)) = 0,$$

Whence it follows that $g_{2\nu} = 0$ for $\nu < m_2$. This could be proven directly from the identity obtained above. However, we use another way. Indeed, $g_{2\nu}$ is a holomorphic function on the "truncated" generating manifold of finite type $Q_1 = \{(z, w_1) : v_1 = \Phi_1(z, \bar{z})\}$ whose imaginary part is equal to zero. Therefore, $g_{2\nu}$ is constant, and, since its weight is greater than zero, it follows that $g_{2\nu}$ is zero.

In the weight m_2 we have $g_{2m_2} = \sum \psi_{\gamma\delta}\,(z, w_1) + \rho_2\, w_2$, where $\psi_{\gamma\delta}$ has the weight γ with respect to z and the degree δ with respect to w_1, ρ_2 is linear, and $\gamma + m_1\,\delta = m_2$; moreover, the following relation holds:

$$\begin{aligned}\operatorname{Im}\Big(\sum \psi_{\gamma\delta}\,(z, u_1 + i\,\Phi_1) + \rho_2(u_2 + i\,\Phi_2(z, \bar{z}, u_1))\Big)\\ = \tilde{\Phi}_2(C\,(z + \tau(z, w_1)), \overline{C\,(z + \tau(z, w_1))}, \rho_1\, u_1).\end{aligned}$$

Separating the terms that are linear in u_2, we obtain $\operatorname{Im}(\rho_2)u_2 = 0$, i.e., the linear mapping ρ_2 is real. Thus, the relation acquires the form

$$\begin{aligned}\operatorname{Im}\Big(\sum \psi_{\gamma\delta}\,(z, u_1 + i\,\Phi_1(z, \bar{z}))\Big) = -\rho_2\,\Phi_2(z, \bar{z}, u_1)\\ +\tilde{\Phi}_2(C\,(z + \tau(z, u_1 + i\,\Phi_1)), \overline{C\,(z + \tau(z, u_1 + i\,\Phi_1(z, \bar{z})))}, \rho_1\, u_1).\end{aligned}$$

By condition (I), the right-hand side of this relation does not contain terms that are holomorphic in z. Setting $\bar{z} = 0$, we see that. if $\gamma \neq 0$, then $\psi_{\gamma\delta}\,(z, u_1) = 0$, and $\psi_{0\delta}\,(u_1)$ is the real form of the weight m_2, which we redenote by $\rho_2\,\theta_2(w_1)$, i.e., $g_{2m_2} = \rho_2(w_2 + \theta_2(w_1))$. Now the relation becomes

$$\begin{aligned}\rho_2\,\Phi_2(z, \bar{z}, u_1) =\\ \tilde{\Phi}_2(C\,(z + \tau(z, u_1 + i\,\Phi_1)), \overline{C\,(z + \tau(z, u_1 + i\,\Phi_1))}, \rho_1\, u_1) + \operatorname{Im}\theta_2(u_1 + i\,\Phi_1)\end{aligned}$$

Note that this relation is equivalent to the fact that the mapping

$$(z \to C\,(z + \tau(z, w)), \quad w_1 \to \rho_1\, w_1, \quad w_2 \to \rho_2\,(w_2 + \theta_2(w_1)))$$

takes the second "truncated" generating surface $Q(2) = \{v_1 = \Phi_1(z, \bar{z}),\, v_2 = \Phi_2(z, \bar{z}, u_1)\}$ $\mathbf{C}^{n+k_1+k_2}$ of the type $((m_1, k_1), (m_2, k_2))$ into another "truncated" surface $\tilde{Q}(2) = \{v_1 = \tilde{\Phi}_1(z, \bar{z}),\, v_2 = \tilde{\Phi}_2(z, \bar{z}, u_1)\}$ of the same type.

And so on, till the last weight group corresponding to w_q. Let us formulate the result.

Statement 2. Let

$$\chi = (z_\alpha \to f_\alpha(z, w),\; w_\beta \to g_\beta(z, w), \quad \alpha = 1, \ldots, p, \quad \beta = 1, \ldots, q)$$

be an invertible holomorphic mapping of the germ (3) onto another such germ (4) such that its action on the complex tangent at zero preserves its decomposition into the components of the weights μ. Then
(a) this mapping has the form

$$\begin{aligned}(z_\alpha \to C_\alpha\,(z_\alpha + \tau_\alpha(z, w)) + o(\mu_\alpha),\; w_\beta \to \rho_\beta\,(w_\beta + \theta_\beta(w_1, \ldots, w_{\beta-1})) + o(m_\beta)),\\ \text{where } C_\alpha \in GL(n_\alpha, \mathbf{C}),\; \rho_\beta \in GL(k_\beta, \mathbf{R}) \text{ and where } [\tau_\alpha] = \mu_\alpha,\; [\theta_\beta] = m_\beta,\\ \text{and, for all } \beta = 1, \ldots, q,\\ \tilde{\Phi}_\beta(C\,(z + \tau(z, u + i\,\Phi)), \overline{C\,(z + \tau(z, u + i\,\Phi))}, \rho\,(u + \operatorname{Re}(\theta(u + i\,\Phi))))\\ = \rho_\beta\,(\Phi_\beta(z, \bar{z},\, u) + \operatorname{Im}\theta_\beta(u + i\,\Phi));\end{aligned} \tag{8}$$

(b) here the "quasilinear" mapping

$$(z \to C\,(z + \tau(z, w)),\; w_\nu \to \rho_\nu\,(w_\nu + \theta_\nu(w_1, \ldots, w_{\nu-1})),\; \nu = 1, \ldots, \beta) \tag{9}$$

takes the μ-model surface Q to the μ-model surface $\tilde{Q}$ (see (19)).

Moreover, for every $\beta = 1, \ldots, q$, the truncated mapping

$$(z \to C\,z + \tau(z, w),\; w_\nu \to \rho_\nu\, w_\nu + \theta_\nu(w_1, \ldots, w_{\nu-1}),\; \nu = 1, \ldots, \beta)$$

takes the truncated model surface

$$Q(\beta) = \{v_\nu = \Phi_\nu,\; \nu = 1, \ldots, \beta\}$$

of the space $\mathbf{C}^{n+k_1+\dots+k_\beta}$ into the corresponding truncated model surface.

$$\tilde{Q}(\beta) = \{v_\nu = \tilde{\Phi}_\nu,\ \nu = 1, \dots, \beta\}.$$

Part (b) of the theorem is a special case of this statement.

If we set $\tilde{Q} = Q$, then the mappings (9) with the condition (8) form some subgroup G_0 of the automorphisms Q preserving the origin and the weighted expansion of the complex tangent at zero. These are the automorphisms Q of the form

$$\begin{aligned} z_\alpha &\to C_\alpha\,(z_\alpha + \tau_\alpha(z, w)), \\ w_\beta &\to \rho_\beta\,(w_\beta + \theta_\beta(w_1, \dots, w_{\beta-1})). \end{aligned} \tag{10}$$

That is, these are the automorphisms such that every coordinate retains its weight. If we use the notion of a component introduced by us in Sec. 4, then G_0 can be characterized as follows: these are the automorphisms of Q whose decomposition into components contains only the 0-component. This is just the fact fixed by (10). The fact that such a mapping is an automorphism of Q is given by conditions (8) (Statement 2) for $\tilde{\Phi} = \Phi$.

The family of transformations of the form (10) without conditions (8) forms a subgroup of polynomial automorphisms of the space

$$\mathbf{C}^N = \mathbf{C}^{n_1} + \dots + \mathbf{C}^{n_p} + \mathbf{C}^{k_1} + \dots + \mathbf{C}^{k_q}.$$

This subgroup $\mathcal{G}_0$ is a semidirect product of the subgroup of triangular transformations of the form

$$z_\alpha \to z_\alpha + \tau_\alpha(z, w), \quad w_\beta \to w_\beta + \theta_\beta(w_1, \dots, w_{\beta-1}),$$

and linear transformations of the form

$$z_\alpha \to C_\alpha\, z_\alpha, \quad w_\beta \to \rho_\beta\, w_\beta.$$

In the standard (unweighted) version [2], there is an assertion (Theorem 5, part (f)) claiming that an element of G_0 is uniquely determined by its action on the coordinate z. Here is an analog of this assertion.

Statement 3. If the model surface Q has a finite μ-type and there is an automorphism $(f(z,w), g(z,w)) \in G_0$ such that $f(z,w) = z$, then $g(z,w) = w$, i.e., this is the identity mapping.

Proof. Set $\tau(z,w) = 0$ and $C\,z = z$ and write out relation (8) for $\beta = 1$; we obtain $\rho_1 \Phi_1(z, \bar{z}) = \Phi_1(z, \bar{z})$. Whence, by the linear independence of the coordinates of Φ_1, it follows that $\rho_1\, w_1 = w_1$. Let us write out relation (8) for $\beta = 2$; we obtain

$$\Phi_2(z, \bar{z}, u_1) = \rho_2\, \Phi_2(z, \bar{z}, u_1) + \operatorname{Im} \theta_2(u_1 + i\, \Phi_1)$$

Since Φ_2 is written out in the reduced form, it follows that $\theta_2 = 0$ and $\rho_2\, w_2 = w_2$. And so on, up to $\beta = q$. The statement has been proven.

This statement can be interpreted as the presence of some parametrization of the group G_0. Every element $\chi \in G_0$ is uniquely determined by the family of parameters (C, τ, ρ, θ), where the family of parameters is connected by the algebraic relations (8). Statement 3 means that the parameters (ρ, θ) are uniquely determined by the parameters (C, τ), i.e., $\rho = \rho(C, \tau)$, $\theta = \theta(C, \tau)$. The algebraic subset in the space (C, τ, ρ, θ) defined by the relations (8) has a one-to-one projection to some algebraic subset $\mathcal{C}$ of the space (C, τ). As a result, $\mathcal{C}$ obtains the structure of an algebraic group acting on $\mathbf{C}^N$ and isomorphic to G_0.

