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Real Submanifolds of $\mathbf{C}^2$ With Singularities

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Abstract — We consider real submanifolds of $\mathbf{C}^2$ with singularities of three types: RC-singular 2 - dimensional surfaces, real quadratic cones, and hypersurfaces with degeneration of the Levi form. The holomorphic automorphisms of singular germs are evaluated. We also discuss resolution of singularities in the context of CR geometry.

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1. A TWO-DIMENSIONAL SURFACE NEAR AN RC-SINGULAR POINT

Let M be a two-dimensional manifold real-analytic near the origin in $\mathbf{C}^2$ (in the coordinates (z,w)). Next, let $(0,0) \in M$; that is, M be an RC-singular point in the language of [1] and [2]. In this case, $T_0\, M$ is a one-dimensional complex subspace of the space $\mathbf{C}^2$. By choosing a local holomorphic frame, we can assume that $T_0\, M = \{w = 0\}$, and represent M as the graph

$$w = \Phi(z,\bar z), \quad \text{where } \Phi \text{ is analytic, and } \ \Phi(z,0) = 0,\ \Phi'_{\bar z}(0,0) = 0.$$

Indeed, we write the function Φ as $\Phi(z,\bar z) = \phi(z) + \bar z\, \psi(z,\bar z)$. After the biholomorphic change $(z \to z,\ w \to w - \phi(z))$, the equation of M assumes the form $w = \bar z\, \psi(z,\bar z) = \tilde\Phi(z,\bar z)$. So, in the old notation it can be assumed that $\Phi(z,0) = 0$.

It is clear that the RC-singular points of M are the points of the graph lying above the nonempty real-analytic set

$$\sigma = \{z \in \mathbf{C} : \Phi'_{\bar z}(z,\bar z) = 0\}.$$

If its dimension is 2, then $\Phi'_{\bar z}$ is identically zero. By the above condition, $\Phi(z,0) = 0$, and, therefore, $\Phi(z,\bar z) = 0$. The surface M is the complex line $w = 0$. We will be interested in cases where the dimension of σ is either 1 or 0.

Let $\mathrm{aut}\, M_0$ be the Lie algebra of holomorphic infinitesimal automorphisms of the germ M_0. And let $\mathrm{Aut}\, M_0$ be the corresponding local Lie group. The elements of $\mathrm{aut}\ M_0$ are germs of real vector fields at the origin of the form

$$X = 2\,\mathrm{Re}\,\left(f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w} \right),$$

where f and g are holomorphic near the origin, and at $p \in M$ the tangency condition $X_p \in T_pM$ is satisfied, which can be written as

$$\begin{gathered} -g(z,w) + \Phi'_z(z,\bar z)\, f(z,w) + \Phi'_{\bar z}(z,\bar z)\, \bar f(\bar z,\bar w) = 0 \ \text{ for } \ w = \Phi(z,\bar z), \ \text{that is,} \\ E(z,\bar z) = -g(z,\Phi(z,\bar z)) + \Phi'_z(z,\bar z)\, f(z,\Phi(z,\bar z)) + \Phi'_{\bar z}(z,\bar z)\, \bar f(\bar z,\bar\Phi(\bar z,z)) = 0. \end{gathered} \tag{1}$$

The automorphisms of the germ can be dealt with the model surface method. Let the weights be defined by $[z] = [\bar z] = 1$. Then each power series of $(z,\bar z)$ can be written as the sum of its homogeneous components of a fixed degree. In particular,

$$\Phi(z,\bar z) = \Phi_m(z,\bar z) + \Phi_{m+1}(z,\bar z) + \ldots,$$

where $m \geqslant 2$ is the degree of the first nonzero component; this number m will be called the *o*rder of the surface M at the origin.

Next, let the weights be $[w] = [\bar w] = m$. This convention allows us to expand holomorphic expressions of (z,w) in the sum of weighted components. By $o(j)$ we denote the sum of the monomials of weight $> j$. In this case, the surface $Q_0 = \{w = \Phi_m(z,\bar z)\}$ is a model surface relative to the germ of $M_0 = \{w = \Phi_m(z,\bar z) + o(m)\}$. The weight convention can be augmented as follows:

$$\left[\frac{\partial}{\partial z}\right] = \left[\frac{\partial}{\partial \bar z}\right] = -1, \quad \left[\frac{\partial}{\partial \bar w}\right] = \left[\frac{\partial}{\partial w}\right] = -m.$$

With this convention, the set of germs of vector fields becomes the graded Lie algebra $\mathcal{G}$. In this algebra, one can define the subspaces $\mathcal{G}_j$ consisting of the fields of weight $\geqslant j$. If $j \geqslant 0$, then $\mathcal{G}_j$ is a subalgebra in $\mathcal{G}$. Correspondingly, the algebras $\operatorname{aut} M_0$ and $\operatorname{aut} Q_0$ are also graded Lie algebras. One can define two sequences of subspaces: $g_j = \operatorname{aut} M_0 \cap \mathcal{G}_j$ and $G_j = \operatorname{aut} Q_0 \cap \mathcal{G}_j$. Note that $\operatorname{aut} Q_0$ contains the graded field

$$X = 2\,\mathrm{Re}\left(z\,\frac{\partial}{\partial z} + m\,w\,\frac{\partial}{\partial w}\right).$$