4. Weighted Poincaré construction

In the linear space $\mathcal{V}$ of families of holomorphic germs of the form $\chi = (f_1, \dots, f_p; g_1, \dots, g_q)$, in a neighborhood of zero, we introduce the direct decomposition into the *components* $\mathcal{V} = \sum \mathcal{V}_\nu$, where $\mathcal{V}_\nu$ consists of families of the following weights: $((\mu_1 + \nu, \dots, \mu_p + \nu), (m_1 + \nu, \dots, m_q + \nu))$. Accordingly, we can write $\chi = \sum \chi^{(\nu)} = (f, g) = \sum (f^{(\nu)}, g^{(\nu)})$, where

$$f^{(\nu)} = (f_{1(\mu_1+\nu)}, \dots, f_{p(\mu_p+\nu)}), \quad g^{(\nu)} = (g_{1(m_1+\nu)}, \dots, g_{q(m_q+\nu)}).$$

Let the lower (model) terms of the equations (3) and (4) coincide, i.e., $\tilde{\Phi} = \Phi$ and $\tilde{Q} = Q$. As shown above, the lowest component of the mapping $\chi = \sum \chi^{(\nu)}$ of the surface (3) onto (4) provided that the origin and the weight expansion of the complex tangent are preserved is χ_0, i.e., the family of coordinate functions has weights of the form $((\mu_1, \dots, \mu_p), (m_1, \dots, m_q))$. Here χ_0 is an element of the linear algebraic group G_0 and, as has been shown, is given by its own system of parameters $\lambda_0 = (C, \tau, \rho, \theta)$, i.e., $\chi_0 = \chi_0(\lambda_0)$. It is clear that any such mapping $\chi = \chi_0 + \chi_1 + \dots$ can be represented as a composition $\varphi \circ \psi$, where $\psi = \varphi^{-1} \circ \chi$ is a mapping whose 0-component is the identity mapping, i.e., $\psi = Id + \psi^{(1)} + \psi^{(2)} + \dots$.

Let us apply the Poincaré construction to estimate the dimension of the family of mappings of the form $\psi = Id + \psi^{(1)} + \psi^{(2)} + \dots$. Denote the space $\mathcal{V}_1 + \mathcal{V}_2 + \dots$ by $\mathbf{V}_+$. Let us now consider identity (6) and select the ν-th component in this identity. We obtain the relation

$$\begin{aligned} -\operatorname{Im} g^{(\nu)} + d\,\Phi(z, \bar{z}, u)\,(f^{(\nu)}, \bar{f}^{(\nu)}, \operatorname{Re} g^{(\nu)}) = {}& \\ \text{terms depending on the components } f^{(\iota)},\ g^{(\iota)} \text{ for } \iota < \nu,& \\ \text{where } w = u + i\,\Phi(z, \bar{z}, u).& \end{aligned} \tag{11}$$

Let $\mathcal{K}$ be the kernel of the linear operator

$$\mathcal{L}(f, g) = -\operatorname{Im} g + d\,\Phi(z, \bar{z}, u)\,(f, \bar{f}, \operatorname{Re} g), \text{ where } w = u + i\,\Phi(z, \bar{z}, u), \tag{12}$$

acting on $\mathbf{V}_+$. This is a linear subspace of $\mathbf{V}_+$ which can be expanded into components, $\mathcal{K}^{(1)} + \mathcal{K}^{(2)} + \dots$. Regardless of the finite-dimensionality of the kernel $\mathcal{K}$, each of its components separately is finite-dimensional, since its coordinate projections are subspaces of polynomials of a chosen weight.

The main observation on which the applications of the Poincaré construction in CR-geometry is based is as follows. The condition that the vector field in a neighborhood of the origin

$$X = 2\operatorname{Re}\left(f\,\frac{\partial}{\partial z} + g\,\frac{\partial}{\partial w}\right)$$

belongs to the Lie algebra $\operatorname{aut} Q$ of infinitesimal holomorphic automorphisms of the model surface Q is $\mathcal{L}(f, g) = 0$.

Consider relation (11) as a recursive relation for calculating successive components of the mapping ψ.

The first step is to extract the 1-component of (11). It has the form $\mathcal{L}(f^{(1)}, g^{(1)}) = T_1(\lambda_0)$. We see that, for a unique definition of $(f^{(1)}, g^{(1)})$, it suffices to choose an element $\lambda_1 \in \mathcal{K}^{(1)}$, and we can write $(f^{(1)}, g^{(1)}) = (f^{(1)}(\lambda_1), g^{(1)}(\lambda_1))$. However, for the solvability of the resulting inhomogeneous linear system, the solvability condition should be added (the condition that $T_1(\lambda_0)$ falls into the image $\mathcal{L}(f^{(1)}, g^{(1)})$. This condition $C_1(\lambda_0, \lambda_1) = 0$ is a real algebraic relation between $\lambda_0 \in G_0$ and $\lambda_1 \in \mathcal{K}^{(1)}$.

The second step is to extract the 2-component (11). It has the form $\mathcal{L}(f^{(2)}, g^{(2)}) = T_2(\lambda_1)$, where T_2 is a real polynomial vector in λ_1. We see that, for a chosen λ_1, for the unique definition of $(f^{(2)}, g^{(2)})$, it suffices to choose an element $\lambda_2 \in \mathcal{K}^{(2)}$. And also there is a new solvability condition $C_2(\lambda_0, \lambda_1, \lambda_2) = 0$. And so on ad infinitum.

If $F(z, \bar{z}, u) = \tilde{F}(z, \bar{z}, u) = 0$, i.e., the mapping is an automorphism of the model surface, then all matching conditions vanish, i.e., $C_\nu = 0$ for all ν. Thus, the system of parameters that defines an automorphism of Q of the form $\chi = \chi_0 + \chi_1 + \dots$ coincides with $G_0 \cup \mathcal{K}$.

In the general case, we obtain the following description of the system of parameters:

$$\Lambda(M_0) = \{(\lambda_0, \lambda_1, \lambda_2, \dots) \in G_0 \cup \mathcal{K} : C_\nu(\lambda_0, \lambda_1, \dots . \lambda_\nu) = 0,\ \nu = 1, 2, \dots\}. \tag{13}$$

If the kernel $\mathcal{K}$ turns out to be finite-dimensional and, therefore, finitely graded, i.e., $\mathcal{K}_\nu = 0$ for $\nu > d$, then Λ, as we see, is a real algebraic subset of a finite-dimensional real space.

If $\tilde{M}_0 = M_0$, i.e., the family of mappings that is parametrized by the set of parameters Λ consists of automorphisms of M_0, then the set Λ is naturally identified with some subgroup of automorphisms M_0. We denote this subgroup, which consists of automorphisms of the form $\chi = \chi_0 + \chi_1 + \dots$, by $\operatorname{Aut}^\mu_0 M_0$. These are exactly the automorphisms of M_0 that preserve the origin and the weight expansion of the complex tangent at zero. We call this subgroup of the stabilizer of zero the *μ-stabilizer*. Using the correspondence $\lambda \to \chi(\lambda)$, we can write

$$\Lambda \approx \operatorname{Aut}^\mu_0 M_0.$$

Since the group $\operatorname{Aut}^\mu_0 M_0$ acts transitively on itself by left shifts, it follows that this action induces a transitive action on Λ. If $\mathcal{K}$ is finite-dimensional, then Λ is a real algebraic variety without singularities.

Denote the corresponding Lie algebra by $\mathrm{aut}_0^\mu M_0$. This is a subalgebra of the complete stabilizer $\mathrm{aut}_0 M_0$, which, in turn, is a subalgebra in the complete algebra $\mathrm{aut}\, M_0$.

Denote the subgroup of automorphisms of M_0 consisting of the automorphisms of the form $\chi = Id+\chi_1+\dots$ by $G_+(M_0)$. Denote the subgroup of automorphisms of M_0 consisting of automorphisms of the form $\chi = \chi_0$ by $G_0(M_0)$.

Before the general formulation of the result thus obtained, we present particular statements related to the model surface Q.

Statement 4.
(a) $G_+(Q_0) \approx \mathcal{K}$, where $\mathcal{K}$ is a linear space.
(b) $\mathrm{Aut}_0^\mu Q_0 \approx \Lambda(Q_0) = G_0 \sqcup \mathcal{K}$.
(c) $\mathrm{Aut}_0^\mu Q_0 = G_0(M_0) \ltimes G_+(Q_0)$ is a semi-direct product.

Thus,
– let μ be an arbitrary grading of the variable z,
– let there be two germs at the origin M_0 and $\tilde{M}_0$ of finite type m that are given by equations in the reduced form

$$\begin{gathered} M_0 = \{v_\beta = \Phi_\beta(z,\bar{z},u) + F_\beta(z,\bar{z},u)\}, \\ \tilde{M}_0 = \{v_\beta = \Phi_\beta(z,\bar{z},u) + \tilde{F}_\beta(z,\bar{z},u)\}, \\ j = 1,\dots,q, \quad F_\beta = o(m_\beta), \;\; \tilde{F}_\beta = o(m_\beta). \end{gathered}$$

Our reasoning proves the following theorem.

Theorem 5: Let μ be an arbitrary grading of the variable z and let Q_0 be a μ-model surface of the germ M_0; then
(a) $G_0(M_0)$ is a subgroup of $G_0(Q_0)$;
(b) the set $\Lambda(M_0)$ parameterizing the family of invertible mappings of M_0 on $\tilde{M}_0$ of the form $\chi = \chi_0 + \chi^{(1)} + \chi^{(2)} + \dots$ has the form (13);
(c) if $\mathrm{aut}_0^\mu Q_0 < \infty$, then $\Lambda(M_0)$ is a nonsingular real algebraic set and $\dim \Lambda(M_0) \leqslant \dim \mathrm{aut}_0^\mu Q_0$;
(d) $\dim \mathrm{aut}_0^\mu M_0 \leqslant \dim \mathrm{aut}_0^\mu Q_0$.

In connection with this theorem, it is appropriate to ask the following question. When $\dim \mathrm{aut}_0^\mu Q_0$ is finite?

Statement 6: $\dim \mathrm{aut}_0^\mu Q_0 < \infty$ if and only if the μ-model surface Q is of finite type and is holomorphically nondegenerate.

Proof. The sufficiency follows from the well-known theorem (see [11]). Let us show the necessity. The necessity of the holomorphic nondegeneracy is obvious. If the μ-model surface Q is of infinite type, then this means that there is a linear dependence among the coordinate forms of some weight. After a linear transformation of the corresponding group of variables w, identical zeros occur among the coordinate forms. This immediately gives the infinite-dimensionality. The assertion has been proven.

This assertion, as well as its proof, are quite similar to those that are available in the ordinary unweighted approach [2]. Note that the conditions of assertion 6 are also a criterion for the finite-dimensionality of the complete algebra $\mathrm{aut}\, Q_0$.

In connection with this assertion, we recall the following definition. A germ is said to be *nondegenerate* if it is of finite type and holomorphically nondegenerate. This definition does not depend on the choice of μ. Therefore the nondegeneracy with respect to one weight means the nondegeneracy with respect to all weights. The nondegeneracy of a μ-model surface Q implies the nondegeneracy of the corresponding germ. The converse assertion fails.

5. Weighted model surface and its automorphisms

The weights introduced for the variables in the groups z and w can naturally be extended to the differentiations. As a result, the Lie algebra of all infinitesimal holomorphic automorphisms of any germ becomes a graded Lie algebra. We denote this graded Lie algebra by $\mathrm{aut}^\mu Q_0$. The introduction of the superscript μ in the notation for the automorphism algebra stresses that different gradings of the variables of the group z transform the same automorphism algebra into different *graded* Lie algebras.

To the variables of the group w_q of maximum weight m_q there correspond the differentiations of the highest negative weight $-m_q$. Therefore, we can write

$$\mathrm{aut}^{\mu}\, Q_0 = \sum_{\nu=-m_q}^{\infty} g_{\nu}.$$

We can consider three subalgebras whose sum decomposes the complete algebra: $g_- = \sum_{\nu<0} g_{\nu}$, g_0, $g_+ = \sum_{\nu>0} g_{\nu}$,

$$\mathrm{aut}^{\mu}\, Q_0 = g_- + g_0 + g_+.$$

We directly verify the validity of the following assertion.

Statement 7.
(a) g_0 is the Lie algebra of the group G_0.
(b) The subgroup G_0 contains a 1-parameter (grading) subgroup

$$(z_{\alpha} \to t^{\mu_{\alpha}}\, z_{\alpha},\ \ w_{\beta} \to t^{m_{\beta}}\, w_{\beta}), \quad t \in \mathbf{R}^* \tag{14}$$

To this subgroup, there corresponds a vector field of weight zero,

$$X_0 = 2\,\mathrm{Re}\,(\sum \mu_{\alpha}\, z_{\alpha}\, \frac{\partial}{\partial\, z_{\alpha}} + \sum m_{\beta}\, w_{\beta}\, \frac{\partial}{\partial\, w_{\beta}}) \tag{15}$$

(c) If $X = \sum_{\nu=-m_q}^{\infty} X_{\nu} \in \mathrm{aut}^{\mu}\, Q_0$, then $X_{\nu} \in \mathrm{aut}^{\mu}\, Q_0$ for every ν.
(d) The μ-stabilizer of zero is the semidirect product $G_0 \rtimes G_+$; correspondingly, $\mathrm{aut}_0^{\mu}\, Q_0 = g_0 + g_+$.
(e) $\mathrm{aut}^{\mu}\, Q_0$ is finite-dimensional if and only if $\mathrm{aut}^{\mu}\, Q_0$ is finitely graded, i.e., only finitely many weight components g_{ν} are nonzero.

Thus, if Q is nondegenerate, then

$$\mathrm{aut}^{\mu}\, Q_0 = \sum_{\nu=-m_q}^{d} g_{\nu},$$

where d is the highest nonnegative nonzero component.

Let $\xi = (z = a,\ w = b)$ be a point of Q. Consider the first group of equations Q, namely, $v_1 = \Phi_1(z, \bar{z})$; make the substitution $z \to a + z,\ \ w_1 \to (b_1 + i\,\Phi_1(a, \bar{a})) + w_1$ and write out the resulting equations:

$$v_1 = \Phi_1(z + a, \bar{z} + \bar{a}) - \Phi_1(a, \bar{a}) = \Phi_1(z, \bar{z}) + d\ , \Phi_1(z, \bar{z})(a, \bar{a}) + \dots.$$

If we assign the corresponding μ-weights to the parameters a, then all summands have the weight m_1. If we consider the weights only with respect to the variables z, then the first summand has the weight m_1, and all other ones have strictly lesser weights. Using the known procedure, we can write out this "truncated" surface $Q(1)$ in the reduced form. For this first step, the procedure is directed only to remove the pluriharmonic terms. If summands of weight less than m_1 remain, then we can recognize that the μ-type has changed and can introduce new weights for the variables of the w_1 group. If this did not happen, i.e., the equation $Q(1)$ returned to its previous form after reduction, then this means that there exists a holomorphic automorphism $Q(1)$ of the form

$$z \to a + z,\ \ w_1 \to b_1 + w_1 + P_1(z, a, \bar{a}),$$

taking $(z = 0,\ w_1 = 0)$ into $(z = a,\ w_1 = b_1 + i\,\Phi_1(a, \bar{a}))$, where the weight of the polynomial P_1 is strictly less than m_1.

Further, we pass to the consideration of the second "truncated" surface $Q(2)$ given by the equations of $Q(1)$ and the group of equations of the form $v_2 = \Phi_2(z, \bar{z}, u_1)$ at the point $(z = a,\ w_1 = b_1 + i\,\Phi_1(a, \bar{a}),\ w_2 = b_2 + i\,\Phi_2(a, \bar{a}, b_1))$. Here the equations of $Q(1)$ are new reduced equations. Now the construction procedure of the reduced equations of $Q(2)$ and the calculation of the μ-type at a new point are repeated again. If the type is not changed, then we obtain a polynomial-triangular automorphism of $Q(2)$ of the form

$$\begin{aligned} z \to a + z,\ \ w_1 &\to (b_1 + i\,\Phi_1(a, \bar{a})) + w_1 + P_1(z, a, \bar{a}),\\ w_2 &\to (b_2 + i\,\Phi_2(a, \bar{a}, b_1)) + w_2 + P_2(z, w_1, a, \bar{a}, b_1), \end{aligned}$$

such that it takes $(z=0,\ w_1=0,\ w_2=0)$ to $(z=a,\ w_1=b_1+i\,\Phi_1(a,\bar a),\ w_2=b_2+i\,\Phi_2(a,\bar a,b_1))$, and the weight P_2 is strictly less than m_2. And so on, until the last group of variables. We denote by S_ξ the uniquely defined polynomial-triangular "shift" constructed as the result of this process.

Let Q be a weighted model surface whose μ-type at the origin is equal to m. Let Q^m be the collection of points of Q such that, at these points, the μ-type is equal to m. Thus, for every point $\xi \in Q^m$, there exists a unique automorphism S_ξ that is constructed as a result of the reduction of equations and takes the origin to ξ. The totality of these "shifts" is a subgroup, which we denote by GS; we denote its algebra by gs.

To explicitly describe the fields in gs, we are to differentiate the triangular-polynomial replacements obtained above with respect to the parameters (to select summands linear in parameters).

Let St be the stabilizer of the origin in the group $\operatorname{Aut} Q_0$ of holomorphic automorphisms of Q, and let st be the corresponding Lie algebra. It is clear that St is a subgroup and st is a subalgebra. Under the standard (unweighted) approach, st is the sum of all nonnegative components of the algebra (i.e., g_0+g_-). Under the weighted approach, this is not the case in general. In this connection, we denote by st_- the subalgebra consisting of the fields included in to g_- that vanish at the origin. The fields

$$X=(f,g)=(f_1,\dots,f_p;\,g_1,\dots,g_q),$$

belonging to st_- are the fields that, in addition to the tangency condition for Q,

$$\operatorname{Im} g = d\,\Phi(z,\bar z,u)\,(f,\bar f,\operatorname{Re} g),\ \text{ where } w=u+i\,\Phi(z,\bar z,u),$$

satisfy also the following conditions

$$f(0,0)=0,\ g(0,0)=0,\ \text{the weight: } f_\alpha<\mu_\alpha,\ \text{the weight: } g_\beta<m_\beta. \tag{16}$$

On the other hand, the gs algebra is formed by the fields with the tangency condition such that

$$f(z,w)=a=const,\ g_1=b_1+p_1(z,a),\ g_2=b_2+p_2(z,w_1,a,b_1),\dots, \tag{17}$$

where the polynomials p_β are uniquely recovered from the tangency condition; they have the weights less than m_β and vanish at $a=0,\ b=0$.

This, we obtain the following assertion.

Statement 8.
(a) There is a decomposition of the subalgebra $g_- = gs + st_-$.
(b) gs is the subalgebra corresponding to the subgroup of shifts GS.
(c) The orbit of the origin with respect to the full automorphism group $\operatorname{Aut} Q_0$ coincides with the orbit of the origin with respect to GS.

Let G_+ be the subgroup of the automorphism group Q consisting of the automorphisms whose expansions in terms of components have the form

$$\chi = Id + \chi^{(1)} + \chi^{(2)} + \dots.$$

These are the coordinate changes that we used when describing the Poincaré construction. Similarly, denote by G_- the subgroup of the automorphism group Q consisting of automorphisms whose component expansions have the form

$$\chi = Id + \chi^{(-1)} + \chi^{(-2)} + \dots + \chi^{(-m_q)}.$$

While the decomposition of elements of G_+ can, generally speaking, contain components with an arbitrarily large number, note that the decomposition of G_- is bounded from below by the number $-m_q$, where m_q is the highest weight in the group of variables w. This automatically implies that the transformations in G_- are polynomial. Write $St_- = G_- \cap St$.

We have the following assertion.

Statement 9. (a) To the subgroup G_+ of automorphisms, there corresponds the Lie algebra g_+.
(b) To the subgroup G_- of automorphisms, there corresponds the Lie algebra g_-.
(c) To the subgroup St_- of automorphisms, there corresponds the Lie algebra st_-.

Thus, we have the decomposition $\mathrm{aut}^{\mu} Q_0 = gs + st_- + g_0 + g_+$. These four subalgebras correspond to four subgroups

$$GS, \quad St_-, \quad G_0, \quad G_+,$$

which were described above.