This field generates the 1-parametric group of transformations of the form

$$z \to t\,z, \quad w \to t^m\,w, \quad t > 0.$$

It easily follows that if $X = \sum X_j \in \operatorname{aut} Q_0$ (X_j is a component of weight j), then $X_j \in \operatorname{aut} Q_0$. So, $\operatorname{aut} Q_0$ is finite-dimensional if and only if it is finite graded, that is, $G_j = 0$ for j exceeding some d.

Assume that we have two germs $M_0 = \{w = \Phi(z, \bar z)\}$ and $N_0 = \{w = \Psi(z, \bar z)\}$ of such surfaces at the origin, where we assume that Φ and Ψ do not contain holomorphic terms, that is, $\Phi(z, 0) = \Psi(z, 0) = 0$. Let $\Phi_m(z, \bar z)$ be the lowest term of the weight decomposition of Φ. And let these germs be holomorphically equivalent, that is, near the origin there is a locally invertible holomorphic mapping

$$z \to Z(z, w) = \sum Z_j, \;\; w \to W(z, w) = \sum W_j \tag{2}$$

which maps M_0 to N_0 and fixes the origin.

That this mapping sends M_0 to N_0 can be written as

$$W(z, w) = \Psi(Z(z, w), \overline{Z(z, w)}) \quad \text{for } w = \Phi(z, \bar z). \tag{3}$$

Let $Z_1 = \lambda\,z$, where $\lambda \neq 0$ by invertibility of the mapping. Hence the component in (3) of weight 1 has the form $W_1(z) = \Psi_1(\bar\lambda \bar z)$. Since $W_1(z)$ is holomorphic, this implies that $W_1 = \Psi_1 = 0$. Separating the components of (3) of weights $(2, 3, \dots, (m-1))$, we obtain in the same way that $W_j = \Psi_j = 0$ for all $j \leqslant m-1$. The component W_m has the form $W_m = \tilde W_m(z) + \mu\,w$. Separating the component in (3) of weight m, we get $\tilde W_m(z) + \mu\,\Phi_m(z, \bar z) = \Psi_m(\lambda z, \bar\lambda \bar z)$, that is, $\tilde W_m(z) = 0$. We also have

$$\mu\,\Phi_m(z, \bar z) = \Psi_m(\lambda z, \bar\lambda \bar z). \tag{4}$$

So, the action of the pseudogroup of locally invertible holomorphic changes near the origin generates the linear action in the space of complex polynomials of homogeneous degree m of the form

$$\Phi_m(z, \bar z) \to \mu^{-1}\,\Phi_m(\lambda z, \bar\lambda \bar z). \tag{5}$$

This, in particular, means that the holomorphic equivalence of germs generates a linear equivalence of the model surfaces of the form $(z \to \lambda\,z, \;\; w \to \mu\,w)$.

Summarizing, we have the following result.

Proposition 1.1. *(a) The order m is a biholomorphic invariant.*

(b) If $(z \to Z(z, w), \; w \to W(z, w))$ is a holomorphic mapping of the germ

$$\{w = \Phi(z, \bar z) = \Phi_m(z, \bar z) + o(m)\} \;\; \textit{on the germ}$$
$$\{w = \Psi(z, \bar z) = \Psi_m(z, \bar z) + o(m)\}, \;\; \textit{so that} \;\; \Phi(z, 0) = \Psi(z, 0) = 0,$$

then this mapping has the form $(Z = \lambda\,z + o(1), \;\; W = \mu\,w + o(m))$; their model surfaces are also equivalent, and the equivalence is established by a linear mapping of the form $(z \to \lambda\,z, \;\; w \to \mu\,w)$.

(c) The action of the pseudogroup of locally invertible holomorphic changes near the origin generates the linear action in the space of complex polynomials of homogeneous degree m of the from

$$\Phi_m(z, \bar z) \to \mu^{-1}\,\Phi_m(\lambda z, \bar\lambda \bar z), \tag{6}$$

and all invariants of this action are holomorphic invariants of the manifold M.

Note that the parameters (λ, μ) are related to the mapping via $\lambda = Z'_z(0, 0), \;\; \mu = W'_w(0, 0)$.

Let a mapping $H = (Z, W)$ be an automorphism, that is, $\Phi = \Psi$, and let τ be the mapping $\tau(H) = (Z'_z(0,0),\ W'_w(0,0))$. Then from Proposition 1.1 we conclude that τ is a homomorphism from the pseudogroup of holomorphic automorphisms of M_0 which fixes the origin to the subgroup $(z \to \lambda\, z,\ \ w \to \mu\, w)$, so that

$$\Phi_m(\lambda z, \bar{\lambda}\bar{z}) = \mu\, \Phi_m(z, \bar{z}). \tag{7}$$

The kernel of this representation is the subgroup of automorphisms of M_0 of the form $(z \to z + o(1),\ w \to w + o(m))$. If this kernel is trivial, the above representation is exact.