Concerning the subgroup St_-, a statement holds that is completely analogous to Statement 3.

Statement 10/ If the model surface Q has finite μ-type and there is an automorphism $(f(z,w), g(z,w)) \in St_-$ such that $f(z,w) = z$, then $g(z,w) = w$, i.e., this is the identity mapping.

The proof carries over from Statement 3 without changes. .

Above (Theorem 5), using the recurrent Poincaré construction, we have showed that the dimension of the μ-stabilizer of a point ξ in the automorphism group of the germ is estimated in terms of the dimension of the μ-stabilizer of zero in the automorphism group of its model surface. However, in order to estimate the entire stabilizer, it is necessary to estimate the dimension of $St_-(M_\xi)$. To estimate the dimension of st_-, we have a complete analog of Theorem 5. However, there is a difference in the scheme of the proof. While the proof of Theorem 5 is the application of the Poincaré construction to a family of mappings, the proof of Theorem 11 is the Poincaré construction applied to vector fields. Moreover, the application of Poincaré's construction to estimate the dimensions of spaces of vector fields rather than a family of mappings enables us to estimate the dimension of the algebra of the perturbed surface using the corresponding dimension for the model one not only for $st_-(M_\xi)$, but also for $g_-(M_\xi)$ and for the entire $\mathrm{aut}\,(M_\xi)$.

Theorem 11. Let μ be an arbitrary grading of the variable z and let $Q_0(\mu)$ be the μ-model surface of M_0. Then

$$\dim \mathrm{aut}\, M_0 \leqslant \dim \mathrm{aut}\, Q_0(\mu).$$

Proof. Let $\{v = \Phi(z,\bar z,u) + \Psi(z,\bar z,u)\}$ be a germ, let $\{v = \Phi(z,\bar z,u)\}$ be the model surface, and let $\Psi(z,\bar z,u)$ be the perturbation. That is, if Φ_j and Ψ_j are the coordinates of Φ and Ψ corresponding to w_j, then Φ_j has the weight m_j, and Ψ_j is the sum of components of greater weights, $\Psi_j = \Psi_{j(m_j+1)} + \dots$. We can write out the equation of the perturbed surface as an expansion in terms of components, namely, $v = \Phi + \Psi^{(1)} + \Psi^{(2)} + \dots$, where $\Psi^{(\nu)} = (\Psi_{1(m_1+\nu)}, \dots, \Psi_{q(m_q+\nu)})$. It' is clear that the 0-component of the right-hand side of the equation is Φ. Let

$$X = 2\,\mathrm{Re}\left(\sum f_i(z,w)\frac{\partial}{\partial z_\nu} + \sum g_j(z,w)\frac{\partial}{\partial w_j}\right) = (f(z,w), g(z,w))$$

be a field in $\mathrm{aut}\, M_0$ and let $X = \sum X^{(\nu)}$ be its expansion in the weight components. Then

$$X^{(\nu)} = (f^{(\nu)}, g^{(\nu)}) = (f_{1(\mu_1+\nu)}, \dots, f_{q(\mu_q+\nu)}; g_{1(m_1+\nu)}, \dots, g_{q(m_q+\nu)})$$

is the set of coefficients of $X^{(\nu)}$ (the ν-component of the field). Writing out the tangency condition, we obtain

$$\begin{aligned}\mathcal{L}(f,g;\Phi,\Psi) = -\mathrm{Im}\, g(z,w) + 2\,\mathrm{Re}\,(\partial_z(\Phi(z,\bar z,u) + \Psi(z,\bar z,u))(f(z,w)))\\ +\partial_u(\Phi(z,\bar z,u) + \Psi(z,\bar z,u))(\mathrm{Re}\, g(z,w)) = 0,\\ \text{where } w = u + i\,(\Phi(z,\bar z,u) + \Psi(z,\bar z,u)).\end{aligned} \tag{18}$$

This linear expression (AN operator) $\mathcal{L}(f,g;\Phi,\Psi)$ is the sum

$$\mathcal{L}(f,g;\Phi,\Psi) = L(f,g;\Phi) + L'(f,g;\Phi,\Psi),$$

where $L(f,g;\Phi) = \mathcal{L}(f,g;\Phi,0)$. That is, $L(f,g;\Phi) = 0$ is the record of the fact that the field (f,g) is tangent to the model surface Q, and all terms $\mathcal{L}$ depending on Ψ are collected in L'. Let us write relation (18) as the vanishing condition of its components. It is clear that, if the exponent ν is small enough ($\nu < -m_q$), then $X^{(\nu)} = 0$, and the relation in this component gives no relations for the field coefficients. We obtain the first meaningful relation for $\nu = -m_q$. We have $X^{(-m_q)} = (0,\dots,0;0,\dots,0,g_{q0})$, where $[g_{q0}] = 0$, i.e., g_{q0} is constant. A nontrivial relation for the $(-m_q)$-component in (18) is $\mathrm{Im}\, g_{q0} = 0$. The next $(-m_q+1)$-th component involves $X^{(-m_q+1)} = (f^{(-m_q+1)}, g^{(-m_q+1)})$ and $X^{(-m_q)}$, etc., and the ν-th component of (18) has the form

$$L(f^{(\nu)}, g^{(\nu)}) + \text{terms depending on } (f^{(\nu-1)}, g^{(\nu-1)}), (f^{(\nu-2)}, g^{(\nu-2)}), \dots = 0.$$

That is, this system of linear relations has the property of triangularity. This enables us to estimate the rank of the complete (perturbed) system $\mathcal{L}(f, g; \Phi, \Psi) = 0$ using the rank of the model system $L(f, g; \Phi) = 0$. As a result, we obtain an estimate for the dimension. The theorem has been proven.

Remark 12. (a) The property of triangularity of system (18) noted in the proof enables us to make a more precise statement. Let $\nu \in \mathbb{Z}$; denote by $\mathcal{V}^\nu$ the subspace of the space vector fields such that their weight decomposition does not contain components of weight less than ν. Then we can claim that

$$\forall\, \nu \quad \dim(\mathrm{aut}\, M_0 \cap \mathcal{V}^\nu) \leqslant \dim\left(\mathrm{aut}\, Q_0 \cap \mathcal{V}^\nu\right).$$

(b) The reasoning given in the proof applies not only to the full automorphism algebra but also to any of its subalgebras. In particular, for these subalgebras, we can take gs and st_-. Then we obtain

$$\dim\, gs(M_0) \leqslant \dim\, gs(Q_0), \quad \dim\, st_-(M_0) \leqslant \dim\, st_-(Q_0).$$

Since the condition for the holomorphic homogeneity of the germ is the relation $\dim\, gs(M_0) = \dim M = 2n + K$, it follows from the first inequality that any model surface of a holomorphically homogeneous germ is holomorphically homogeneous. This assertion was previously proven in [20].
(c) The dimension of the automorphism algebra of the germ does not depend on the μ-gradings and, therefore, the inequality in Theorem 11 can be replaced by

$$\dim\, \mathrm{aut}\, M_0 \leqslant \min \dim\, \mathrm{aut}\, Q_0(\mu) \text{ over all } \mu.$$

As in [2], introduce the concept of highest weight for the μ-type $m = ((m_1, k_1), \ldots, (m_q, k_q))$, namely, we set $\lambda = \lambda(m) = m_q$. To have the ability to speak about the μ-type of a surface at a point ξ of this surface, we introduce the notation $m(\xi)$; then the highest weight at the point ξ is $\lambda(\xi) = \lambda(m(\xi))$. Next, we define the following subsets of Q:

$$Q^{m\prime} = \{\xi \in Q : m(\xi) = m'\}, \quad Q^{\lambda'} = \{\xi \in Q : \lambda(\xi) = \lambda'\}$$

A subset of a real affine space is said to be *semi-algebraic* if it is given by conditions of the form $\{R^1(x) = 0,\ R^2(x) \neq 0\}$, where R^1 and R^2 are two finite sets of real polynomials.

Theorem 13.
(a) Let $\xi \in Q$; then the highest weight at ξ does not exceed the highest weight at zero, i.e., $\lambda(\xi) \leqslant \lambda(0)$.
(b) The sets $Q^{m\prime}$ and $Q^{\lambda'}$ are semi-algebraic for any m' and λ'.
(c) The set of values of the functions $m(\xi)$ and $\lambda(\xi)$ is finite.

Proof. Part (a) follows from our consideration of the procedure of constructing the reduced form of equations at a point of the model surface. Part (b) is obvious. The assertion of point (c) concerning the function $\lambda(\xi)$ follows from (a). The assertion concerning $m(\xi)$ follows from the fact that, for chosen highest weight and codimension, there are only finitely many μ-types m'.

Theorem 14. Let the μ-model surface Q be nondegenerate and holomorphically homogeneous. Then the group of its holomorphic automorphisms $\mathrm{Aut}\, Q$ is a subgroup of the birational automorphism group $\mathbf{C}^N$ (the Cremona group) consisting of mappings of uniformly bounded powers $d(\chi)$. The constant bounding the powers depends only on N,

$$d(\chi) \leqslant C(N).$$

Proof. The scheme of proving such assertions, going back to V. Kaup [12], was used many times (see[14]). Since the holomorphic homogeneity is realized by polynomial-triangular shifts, it suffices to prove this assertion for the element of the stabilizer. To this end, the polynomiality of fields with a bound for the degree and the presence of a grading field are required. All this is present in our situation. Thus, the theorem is proved.

The question of the holomorphic homogeneity of a manifold is very subtle. In [16], for model surfaces (regular, unweighted), a simple criterion was given. It was shown that a point of the model surface falls into orbit of the origin if and only if its Bloom–Graham-type is the same as the Bloom–Graham type of the origin. This criterion remains valid for weighted model surfaces as well.

Theorem 15: Let Q be the μ-model surface and let ξ be a point of Q. This point belongs to the orbit of the origin in the automorphism group Q if and only if the Bloom–Graham–Stepanova μ-type of Q at the point ξ coincides with the μ-type of Q at the origin.

Proof. If the μ-type of Q at the point ξ coincides with the μ-type Q at the origin, then the automorphism is constructed as follows. We move the origin to the point ξ, re-expand the defining polynomials in the new coordinates, and start the process of constructing the reduced form of new equations. As a result, the equations take the same form (otherwise the type changes) as that at the origin. In this way, a triangular-polynomial automorphism is obtained that sends the origin to ξ. Conversely. Let there be an automorphism χ taking the origin to ξ. It can be represented as a composition of two automorphisms $\chi = \chi_0 \circ \tilde{\chi}$, where χ_0 is an element of the zero stabilizer and $\tilde{\chi} \in GS$ is an element of G_-, which is identical on the complex tangent. Then $\tilde{\chi} = \chi = (\chi_0)^{-1} \circ \chi$. This is an automorphism taking the origin to ξ with the identical action on the complex tangent. Such mappings preserve the type. Hence the μ-types at zero and ξ are the same. The theorem has been proven.

Corollary 16. The μ-model surface Q is holomorphically homogeneous if and only if all its points have the same μ-type (the Bloom–Graham–Stepanova type with the weight μ).

6. Examples of holomorphically homogeneous μ-nondegenerate model surfaces

As the **first example** of a holomorphically homogeneous μ-model surface we propose the hypersurface (1) as mentioned in the introduction. In this example, the type has the following form:

$$\Big((\mu_1 = 1, n_1 = 2), (\mu_2 = 2, n_2 = 1), \dots, (\mu_p = p, n_p = 1);\ (m_1 = p+1,\ k_1 = 1)\Big),$$
$$\text{where } p \geqslant 1,\ q = 1,\ n = p+1,\ N = p+2.$$

For the convenience of the reader, and as an illustration of the general considerations made above, we give here a description of the automorphisms of the first hypersurface in this series [14] (Theorem 15). Let $p = 2,\ n = 3,\ N = 4$. We obtain a hypersurface of the space $\mathbf{C}^4$,

$$Q = \{v = 2\,\mathrm{Re}(z_1\,\bar{\zeta} + z_2\,\bar{\zeta}^2)\} \tag{19}$$

The weights are assigned as follows:

$$[z_2] = [\zeta] = 1,\ [z_1] = 2,\ [w] = [u] = 3.$$

For brevity, we write out the vector field of the form

$$X = 2\,\mathrm{Re}\left(f_1\,\frac{\partial}{\partial z_1} + f_2\,\frac{\partial}{\partial z_2} + h\,\frac{\partial}{\partial \zeta} + g\,\frac{\partial}{\partial w}\right)$$

in the form (f_1, f_2, h, g). The algebra $\mathrm{Aut}\,Q$ has the form $g_{-3} + g_{-2} + g_{-1} + g_0 + g_1$, and

$$\begin{aligned}
g_{-3} &= \{(\ 0,\quad 0,\quad 0,\quad d\ )\},\\
g_{-2} &= \{(\ a,\quad 0,\quad 0,\quad 2\,i\,\bar{a}\,\zeta\ )\},\\
g_{-1} &= \{(\ -2\,\bar{c}\,z_2 + i\,e\,\zeta,\quad b,\quad c,\quad 2\,i\,\bar{c}\,z_1 + 2\,i\,\bar{b}\,\zeta^2\ )\},\\
g_0 &= \{(\ \alpha_1\,z_1 - \bar{\alpha}_2\,\zeta^2,\quad (2\,\alpha_2 - \alpha_3)\,z_2 + \alpha_2\,\zeta,\quad (\alpha_3 - \bar{\alpha}_1)\,\zeta,\quad \alpha_3\,w\ )\},\\
g_1 &= \{(\ 2\,i\,\bar{\beta}_1\,z_1\,\zeta + \beta_1\,w,\quad 2\,i\,\bar{\beta}_1\,z_2\,\zeta - i\,\beta_1\,z_1 + i\,\beta_2\,\zeta^2,\quad i\,\bar{\beta}_1\,\zeta^2,\quad 2\,i\,\bar{\beta}_1\,\zeta w\ )\},\\
& a,\ b,\ c,\ \alpha_1,\ \alpha_2,\ \beta_1 \in \mathbf{C},\quad d,\ e,\ \alpha_3,\ \beta_2 \in \mathbf{R}.
\end{aligned}$$

We write g_{-1} as a direct sum $g'_{-1} + st_-$, where

$$g'_{-1} = \{(-2\,\bar{c}\,z_2,\ b,\ c,\ 2\,i\,\bar{c}\,z_1 + 2\,i\,\bar{b}\,\zeta^2)\},$$
$$st_- = \{(i\,e\,\zeta,\ 0,\ 0,\ 0)\}.$$

Then the algebra gs corresponding to the group GS of "shifts" has the form $gs = g_{-3} + g_{-2} + g'_{-1}$. It is parametrized by the set (a, b, c, d); respectively, $\dim gs = 7$. The subgroup GS itself, which provides the holomorphic homogeneity of Q, consists of transformations of the form

$$\begin{gathered} z_1 \to A + z_1, \qquad z_2 \to B + 2\,\bar{A}\,\zeta + z_2, \qquad \zeta \to C + \zeta, \\ w \to D + 2\,i\,(A\,\bar{B} + A^2\,\bar{C} + (\bar{B} + 2\,A\,\bar{C})\,z_1 + \bar{A}\,z_2 + \bar{A}^2\,\zeta + \bar{C}\,z_1^2) + w, \end{gathered} \tag{22}$$

where (A, B, C, D) is an arbitrary point of Q.

We have $\dim st_- = 1$, the field $(i\,\zeta, 0,\ 0,\ 0)$ generates the group St_- which has the form

$$z_1 \to z_1 + i\,t\,\zeta, \quad z_2 \to z_2, \quad \zeta \to \zeta, \quad w \to w.$$

The algebra g_0 is parametrized by the family $(\alpha_1, \alpha_2, \alpha_3)$; respectively, $\dim g_0 = 5$. To calculate the group G_0 corresponding to g_0, write $\gamma = \alpha_1 + \bar{\alpha}_1 - \alpha_3$. If $\gamma \neq 0$, then we obtain

$$\begin{gathered} z_1 \to \left(z_1 - \bar{\alpha}_2\left(\frac{e^{\gamma\,t} - 1}{\gamma}\right)\zeta^2\right)e^{\alpha_1\,t}, \\ z_2 \to \left(z_2 + \alpha_2\left(\frac{e^{1-\bar{\gamma}\,t}}{\bar{\gamma}}\right)\zeta\right)e^{(2\,\alpha_1 - \alpha_3)\,t}, \\ \zeta \to \zeta\,e^{(\alpha_3 - \bar{\alpha}_1)\,t}, \quad w \to w\,e^{\alpha_3\,t}. \end{gathered}$$

The degenerate directions $\gamma = 0$ are obtained by passing to the limit.

The subalgebra g_+ consists of a single component g_1, which is parameterized by the set (β_1, β_2), respectively, $\dim g_1 = 3$. The field $(0,\ i\,\zeta^2,\ 0,\ 0)$ in g_1 $(\beta_1 = 0,\ \beta_2 = 1)$ generates the transformation

$$z_1 \to z_1,\ z_2 \to z_2 + i\,t\,\zeta^2,\ \zeta \to \zeta,\ w \to w. \tag{23}$$

The transformations in g_1 with $\beta_2 = 0$ have the form

$$\begin{gathered} z_1 \to \frac{z_1}{(1 - i\,\bar{\beta}_1\,\zeta\,t)^2}, \quad z_2 \to \frac{z_2 - i\,\beta_1\,z_1\,t}{(1 - i\,\bar{\beta}_1\,\zeta\,t)^2}, \\ \zeta \to \frac{\zeta}{1 - i\,\bar{\beta}_1\,\zeta\,t}, \quad w \to \frac{w}{(1 - i\,\bar{\beta}_1\,\zeta\,t)^2}. \end{gathered} \tag{24}$$

The transformations (23) and (24) generate the group G_+ corresponding to g_+.

As the **second example** of holomorphically homogeneous μ-model surface, we can propose a hypersurface from [21]. This is a hypersurface in the space $\mathbf{C}^{n+1}$ with the coordinates $(z_1, \ldots, z_n,\ w = u + i\,v)$ given by the equation

$$v = 2\ \mathrm{Re}(z_1\bar{z}_2 + z_1^2\bar{z}_3) + \sum_4^n \pm|z_j|^2.$$

In this example, the type has the following form:

$$\Big((\mu_1 = 2, n_1 = 2), (\mu_2 = 4, n_2 = 1), (\mu_3 = 3, n_3 = n - 3);\ (m_1 = 6,\ k_1 = 1)\Big),$$

This hypersurface is holomorphically homogeneous, 2-nondegenerate, the dimension of its group of automorphisms is equal to $n^2 + 7$, and this is the maximum in the class of such hypersurfaces [21].

Third example. The hypersurface from [22]. This is a hypersurface in the space $\mathbf{C}^{n+1}$ with the coordinates $(z_1, \ldots, z_n,\ w = u + i\,v)$ given by the equation

$$v = 2\ \mathrm{Re}(z_1\bar{z}_2) + |z_1|^4 + \sum_3^n \pm|z_j|^2. \tag{25}$$

In this example, the type has the following form:

$$\Big((\mu_1 = 1, n_1 = 1), (\mu_2 = 3, n_2 = 1), (\mu_3 = 2, n_3 = n - 2);\ (m_1 = 4,\ k_1 = 1)\Big),$$

This hypersurface is holomorphically homogeneous and 1-nondegenerate (Levi nondegenerate). In the class of these hypersurfaces, the group of the nondegenerate standard hyperquadric $\{v =< z, \bar z >\}$, where $< z, \bar z >$ is a nondegenerate Hermitian form, has the maximum dimension $(n+2)^2 - 1$. The hypersurface (25) is submaximal, i.e., realizes the next, after that of the hyperquadric, maximum value of the dimension of the group, equal to $n^2 + 4$.