Let us apply the Poincaré construction to (1). To this end, this relation should be expanded in components in powers, $E = E_0 + E_1 + \dots$, and the field X should be expanded in weights,

$$X = X_{-1} + X_0 + X_1 + X_2 + \dots = \sum (f_{1+j}, g_{m+j}).$$

In this case, E_{m+j} assumes the form

$$E_{m+j} = -g_{m+j}(z, w) + \frac{\partial \Phi_m}{\partial z}(z, \bar{z})\, f_{1+j}(z, w) + \frac{\partial \bar{\Phi}_m}{\partial \bar{z}}(z, \bar{z})\, \bar{f}_{1+j}(\bar{z}, \bar{w}) + \dots = 0,$$

where the dots denote the terms of the expression depending on $\chi_J = (f_{1+J}, g_{m+J})$ for $J < j$, $w = \Phi_m(z, \bar{z})$. Since the system $E = 0$ is triangular, the dimension of the space of its solutions can be estimated in terms of the dimension of the solution of the model system

$$\tilde{E} = -g(z, w) + \frac{\partial \Phi_m}{\partial z}(z, \bar{z})\, f(z, w) + \frac{\partial \bar{\Phi}_m}{\partial \bar{z}}(z, \bar{z})\, \bar{f}(\bar{z}, \bar{w}) = 0 \tag{8}$$

for $w = \Phi_m(z, \bar{z})$. Clearly, the latter system is equivalent to the fact that $X = (f, g) \in \operatorname{aut} Q_0$.

This establishes the following result.

Proposition 1.2. *For each integer j,*

$$\dim\, g_j \leqslant \dim\, G_j.$$

In particular,
– $\dim \operatorname{aut} M_0 \leqslant \dim \operatorname{aut} Q_0$ *(an estimate of the full algebra),*
– *if* $\dim \operatorname{aut} Q < \infty$*, then* $\dim \operatorname{aut} M_0 < \infty$,
– $\dim\, g_0 \leqslant \dim G_0$ *(an estimate of the stabilizer),*
– *if $G_j = 0$, then $g_j = 0$.*

The last particular assertion can be reformulated as follows. Let $\operatorname{aut} Q_0$ be finite graded and let G_d be the last nonzero subalgebra. Also let $X(a) = X_{-m}(a) + \dots + X_d(a) + \dots$ be an arbitrary field from $\operatorname{aut} M_0$ depending on some set a of parameters (X_j is the component of the weight j). If $X(a_1)$ and $X(a_2)$ are two such fields, where $X_j(a_1) = X_j(a_2)$ for $-m \leqslant j \leqslant d$, then $X(a_1) = X(a_2)$. In other words, any field, and hence, any automorphism, is uniquely defined by the values of its weight d-jet.

Proposition 1.2 follows from the Poincaré construction for estimation of the dimension of a Lie algebra. But it can also be applied for estimating the dimension of the stabilizer of the origin in the pseudogroup of automorphisms. Consider the mapping

$$z \to z + F_2 + \dots, \quad w \to w + G_{m+1} + \dots \tag{9}$$

of the surface $M^1 = \{w = \Phi_m(z, \bar{z}) + \Phi_{m+1}(z, \bar{z}) \dots\}$ to the surface $M^2 = \{w = \Phi_m(z, \bar{z}) + \Psi_{m+1}(z, \bar{z}) \dots\}$. We write identity (3) for this mapping and apply the Poincaré construction. As a result, we get the following result.

Proposition 1.3. *Let Q be the model surface of the germ M^1, G_j be the graded components of autQ and $G_j = 0$ for $j \geqslant 1$. Then the family of holomorphic mappings of M^1 in M^2 is parameterized by the subset G_0.*

Under these assumptions, any automorphism of M^1 of the form (9) *is the identical mapping, and the representation alluded to after Proposition 1.1 is exact.*

Proposition 1.4. *(a) Let a surface M be given by $w = \Phi(z, \bar{z}, w, \bar{w})$, where Φ is real-valued. Then the dimension* $\operatorname{aut} M_0$ *is either* 0 *or* ∞.

(b) Let the model surface Q be given by $w = \Phi_m(z, \bar{z})$, where Φ_m is a homogeneous real-valued polynomial of degree m. Then $\dim \operatorname{aut} Q_0 = \infty$.

The proof is based on an observation from [6]. That Φ is real means that $v = 0$, $w = \bar{w} = u$ on the surface. If the algebra contains a nonzero field, it can be multiplied by $u^k = w^k$ (k is an arbitrary nonnegative integer). The algebra of the model surface $\operatorname{aut} Q_0$ contains a field (a graded field ($f = z$, $g = m\,w$)) satisfying (1)). Multiplying this relation by u^k (k is an arbitrary nonnegative integer) and since w is real, we find that (1) is satisfied also by all fields of the form ($f = z\,w^k$, $g = m\,w^{k+1}$). This proves the proposition.

Compact 2-dimensional submanifolds $\mathbf{C}^2$ were studied in [3]. The main result of [3] asserts that near an elliptic RC-singular point the manifold can be implicitly given by

$$u = |z|^2 + 2\,\gamma\,\operatorname{Re}(z^2) + 2\,\kappa\,u^s\,\operatorname{Re}(z^2), \quad \text{if } v = 0,$$
$$s \in \{1, 2, \dots\}, \ \ 0 < \gamma < 1/2, \ \ \kappa \in \{-1, 0, 1\}.$$

Proposition 1.4 establishes that the dimension of the algebra of automorphisms of this surface is either 0 or ∞. For $\kappa = 0$, the right-hand side of the equation is homogeneous, and hence there is one (graded) field. Therefore, the dimension is infinite. If $\kappa = \pm 1$, then it seems that the dimension is 0.

It is clear that the set of RC-singular points is holomorphically invariant, and hence, if $\sigma = \{0\}$, then each automorphism fixes the origin. In this case, all fields from $\operatorname{aut} Q_0$ vanish at the origin, that is, $\operatorname{aut} Q_0 = G_0 + G_1 + \dots$. If $\dim \sigma = 1$, then the decomposition of $\operatorname{aut} Q_0$ may also contain negative components. Let us give the corresponding example.

Example 1.5. Let $Q = \{w = (z - \bar{z})^2\}$. Here, $[w] = 2$, and the singular set is $\sigma = \{\operatorname{Im} z = 0\}$. For each $r \in \mathbf{R}$, the transformation ($z \to z + r$, $w \to w$) maps Q into Q. The corresponding field has the form $X = 2\operatorname{Re}\left(\frac{\partial}{\partial z}\right)$. Its weight is (-1). By Proposition 1.4, the dimension of $\operatorname{aut} Q_0$ is infinite.

Let Q be a monomial model surface, that is, a surface of the form

$$Q = \{w = z^\alpha\,\bar{z}^\beta\}, \tag{10}$$

where (α, β) are nonnegative integer exponents. If $\beta = 0$, then the surface is equivalent to the complex line and the algebra is infinite-dimensional. So, we can assume that $\beta \geqslant 1$.

Consider the field

$$X = (f(z, w), g(z, w)) \in \operatorname{aut} Q_0.$$

In this case, relation (1) assumes the form

$$g(z, w) = \alpha\,z^{\alpha-1}\,\bar{z}^\beta\,f(z, w) + \beta\,z^\alpha\,\bar{z}^{\beta-1}\,\bar{f}(\bar{z}, \bar{w}), \quad (z, w) \in Q. \tag{11}$$

From the equation of the surface Q, we find that, on the surface,

$$\bar{z} = z^{-\frac{\alpha}{\beta}}\,w^{\frac{1}{\beta}}, \quad \bar{w} = z^{\frac{\beta^2-\alpha^2}{\beta}}\,w^{\frac{\alpha}{\beta}}.$$

Correspondingly, (11) can be written as

$$g(z, w) = \alpha\,z^{-1}\,w\,f(z, w) + \beta\,z^{\frac{\alpha}{\beta}}\,w^{1-\frac{1}{\beta}}\,\bar{f}(z^{-\frac{\alpha}{\beta}}\,w^{\frac{1}{\beta}}, z^{\frac{\beta^2-\alpha^2}{\beta}}\,w^{\frac{\alpha}{\beta}}). \tag{12}$$

Assume that near the origin f is represented by the power series $\sum_{k,l\geqslant 0} c_{kl}\,z^k\,w^l$. Then from (12) it follows that the sum of the series $\sum_{k,l\geqslant 0} \bar{c}_{kl}\,z^K\,w^L$, where

$$K(k, l) = -k\,\frac{\alpha}{\beta} + l\,\frac{\beta^2 - \alpha^2}{\beta} + \frac{\alpha}{\beta} + 1, \quad L(k, l) = k\,\frac{1}{\beta} + l\,\frac{\alpha}{\beta} - \frac{1}{\beta} + 1, \tag{13}$$

is a function which coincides on Q with the function $\frac{1}{\beta}\,(z\,g(z, w) - \alpha\,w\,f(z, w))$ holomorphic near the origin in $\mathbf{C}^2$.

Let us formulate the following clear uniqueness theorem in a form convenient for us.