Fourth example. A family of hypersurfaces from the list of the paper [23]. These are hypersurfaces in the space $\mathbf{C}^3$ with the coordinates $(z_1, z_2,\ w = u + i\,v)$ given by the equations $v = 2\,\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) + \mathrm{Re}(z_1)^m$ (m is an arbitrary positive integer),

$$\Big((\mu_1 = 1, n_1 = 1), (\mu_2 = m-1, n_2 = 1);\ (m_1 = m,\ k_1 = 1)\Big),$$

These tubular hypersurfaces are holomorphically homogeneous and Levi nondegenerate. The hypersurfaces in the examples three and four are some generalizations of the Winkelmann hypersurfaces $v = 2\,\mathrm{Re}(z_1\,\bar z_2) + |z_1|^4$.

All the examples considered here are hypersurfaces. Using hypersurfaces, one can obtain examples of higher codimension by considering their direct products. But this is far from the only possibility.

Since all these μ-model hypersurfaces are nondegenerate and holomorphically homogeneous, it follows that Theorem 11 applies to them, i.e., their automorphisms are birational.

7. Completely μ-nondegenerate model surfaces and the Tanaka theory

It is also possible to offer a fairly wide class of examples of homogeneous model hypersurfaces for an arbitrary weight μ. These μ-model surfaces are natural generalizations of ordinary completely nondegenerate model surfaces [15].

Let us choose an arbitrary grading of the variable z

$$\mu = ((\mu_1, n_1), (\mu_2, n_2), \dots, (\mu_p,\ n_p))$$

and consider the sequence of spaces of reduced polynomials $\mathcal{N}_\nu$ of all positive integer weights. We do not claim, however, that all $\kappa_\nu = \dim\,\mathcal{N}_\nu > 0$. Renumber the nonzero spaces by $\mathcal{N}$. We index them by the step number at which they occur (i.e., we skip the null spaces) rather that by the weight.

It is clear that the first (minimum) possible weight for a nonzero space is $2\,\mu_1$. Here $\mathcal{N}_1$ is the space of Hermitian forms in z_1. The dimension of this space is $\kappa_1 = n_1^2$. Correspondingly, we introduce the variable w_1 and write $m_1 = [w_1] = 2\,\mu_1$ and $k_1 = \kappa_{m_1} = n_1^2$.

Further, among the given polynomials in z_1, z_2, u_1, one should choose a subspace of polynomials of the weight that is minimal after $2\,\mu_1$. If $\mu_2 < 2\,\mu_1$, then this weight is equal to $\mu_1 + \mu_2$; if not, then to $3\,\mu_1$. After this, we introduce the variable w_2 and set $m_2 = [w_2]$ equal to the second weight and write $k_2 = \kappa_{t_2}$. And so on. We stop at an arbitrary q-th step. Now, for $\beta = 1, \dots, q-1$, we set Φ_β to be equal to the family of basic forms of the space $\mathcal{N}_\beta$ and Φ_q to an arbitrary linear independent family $(\Phi_q^1, \dots, \Phi_q^k)$ of elements of $\mathcal{N}_q$, i.e., $1 \leqslant k \leqslant k_q$. As a result, in the space $\mathbf{C}^{n+K},\ n = \sum_1^p n_\alpha,\ K = (\sum_1^{q-1}\ k_\beta + k)$ we obtain the surface

$$Q = \{v_\beta = \Phi_\beta(z, \bar z, u),\ \ \beta = 1, \dots, q\} \tag{26}$$

Note that the arbitrariness in the choice of such a surface is the arbitrariness in the choice of q and of the family of coordinates of the highest weight form Φ_q.

Statement 17:

(a) Every such surface Q is of finite μ-type.
(b) Every such surface is holomorphically homogeneous.
(c) If $m_q \geqslant 2\,\mu_p + 1$, then Q is holomorphically nondegenerate.
(d) Two such surfaces with the same parameter q and the same highest forms Φ_q are linearly equivalent.
(e) If the multiplicity of the highest weight k_q takes the maximum value $k_q = \dim\mathcal{N}_q$, then all model surfaces are linearly equivalent (i.e., such a surface is unique).

Proof/ (a) By construction, Q has a finite μ-type at zero. (b) holds because the re-expansion of equations at an arbitrary point by triangular-polynomial transformations reduces to the original form. (c) holds since this is a consequence of the fact that, under this condition, the space $\mathcal{N}_1+\cdots+\mathcal{N}_{q-1}$ contains the space of all Hermitian forms in z. (d) holds since one of these surfaces is transformed into another by the transformation

$$z \to z, \quad w_\beta \to \rho_\beta\, w_\beta, \;\; \beta = 1,\dots,q-1, \;\; w_q \to w_q,$$

where ρ_β is a real nondegenerate linear transformation that makes the transition from one basis to another in the corresponding space. (e) If $k_q = \dim \mathcal{N}_q$ then the coordinates of Φ_q form a basis of $\mathcal{N}_q$, and any two bases are connected by a nondegenerate linear transformation. The assertion has been proven.

Definition 18. If the surface Q constructed above is holomorphically nondegenerate, then we call it a *completely μ-nondegenerate* model surface of the highest weight m_q.

To any completely μ-nondegenerate model surface Theorem 12 applies, i.e., the group of holomorphic automorphisms of the surface consists of birational transformations of uniformly bounded powers.

The ordinary, unweighted, completely nondegenerate model surfaces have the following property. If the highest degree of the model completely nondegenerate surface Q is greater than two, then $g_+ = 0$.

Thus, the completely nondegenerate model surfaces are divided into two classes: the nondegenerate quadrics for which the highest degree is $l = 2$ and for which g_+ may be nontrivial, and the others, which have $l \geqslant 3$ and no g_+ component. There is another important difference between these two classes. The criterion for the finite-dimensionality of the algebra of a quadric is given by two conditions: the linear independence of the Hermitian forms and the absence of their common kernel. Two general requirements turn into these simple conditions for a quadric: the finiteness of the Bloom–Graham-type and the holomorphic nondegeneracy. Here the holomorphic nondegeneracy does not follow from the single finiteness condition for the type. However, for $l \geqslant 3$, the requirement of complete nondegeneracy is a condition only on the Bloom–Graham-type. The holomorphic nondegeneracy follows from this condition on the type.

For our weighted analog of complete nondegeneracy, we can also introduce a partition into such classes. Consider the sequence $\{\mathcal{Q}_s\},\ s = 2, 3, \dots$, where $\mathcal{Q}_s$ is the last (i.e., for which all multiplicities are maximal) model completely μ-nondegenerate surface of the highest weight s.

Lemma 19: There exists an $\mathbf{s}$ such that all $\mathcal{Q}_s$ are holomorphically nondegenerate for $s \geqslant \mathbf{s}$.

Proof. Starting from some s, all spaces $\mathcal{N}_s$ contain the space of all Hermitian forms in z. Therefore, if the multiplicity of the weight s is equal to the dimension of $\mathcal{N}_s$, then, among the right-hand sides of the equations defining $\mathcal{Q}_s$, a basis of the space of Hermitian forms is contained. This ensures the holomorphic nondegeneracy. The lemma is proven.

Definition 20. We call the least of the weights $\mathbf{s}$ whose existence is proved in the lemma the *critical* weight.

It is clear that the critical weight depends on the choice of the basic set of weights μ, i.e., $\mathbf{s} = \mathbf{s}(\mu)$.

Now we are ready to formulate a weighted analog of the g_+-conjecture.

The weighted g_+-conjecture. Let Q be a completely μ-nondegenerate model surface of highest weight $l = m_q$ such that $l > \mathbf{s}(\mu)$; then $g_+ = 0$.

In the proof of the standard g_+-conjecture ([17], [18], [19]), N. Tanaka's theory plays a significant role. The central concepts of this theory are a fundamental graded algebra, the Tanaka continuation, and the standard model. The bridge between the theory of holomorphically homogeneous model surfaces and Tanaka's theory is the algebra g_-, which is fundamental in the Tanaka theory.

Tanaka's theory is not directly applicable to weighted model surfaces. In this new situation, g_- is not fundamental. However, after some editing of Tanaka's theory, a connection with the theory of holomorphically homogeneous weighted model surfaces is restored.

How should Tanaka's theory be changed? Here is a small sketch of these changes.

The first modification concerns the notion of a fundamental graded algebra. Let $\mathfrak{g} = \sum_{-l}^{-1} \mathfrak{g}_j$ be a finite-dimensional graded Lie algebra. Let it be generated (as a Lie algebra) by the following family of its components:

$$\mathfrak{g}_{-\mu_1},\ \mathfrak{g}_{-\mu_2}, \dots, \mathfrak{g}_{-\mu_p}.$$

It is not assumed here that all $\mathfrak{g}_{-\nu}$ are nonzero for all $1 \leqslant \nu \leqslant l$. We call such an algebra a *μ-fundamental graded Lie algebra*. If, in addition, $\mathfrak{g}_{-l} \neq 0$, then we say that this is an algebra of the highest weight l. It is clear that such an algebra is an analog of our subalgebra gs.

We say that such a μ-fundamental algebra $\mathfrak{g}$ is *nondegenerate* if it follows from the fact that $Y \in g$ commutes with all generators $\mathfrak{g}_{-\mu_1},\ \mathfrak{g}_{-\mu_2}, \dots, \mathfrak{g}_{-\mu_p}$ that $Y = 0$.

As for an analog of the Tanaka continuation, here a new phenomenon occurs. This is the ability to extend $\mathfrak{g}$ by adding new negative components. From the algebraic point of view, every such field defines an endomorphism of the algebra $\mathfrak{g}$. Therefore, these (negative) weight components of the extension are constructed as spaces of endomorphisms with the necessary family of conditions (the Jacobi identity). We thus obtain a family of new negative components $\mathfrak{st}_-$. This is an analog of our component st_-. After this, just as above, we can define the component $\mathfrak{g}_0$ as the space of derivations on $\mathfrak{g}_- = \mathfrak{g} + \mathfrak{st}_-$ (an analog of our g_-). Further, as in the unweighted case, the sequence of components of positive weights is recursively constructed as spaces of operators on the already constructed components with a condition imitating the Jacobi identity. That is, we obtain $\mathfrak{g}_+ = \mathfrak{g}_1 + \mathfrak{g}_2 + \dots$ (an analog of our g_+).