Proposition 1.6. *Let $\sum c_{kl}\,z^{\frac{k}{m}}\,w^{\frac{l}{m}}$ be a Puiseux series converging in the product of punctured disks $\{0 < |z| < \varepsilon,\ \ 0 < |w| < \varepsilon\}$ and let M be a real-analytic 2-dimensional surface, $(0, 0) \in M$, which is totally real outside the singular set σ of dimension $\leqslant 1$. Assume that this series is zero on M. Then this series is identically zero.*

Lemma 1.7. *The affine mapping of the* (k,l)*-plane given by* (13) *is nondegenerate, that is,*

$$\text{if } \ ((K(k_1,l_1),L(k_1,l_1)) = ((K(k_2,l_2),L(k_2,l_2)), \quad \text{then } \ (k_1,l_1) = (k_2,l_2).$$

Proof. Subtracting (13) for (k_1,l_1) from the relations for (k_2,l_2), we conclude that the pair (k_2-k_1,l_2-l_1) satisfies a system of two homogeneous linear equations whose determinant is identically equal to 1. This proves the lemma.

From Proposition 1.6 and Lemma 1.7 it readily follows that if $\sum_{k,l\geqslant 0} \bar{c}_{kl}\, z^K\, w^L$ is holomorphic near the origin, then among the coefficients c_{kl} only those nonzero which correspond to nonnegative integer values of (K,L).

System (13) can be written as

$$\begin{gathered} \alpha\,(k-1) + (\alpha^2-\beta^2)\,l + \beta\,(K-1) = 0, \\ k = \beta\,(L-1) - \alpha\,l + 1, \end{gathered} \tag{14}$$

where all the four unknowns (k,l,K,L) are nonnegative integers.

Case 1. $\beta < \alpha$, that is, $\alpha^2-\beta^2 > 0$. From (14) we have several possibilities.

Case 1.1. $k=0$. We have $(\alpha^2-\beta^2)\,l + \beta\,(K-1) = \alpha$, and so $(\alpha+\beta)((\alpha-\beta)l-1) + \beta\,K = 0$.

Case 1.1.1. $k=0, l=0$. Then $\beta(K-1)=\alpha$. This is possible only if α is divisible by β, that is, $\alpha = m\beta$. We also have $\beta\,(1-L)=1$. This is possible only if $\beta=1$ (that is, $m=\alpha$) and $L=0$. So, $\beta=1$, and also $K=\alpha+1, L=0$.

Case 1.1.2. $k=0, l\geqslant 1$. Then $l=1,\ \ \alpha=\beta+1,\ \ K=0,\ L=2$.

Case 1.2. $k\geqslant 1,\ K=0$. We have $\frac{\alpha}{\beta}\,(k-1) + (\alpha\,\frac{\alpha}{\beta}-\beta)\,l = 1$. Note that $(\alpha\,\frac{\alpha}{\beta}-\beta) > 1$, and hence, if $l\neq 0$, then $k=0$, which is impossible. Hence $l=0$, and, therefore, $(\alpha/\beta)\,(k-1)=1$, which gives $\alpha=\beta$, a contradiction.

Case 1.3. $k\geqslant 1,\ K\geqslant 1$. In this case, we readily get $k=1,\ l=0,\ K=1,\ L=1$.

The above analysis produces three situations in case 1: (I) $(k=1,\ l=0)$, and (α,β) are arbitrary; (II) $(k=0,\ l=0)$, and $\beta=1$, (III) $(k=0,\ l=1)$, and $\alpha=\beta+1$.

In the above analysis the necessary condition was used, and so let us find to which of the obtained f there corresponds holomorphic g:

$$\begin{aligned} (I)\ \ f(z,w) &= c_{10}\,z, \quad g(z,w) = \frac{1}{z}(\alpha\,w\,c_{10}\,z + \beta\,\bar{c}_{10}\,z\,w), \\ (II)\quad f(z,w) &= c_{00}, \quad g(z,w) = \frac{1}{z}(\alpha\,w\,c_{00} + \beta\,\bar{c}_{00}\,z^{\alpha+1}), \\ (III)\ \ f(z,w) &= c_{01}\,w, \quad g(z,w) = \frac{1}{z}(\alpha\,w^2\,c_{01} + \beta\,\bar{c}_{01}\,w^2). \end{aligned}$$

In the first case, c_{10} is an arbitrary complex number, and its dimension is 2. In all the remaining cases, the condition that g is holomorphic implies that all the parameters vanish. So, in all cases, the algebra is of dimension 2 and consists of the fields of the form

$$X = 2\,\mathrm{Re}\left(\Lambda\,z\,\frac{\partial}{\partial z} + (\alpha\,\Lambda + \beta\,\bar{\Lambda})\,\frac{\partial}{\partial w}\right), \quad \Lambda\in\mathbf{C}, \tag{15}$$

and the corresponding group of automorphisms has the form

$$\{z\to\lambda\,z, \quad w\to\lambda^{\alpha}\,\bar{\lambda}^{\beta}\,w, \quad \lambda\in\mathbf{C}^*\}.$$

Note that such a subalgebra is contained in the algebra of any monomial surface.

Case 2. If $\alpha=\beta$, then the monomial is real, and, by Proposition 1.4, the algebra is infinite-dimensional.

Case 3. $\beta>\alpha$, that is, $(\beta^2-\alpha^2)>0$. We claim that in this case the algebra is infinite-dimensional.

Lemma 1.8. *If* $\beta>\alpha$*, then the algebra is infinite-dimensional.*

Proof. Note that if $k\geqslant 1$ and $K\geqslant 1$ in any family of solutions of (14), then g is holomorphic. If $\alpha=0$, then to satisfy this condition it suffices to take any $(L\geqslant 1,\ l\geqslant 0)$. If $\alpha>0$, then substituting the second of (14) into the first one, we have

$$k-1 = \beta\,(L-1) - \alpha\,l, \quad K-1 = \beta\,l - \alpha\,(L-1).$$

As a result, the conditions on k and K assume the form

$$\frac{\alpha}{\beta} \leqslant \frac{L-1}{l} \leqslant \frac{\beta}{\alpha}$$

Since $\beta > \alpha > 0$, there exist infinitely many pairs (L, l) satisfying this condition. This proves the lemma.

Thus, we have established the following theorem.

Theorem 1.9. *Let $Q = \{(z, w) \in \mathbf{C}^2 : w = z^\alpha \bar{z}^\beta\}$, where α and β are nonnegative integers. Let* $\operatorname{aut} Q_0$ *be the Lie algebra of the germs of holomorphic at the origin vector fields tangent to Q, and d be the dimension of this algebra. Then if $0 < \beta < \alpha$, we have $d = 2$, and the algebra and the group are of the form* (15). *In the other case, $d = \infty$.*

Let M_0 be a germ of the form

$$M_0 = \{w = z^\alpha \bar{z}^\beta + o(m), \quad 0 < \beta < \alpha,\ m = \alpha + \beta\}.$$

From (7) we have $\mu = \lambda^\alpha \bar{\lambda}^\beta$. Now from Theorem 1.9 and Propositions 1.1 and 1.2, we get the following result.

Proposition 1.10. *Let $0 < \beta < \alpha$. Then the mapping $\tau : \operatorname{Aut} M_0 \to \mathbf{C}^*$*

$$(Z(z, w), W(z, w)) \to \lambda = Z'_z(0, 0)$$

is an exact representation of the group of automorphisms in $\mathbf{C}^*$. In particular, $\dim \operatorname{Aut} M_0 \leqslant 2$.

Example 1.11. Let $M = \{w = z^\alpha \bar{z}^\beta + z^{\alpha+\gamma} \bar{z}^{\beta+\delta}\}$, where $0 < \beta < \alpha$, $\gamma > 0$, $\delta > 0$. Then by Proposition 1.10 and using simple algebra, we find that

$$\operatorname{Aut} M_0 = \{z \to \lambda z,\ w \to \lambda^{\alpha+\gamma} \bar{\lambda}^{\beta+\delta} w\},$$

where $\lambda^\gamma \bar{\lambda}^\delta = 1$. In other words, $\lambda = e^{i\varphi}$, and $(\gamma - \delta)\varphi \equiv 0 \bmod 2\pi$. As a result:

– if $\gamma = \delta$, then $\operatorname{Aut} M_0 \approx S^1$,
– if $\gamma = \delta + n$, then $\operatorname{Aut} M_0 \approx \mathbf{Z}_n$,
– if $\gamma = \delta + 1$, then $\operatorname{Aut} M_0 = Id$.

2. A QUADRATIC CONE NEAR THE EDGE

Let C be the hypersurfaces defined by the relation $C = \{\rho = 0\}$ near the origin in $\mathbf{C}^2$ with coordinates $(z = x + iy,\ w = u + iv)$, where ρ is a real quadratic form of four variables (x, y, u, v) which is not identically zero. For C, we are interested in its dimension, irreducibility, Levi-nondegeneracy outside the edge, and the structure of the algebra $\operatorname{aut} C$ of infinitesimal holomorphic automorphisms at the origin. We will assume that ρ is defined up to a nonzero constant real factor. The cone itself will be considered up to a nondegenerate complex linear transformation.

The ring of power series of (z, w) can be equipped with grading by assuming that the weight of each coordinate is 1, that is, $[z] = [w] = 1$. Next, this grading can be extended to the Lie algebra of germs of vector fields at the origin by putting in addition

$$\left[\frac{\partial}{\partial z}\right] = \left[\frac{\partial}{\partial w}\right] = -1.$$

As a result, each series and each field is the sum of its graded components. In particular,

$$\operatorname{aut} C = g_{-1} + g_0 + g_1 + \dots$$

All the cones are invariant relative to the 1-parametric group of transformations

$$(z \to e^t z, \quad w \to e^t w).$$

This establishes the following property of the algebra of infinitesimal holomorphic automorphisms of the germ of C at the origin.