One can hope (but this needs a proof) that, if $\mathfrak{g}$ is μ-fundamental and nondegenerate, then
(a) there exists a $j > 0$ such that $\mathfrak{g}_j = 0$;
(b) if, for some $j > 0$, it turns out that $\mathfrak{g}_j = 0$, then $\mathfrak{g}_{j+1} = 0$; i.e., in this case, the extension of the algebra $\mathfrak{g}$ is finitely graded and finite-dimensional;
(c) an analog of the standard model $\mathfrak{G}_(\mathfrak{g})$, as a CR-manifold, is equivalent to G_- for a holomorphically homogeneous weighted model surface for which $gs = \mathfrak{g}$.

If we plan to use this technique for our purposes, then we need to equip the generators $\mathfrak{g}_{-\mu_1},\ \mathfrak{g}_{-\mu_2}, \dots, \mathfrak{g}_{-\mu_p}$ with a complex structure J. In this case, we shall say that we have a CR-fundamental μ-graded Lie algebra.

Analogously, we can construct the standard model, which is a holomorphically homogeneous CR-manifold whose Bloom–Graham type is encoded by the algebra $\mathfrak{g}$. In particular, $\mathfrak{g}_{-\mu_1} + \mathfrak{g}_{-\mu_2} + \dots + \mathfrak{g}_{-\mu_p}$ is its complex tangent. To this end, we should consider the connected Lie group $\mathfrak{G}$ generated by $\mathfrak{g}$ as a real submanifold in the complex Lie group $\mathfrak{G}^c$ generated by the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$.

Completely analogously to the concept of a CR-universal fundamental graded Lie algebra, we introduce the notion of a CR-universal μ-fundamental graded Lie algebra. This is a fundamental graded Lie algebra $\mathfrak{g}$ such that, in the process of its generation by the base set $\{\mathfrak{g}_{-\mu_j}\}$, the growth dimension is maximal under every commutation of two components.

The question concerning the triviality of the components of positive weight for the universal weighted algebra is an algebraic version of the weighted g_+-hypothesis formulated above for weighted completely nondegenerate model surfaces.

The weighted $\mathfrak{g}_+$-hypothesis (an algebraic version): Let $\mathfrak{g}$ be a nondegenerate CR-universal μ-fundamental Lie algebra of highest weight l such that $l > \mathbf{s}$; then $\mathfrak{g}_+ = 0$ (there are no extensions of positive weight).

The definition of critical weight appeals to concepts external for the theory graded algebras. To avoid this, the last condition can be be replaced by the sufficient inequality $l > 2\mu_q$.

8. Multi-weight technique and an estimate for the dimension of the automorphism algebra

The model surfaces are interesting primarily because every model surface is most holomorphically symmetric with respect to its perturbations (see Theorem 5). The multi-weight technique, i.e., the usage of different weights in the problem of constructing a bound for the dimension of a group of local holomorphic automorphisms opens up very broad prospects.

Let M be a real analytic holomorphically nondegenerate generating CR-submanifold of a complex linear space $\mathbf{C}^N$. If M is of infinite Bloom–Graham type everywhere, then the question of dimension of the local groups of automorphisms of germs of such a manifold was recently and quite in detail analyzed in the paper of M. Stepanova [10]. At a point in general position, such a local group has the dimension either zero or infinity. On a singular proper analytic subset, a positive dimension is possible.

Therefore, we can confine ourselves to considering the case of a manifold having a finite type at the point in general position. Moreover, in this case, for the problem to estimate the dimension, we can confine ourselves to an estimate at a point of finite type. This estimate, obviously, remains valid on the singular subset, which is a proper analytic subset.

Thus, let M_ξ be the germ of M and let n be its CR-dimension and K its codimension. Then we can choose local coordinates at the point ξ in such a way that the germ equation takes the form $\operatorname{Im} w = F(z, \bar{z}, \operatorname{Re} w)$, where $(z \in \mathbf{C}^n,\ w = u + i\,v \in \mathbf{C}^K)$ arethe coordinates of the ambient space, and the real analytic vector function F and its first derivatives vanish at the origin.

We arbitrarily split the number n into a sum of positive summands $n = n_1 + \dots + n_p$. Correspondingly, the decomposition of $\mathbf{C}^n$ into a direct sum of summands of the form $\mathbf{C}^{n_\alpha}$ arises. We assign weights, also arbitrarily, for the elements of each direct summand; let $[z_\alpha] = \mu_\alpha$, where $(\mu_1, \dots, \mu_p)$ is an increasing sequence of positive integers. This forms a weight decomposition $\mu = ((\mu_1, n_1), \dots, (\mu_p, n_p))$ of the variable z. In accordance with the procedure described above, a weighted decomposition $w = (w_1, \dots, w_q)$ among the variables of the group w also occurs and, according to the Bloom–Graham–Stepanova type, all weights and multiplicities will be assigned. After some polynomial change of coordinates, the germ equation can be written in a reduced form, and we obtain a model surface Q. If Q' is another model surface of the same germ with the same choice of partition into n_α (multiplicities) and the weights μ_α, then Q' is equivalent to Q. The mapping is carried out by an invertible quasilinear mapping (Statement 2, (b)). In this sense, such a model surface is unique. For the chosen germ of M_ξ, it depends only on μ. Denote it by $Q(\mu)$.

An obstacle to obtaining an estimate is the holomorphic degeneracy of the model surface for any choice of weight μ. Consider, for a chosen germ of finite type M_ξ, a countable collection of weighted model surfaces $\mathcal{Q} = \{Q(\mu, M_\xi)\}$ for all possible μ.

Definition 21: A germ M_ξ is called *regular* if there exists a weight μ such that the corresponding model surface $Q(\mu, M_\xi)$ is nondegenerate (a finite type plus the holomorphic nondegeneracy). The set of such weights $\mathfrak{M}(M_\xi)$, which is not empty in this case, is called the set of *regular weights.*

It is clear that a regular germ is of finite type. For the regular germs, from Theorem 10, we immediately obtain the following assertion.

Statement 22. Let M_ξ be a regular germ, then

$$\dim \operatorname{aut} M_\xi \leqslant \min \dim \operatorname{aut} Q_0(\mu, M_\xi) \text{ over all regular weights.}$$

Note that the germ of general position is regular, although this does not remove all questions.

As an example of an irregular hypersurface in $\mathbf{C}^3$, one can suggest the well-known “light cone″ which, in coordinates $z_j = x_j + i\, y_j,\ j = 1, 2, 3$, is given by the equation $y_3^2 = y_1^2 + y_2^2,\ y_3 > 0$.

The approach we are describing is not directly applicable to incorrect manifolds and their germs. They require a special approach. As such an approach, in [13] and [14], a procedure of estimating based on the modified Poincaré construction was proposed (the recursion on depth greater than one). However, this procedure is technically more complicated.

The use of model surfaces in the study of automorphisms of a germ has obvious motivation. The germ M_ξ is an analytic object, and its model surface $Q(M_\xi, \mu^0)$ is an algebraic object. The model surface and its automorphisms are simpler than the original germ and its automorphisms. It should still be noted here that, if the dimension of the space is large, then the polynomial quasi-homogeneous forms Φ that define the equations of the model surface are polynomials of a large number of variables, and the general model surface is not available for direct analysis. However, replacing the initial regular weight μ^0 by some other regular weight μ^1 (for Q^0), we can assign to the old model surface $Q^0 = Q(M_\xi, \mu^0)$ a new model surface $Q^1 = Q(Q^0, \mu^1)$. Applying Theorem 18 once again, we obtain an estimate for the dimension of automorphisms of the original germ M_ξ in terms of the dimension of automorphisms of Q^1. This operation can be repeated, reducing the number of nonzero monomials in the current quasi-homogeneous forms Φ. In this case, the model surface becomes simpler, but the resulting estimate may be worse. The process ends as soon as we obtain the model surface for which there is no next regular weight. Note that, at each step, starting from zero, we have the right to choose an arbitrary regular weight with respect to the starting germ or to the current model surface. Thus, there can be a lot of such chains. Each of these chains, in finitely many steps, ends with its terminal link Q^∞ obtained by choosing the weight μ^∞. And, as an estimate for the automorphisms of the original germ, we can suggest the dimension of the automorphisms of Q^∞. This last model surface in the chain is a function of the starting germ M_ξ and of the chain of weights

$$\mathfrak{Z} = (\mu^0 \to \mu^1 \to \dots \to \mu^\infty).$$

In this case, we say that the string $\mathfrak{Z}$ is a regular chain for M_ξ and that $Q^\infty = Q^\infty(M_\xi, \mathfrak{Z})$. Here we can write

$$\dim \operatorname{aut} M_\xi \leqslant \dim \operatorname{aut} Q_0^\infty(M_\xi, \mathfrak{Z}).$$

Let us consider more carefully the case of a hypersurface in the space $\mathbf{C}^{n+1}$ with coordinates $z = (z_1, \dots, z_n),\ w = u + i\, v$. The germ of a hypersurface Γ_0 is the graph of a real analytic function of the form

$$\{v = F(z, \bar z) + u\, G(z, \bar z, u)\},\ F(z, \bar z) = \sum c_{\alpha\beta}\, z^\alpha\, \bar z^\beta,\ \alpha \in \mathbf{Z}_+^n,\ \beta \in \mathbf{Z}_+^n.$$

The coefficients $c_{\alpha\beta}$ satisfy the conditions for the convergence of the series and its real values. We may also assume that the decomposition of F does not contain pluriharmonic terms, i.e., $F(z,0)=0$. To the function F one can also assign the support of the series, which is the set of lattice nodes $\mathbf{Z}_+^{2n}$ for which the corresponding monomial has a nonzero coefficient, and the convex hull $\mathfrak{N}$ of the support, the Newton polytope.