Proposition 2.1. *Let field $X = \sum_{-1}^{\infty} X_j \in \operatorname{aut} C$. Then*

$$X_j \in \operatorname{aut} C \quad \forall j.$$

The irreducible quadratic forms which define 3-dimensional cones were described in [4]. The corresponding classification is as follows:

$$\begin{aligned}
\rho_1 &= \operatorname{Re}(A\,z^2 + B\,w^2) + |z|^2 + |w|^2, \quad 0 \leqslant B \leqslant A,\ 1 < A,\\
\rho_2 &= \operatorname{Re}(A\,z^2 + B\,w^2) + |z|^2 - |w|^2, \quad 0 \leqslant B \leqslant A,\ (A,B) \neq (1,1),\\
\rho_3 &= \operatorname{Re}(A\,z^2 + \bar{A}\,w^2) + \operatorname{Im}(z\,\bar{w}), \quad \operatorname{Re} A > 0,\ \operatorname{Im} A \geqslant 0,\\
\rho_4 &= \operatorname{Re}(z^2 + i\,A\,z\,w) + \operatorname{Im}(z\,\bar{w}), \quad A \geqslant 0,\\
\rho_5 &= \operatorname{Re}(A\,z^2 + w^2) + |z|^2, \quad A \geqslant 0,\\
\rho_6 &= \operatorname{Re}(z\,w) + |z|^2,\\
\rho_7 &= \operatorname{Re}(z^2 + w^2).
\end{aligned}$$

Here, to different parameters there correspond nonequivalent cones.

In what follows, the cone $\{\rho_j = 0\}$ is denoted by C_j; the parameters involved in an equation will be mentioned explicitly, that is, $C_j(A,B)$ or $C_j(A)$.

For the sake of completeness, we give the list of “exotic” cones, that is, the zeros of real quadratic forms not contained in the list from [4]. There are two reasons for a cone not to be in the list: the reducibility of the form and the fact that the cone is not 3-dimensional.

Let a quadratic form ρ be reducible. If it is reducible over the field of reals, then $\rho = l_1\,l_2$, where l_1 and l_2 are two nonzero real forms. If the forms are not proportional, then C splits into two irreducible components, which are two 3-dimensional real hyperplanes which intersect in a two-dimensional subspace. This subspace is either a totally real plane (case 8) or a complex line (case 9). Another possibility is where these forms are proportional (case 10), and then $\rho_{10} = l^2$, the cone C_{10} is a 3-dimensional real hyperplane.

Let ρ be reducible over the field of complex numbers. Since ρ is real, we have $\rho = |l|^2$, where $l = A_1\,z + A_2\,w + B_1\,\bar{z} + B_2\,\bar{w}$ is linear. If l is either holomorphic or antiholomorphic, then C is the complex line (case 11). If not (case 12), then the equation $l = 0$ can be written as two real linear relations. These relations are independent (otherwise, there is a real reducibility) and their solution is a totally real 2-dimensional plane.

Next, let ρ be irreducible of rank $1 \leqslant r \leqslant 4$. It is easily checked that if $r \leqslant 2$, then the form is reducible, that is, its rank is either 3 or 4. Assume that the corresponding cone is not 3-dimensional at a generic point. This gives us two more exotic possibilities for the sign-constant form ρ. In addition, if $r = 3$, then the cone is a real line (case 13), and if $r = 4$, then the cone is a point (case 14).

The list of exotic cones (reducibility or absence of 3-dimensionality) is given in the following proposition.

Proposition 2.2. *(a) The complete list of cones of the form $C = \{\rho = 0\}$, where ρ is a reducible real quadratic form in $\mathbf{C}^2$, is as follows: C_8 is a pair of real hyperplanes, a totally real intersection; C_9 is a pair of real hyperplanes, the intersection is the complex line; C_{10} is a real hyperplane; C_{11} is a complex line; C_{12} is a totally real plane.*

(b) The irreducible forms defining the cones, which are not 3-dimensional — these are the sign-constant forms of ranks 3 and 4, to which there correspond C_{13} (a real line) and C_{14} (a point).

(c) All the above cones (with numbers from 8 to 14) are pairwise locally (near the edge) holomorphically nonequivalent.

(d) The Lie algebras of infinitesimal holomorphic automorphisms of all these cones are infinite-dimensional

(e) All the cones for which the Levi form is defined (that is, C_8, C_9, C_{10}) are Levi-flat (outside the edge) ones.

Consider now the 3-dimensional irreducible cones $C_1, \dots, C_7$. From this list, we single out the Levi-flat (outside the edge) cones. If $\rho = 0$ is the equation of the cone, then $d\zeta = (\rho'_w\,dt, -\rho'_z\,dt),\ dt \in \mathbf{C}$ is the parametrization of the complex tangent at a point of the cone. Correspondingly, the Levi form has the form

$$\partial\,\bar{\partial}\,\rho(d\zeta, d\bar{\zeta}) = L(z,\bar{z},w,\bar{w})\,|dt|^2.$$

So, the scalar coefficient L in the Levi form is a Hermitian form on $\mathbf{C}^2$. The condition that a Levi form is identically zero is the divisibility condition $L = k\,\rho$, where k is a real constant. Here we have the following alternative: either $k = 0$ and $L = 0$, or ρ does not contain the bidegrees $(2,0)$ and $(0,2)$. The cone C_7 corresponds to the first alternative, and the cone C_2 with $A = B = 0$ corresponds to the second alternative. This establishes the following result.