At the zero step, we choose the regular weight $\mu^0=(\mu_1^0,\dots,\mu_n^0)$, i.e., we assume that the weights of z_ν and $\bar z_\nu$ are equal to μ_ν^0, and we obtain a nondegenerate polynomial model surface

$$Q^0=\{v=P^0(z,\bar z)=\sum p^0_{\alpha\beta}\, z^\alpha\, \bar z^\beta\},$$

where P^0 is a quasi-homogeneous polynomial in $(z,\bar z)$ of some weight $m^0\geqslant 2$ without pluriharmonic terms. Thus, the multi-indices $(\alpha,\beta)\in\mathbf{Z}_+^{2n}$ are connected by the linear relation

$$\mu_1^0\,(\alpha_1+\beta_1)+\cdots+\mu_n^0\,(\alpha_n+\beta_n)=m^0.$$

This relation singles out some finite-dimensional subspace of the space of monomials, let $\mathfrak{N}^0$ be the Newton polytope of P^0. The following weight $\mu^1=(\mu_1^1,\dots,\mu_n^1)$ gives us a new grading, which, using the relation

$$\mu_1^1\,(\alpha_1+\beta_1)+\cdots+\mu_n^1\,(\alpha_n+\beta_n)=m^1.$$

cuts out some face $\mathfrak{N}^1$ of lesser dimension on the polytope $\mathfrak{N}^0$. And so on, up to the terminal weight μ^∞. Here we obtain a model surface Q^∞ and its polyhedron $\mathfrak{N}^\infty$, which provide the maximum simplification for this chain. It is clear that the number of steps is bounded by the dimension of the space, i.e., by the number $2n$.

Our algorithm works by reducing the complexity of the model surface and stops at the threat of obtaining a holomorphically degenerate model surfaces. An alternative approach can be proposed that starts with the simplest holomorphically degenerate surface and, moving towards the complication, stops when the holomorphic nondegeneracy is reached. Let us start from an arbitrary vertex V_0 on the boundary of the original polytope $\mathfrak{N}$ (strictly speaking, this is a pair of symmetrical vertices). The starting model surface is the graph of the real part of this monomial. The weight μ^0 is chosen in such a way that the support hyperplane passing through V_0 has no other intersections with $\mathfrak{N}$. The verification for the holomorphic nondegeneracy is carried out as a test for finite nondegeneracy. If surface is holomorphically degenerate, then we choose the second monomial, the vertex V_1 adjacent to V_0 on the boundary of $\mathfrak{N}$. The weight μ^1 is chosen in such a way that the support hyperplane passing through the edge $[V_0,V_1]$ has no other intersections with $\mathfrak{N}$. The verification for the holomorphic nondegeneracy. And so on, until achieving the holomorphic nondegeneracy. It should be noted that the model surface whose equation does not relate all coordinates cannot be holomorphically nondegenerate.

It is natural to call the first algorithm described here as the descending one and the other as the ascending one.

A monomial model hypersurface is a hypersurface of the form $\{v=2\,\mathrm{Re}(z^\alpha\,\bar z^\beta)\}$. If z is a variable of dimension $n=1$ or $n=2$, then such a hypersurface can be holomorphically nondegenerate. Here are examples.

$$\{v=|z|^2\}\quad \text{for } n=1,$$
$$\{v=2\,\mathrm{Re}(z_1\,\bar z_2)\}\quad \text{for } n=2.$$

This is impossible if $n\geqslant 3$.

Statement 23. If $n\geqslant 3$, then the hypersurface $\Gamma=\{v=2\,\mathrm{Re}(z^\alpha\,\bar z^\beta)\}$ is holomorphically degenerate.

Proof. The holomorphic degeneracy of a real analytic manifold is equivalent to its holomorphic degeneracy at an arbitrary point. For $\alpha=\beta$, consider the change of coordinates of the form $z_1\to z^\alpha$; for $\alpha\neq\beta$, the change $z_1\to z^\alpha,\ z_2\to z^\beta$ (other coordinates are kept). At the point in general position, this holomorphic change is locally invertible. After this change, the equations of the hypersurface become $v=|z_1|^2$ or $v=2\,\mathrm{Re}(z_1\,\bar z_2)$. The equations do not contain some coordinates, and thus both the hypersurfaces and their preimage Γ are holomorphically degenerate.

Applying similar considerations, one can show that the minimum number of monomials ensuring the holomorphic nondegeneracy is equal to $\{\frac{n+1}{2}\}$ ($\{x\}$ is the integer part of x). In this connection, we can suggest the following improvement of the second (ascending) proposed algorithm. One should start its work from the edge of the dimension $\{\frac{n+1}{2}\}$ rather than from a vertex of the polytope.

There are many degrees of freedom in the operation of this algorithm. This is the freedom to choose faces of growing dimensions, or, which is the same, in the choice of weights defining them. The termination condition of the algorithm is the achievement of the holomorphic nondegeneracy. The holomorphic nondegeneracy is equivalent to the finite nondegeneracy at a point in general position. Therefore, the choice of a face and a weight should be subject to the condition of growth of the dimension of the set of derivatives of the gradient of the defining polynomial (see the definition of finite nondegeneracy).

The result of work of these algorithms, both descending and ascending, is a *minimal* model surface. That is. a holomorphic nondegenerate surface, which is model with respect to the original germ, and such that removing any of its monomials makes it holomorphically degenerate.

9. Questions for the future

In connection with the approach described above to estimating the dimension, the following questions are of interest.

Question 24. Find a constructive criterion for the regularity of a germ.

Further, possibly there are explicit ways to extract a model subset from the original polytope $\mathfrak{N}$ rather than obtaining it as the result of a work of the algorithm. Such a method would be of undoubted interest.

It is impossible to simultaneously simplify the model subset and to minimize the resulting estimate for the dimension of the automorphism algebra. If we refuse the simplicity and are looking for a model surface with the least dimension, then, as we can readily understand, we are to choose model subsets of $\mathfrak{N}$ lying in the boundary hyperfaces. However, there are several hyperfaces.

Question 25. How to find the boundary facet of $\mathfrak{N}$ such that the corresponding model surface gives the minimum dimension of the algebra of automorphisms? Or, which is the same, how to find the corresponding weight?

Question 26. The weighted g_+-conjecture formulated in item 5. The conjecture has two forms: a geometric and an algebraic.

And in general, placing the theory of model surfaces in a new weighted context, we can consider all the old questions from the weight point of view. It makes sense in connection with the both proven statements and hypotheses (see the list in the end of [2]).

REFERENCES

[1] H. Poincaré, "Les fonctions analytiques de deux variables et la representation conforme", *Rend. Circ. Mat*, 1907, 185–220.

[2] V. K. Beloshapka, "*CR*-Manifolds of Finite Bloom-Graham Type: the Method of Model Surface", *Russ. J. Math. Phys*, **27**:2 (2020), 155–174.

[3] V. K. Beloshapka, "Automorphisms of Degenerate Hypersurfaces in $\mathbf{C}^2$ and a Dimension Conjecture", *Russ. J. Math. Phys*, **4**:3 (1996), 393–396.

[4] A. E. Ershova, "Automorphisms of 2-Nondegenerate Hypersurfaces in $\mathbf{C}^3$", *Math. Notes*, **69**:2 (2001), 188–195.

[5] M. Kolar, F. Meylan, and D. Zaitsev, "Chern–Moser Operators and Polynomial Models in *CR* Geometry", *Adv. Math*, **263** (2014), 321–356.

[6] A. S. Labovskii, "On Dimensions of the Groups of Biholomorphic Automorphisms of Real-Analytic Hypersurfaces", *Math. Notes*, **61**:3 (1997), 287–294.

[7] M. A. Stepanova, "A Modification of the Bloom–Graham Theorem: the Introduction of Weights in the Complex Tangent Space", *Trans. Moscow Math. Soc*, **79**:2 (2018), 201–208.

[8] Th. Bloom and I. Graham, "On Type Conditions for Generic Real Submanifolds of C^n", *Invent. Math*, **40** (1977), 217–243.

[9] S. S. Chern and J. K. Moser, "Real Hypersurfaces in Complex Manifolds", *Acta Mathematica*, 1974, 133–219.

[10] M. A. Stepanova, "*CR*-Manifolds of Infinite Type in the Sense of Bloom and Graham", *Trans. Moscow Math. Soc*, **82**:2 (2021), 289–304.

[11] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "*CR* Automorphisms of Real Analytic *CR* in Complex Space", *Comm. Anal. Geom*, **6** (1998), 291–315.

[12] W. Kaup, "Einige Bemerkungen uber polynomiale Vektorfelder, Jordanalgebren und die Automorphismen von Siegelschen Gebieten", *Math. Ann*, **204** (1973), 131–144.

[13] V. K. Beloshapka, “Symmetries of Real Hypersurfaces in Complex 3-Space”, *Math. Notes*, **78**:2 (2005), 156–163.

[14] V. K. Beloshapka, “Modification of the Poincaré construction and its application in the CR-geometry of hypersurfaces in $\mathbf{C}^4$”, *https://arxiv.org/abs/2102.06451v2*.

[15] V. K. Beloshapka, “Universal Models for Real Submanifolds”, *Math. Notes*, **75**:4 (2004), 475–488.

[16] V. K. Beloshapka, “On the Group of Holomorphic Automorphisms of a Model Surface”, *Russ. J. Math. Phys*, **28** (2021), 275–283.

[17] R. V. Gammel and I. G. Kossovskiy, “The Envelope of Holomorphy of a Model Third-Degree Surface and the Rigidity Phenomenon”, *Proc. Steklov Inst. Math*, **253** (2006), 22–36.

[18] M. Sabzevari and A. Spiro, “On the Geometric Order of Totally Nondegenerate CR-manifolds”, *arXiv: 1807.03076v1 [mathCV], 9 Jul 2018.*

[19] J. Gregorovich, “On the Beloshapka’s Rigidity Conjecture for Real Submanifolds in Complex Space”, *arXiv: 1807.03502v1 [mathCV], 10 Jul 2018.*

[20] M. A. Stepanova, “Holomorphically Homogeneous CR-Manifolds and Their Model Surfaces”, *Izv. Math*, **85**:3 (2021), 529–535.

[21] D. Sykes and I. Zelenko, “Maximal Dimension of Groups of Symmetries of Homogeneous 2-Nondegenerate CR Structures of Hypersurface Type with a 1-Dimensional Levi Kernel”, *arXiv:2102.08599.*

[22] B. Kruglikov, “Submaximally Symmetric CR-Structures”, *J. Geom. Anal*, **26** (2016), 3090–3097.

[23] A. V. Loboda, “Holomorphically Homogeneous Real Hypersurfaces in $\mathbb{C}^3$”, *Trans. Moscow Math. Soc*, **81**:2 (2020), 169–228.