Proposition 2.3. *(a) Among $C_1,\dots,C_7$ the Levi-flat (outside the edge) cones are the two cones $|z|^2 - |w|^2 = 0$ (that is, C_2 for $A = B = 0$) and $\operatorname{Re}(z^2+w^2) = 0$ (that is, C_7). (b) The Lie algebras of infinitesimal holomorphic automorphisms of these cones are infinite-dimensional at all points.*

Proof. We need only to verify that $\operatorname{aut} C$ is infinite-dimensional at the origin. The first cone admits any automorphisms of the form $(z \to f(\zeta)\, z,\ w \to f(\zeta)\, w,\ f(0,0) \neq 0)$. The algebra of the second one contains the grading field $X = (z, w)$. But, for any $m = 1, 2, \dots$, the field of the form $Y = i\,(z^2 + w^2)^m X$ also lies in the algebra. This proves the proposition.

The remaining cones (that is, C_1, C_2 (for $(A, B) \neq (0, 0)$), C_3, C_4, C_5, C_6) are the cones which are Levi-nondegenerate at a generic point. We will call these cones *n*ondegenerate.

A germ of a real hypersurface is called *s*pherical if it is equivalent to the germ of the standard 3-dimensional sphere $\{|z|^2 + |w|^2 = 1\}$.

Proposition 2.4. *Among the nondegenerate cones, only $C_2(1, 0)$, $C_5(A)$, and C_6 are spherical at each point of Levi-nondegeneracy. The remaining cones from this list are nonspherical outside the edge.*

Proof. For the cones $C_5(A)$ and C_6 , we write the explicit mapping on $\{\operatorname{Re} z_2 + |z|^2 = 0\}$ (the projective image of the sphere). For $C_5(A)$, we have $(z \to z,\ w \to A\, z^2 + w^2)$, and for C_6, we have $(z \to z,\ w \to z\, w)$. The cone $C_2(1, 0)$ is given by the equation $\operatorname{Re}(\,z^2) + |z|^2 = |w|^2$, which can be rewritten as $2\,(\operatorname{Re} z)^2 = |w|^2$ or, what is the same, $\sqrt{2}\,(\operatorname{Re} z) = \pm|w|$. After the change $(z \to \sqrt{2}\, z,\ \ w \to w^2)$, we get two spherical hypersurfaces $(\operatorname{Re} z) \pm |w|^2 = 0$ that are tangent to each other in the real line.

That the remaining cones are nonspherical is verified via the sphericity criterion from [8]. Let us do this for the family $C_1(A, B)$.

Consider the family of Segre manifolds, which are defined by the equation of the cone C_1, as the family of graphs of the functions $w = W(z, \bar z, \bar w)$. Here, z is considered as an independent complex variable, and $(\bar z, \bar w)$ is a pair of complex parameters. This family is a family of solutions of some second-order ordinary differential equation. Let us find this equation. To this aim, we differentiate the defining relation two times with respect to z. As a result, we have

$$
\begin{gathered}
A\, z^2 + B\, W^2 + A\, \bar z^2 + B\, \bar w^2 + 2\, z\, \bar z + 2\, W\, \bar w = 0,\\
2\, A\, z + 2\, B\, W\, W' + 2\, \bar z + 2\, \bar w W' = 0,\\
2\, A + 2\, B\, \left(W'\right)^2 + 2\, B\, W\, W'' + 2\, \bar w W'' = 0.
\end{gathered}
$$

Expressing $\bar z$ and $\bar w$ from the second and third relations, and substituting the resulting expressions into the first one, we get

$$
\begin{aligned}
(A^3 z^2 + B^3 W^2 - A z^2 - B W^2)\, W''^2 + (-2\, A^2 B\, W'^3 z + 2\, B^3 W\, W'^2 - 2\, A^3\, W'\, z + 2\, B\, W'^3 z)\, W''&\\
+ (2\, A B^2 W - 2\, B\, W\, W'^2 + 2\, A\, W'\, z - 2\, A W)\, W'' + A B^2\, W'^6 + 2\, A^2 B\, W'^4 + B^3\, W'^4 + A^3\, W'^2&\\
+ 2\, A B^2\, W'^2 + A^2 B = 0.&
\end{aligned}
$$

The sphericity criterion from [8] can be formulated as follows: the second derivative W'' depends on W' polynomially (of degree at most three). This criterion is an implementation of the approach dating back to É. Cartan.

A necessary condition for the above relation be linear with respect to W'' is that the coefficient of $(W'')^2$ be identically zero, that is, $A^3 z^2 + B^3 W^2 - A z^2 - B W^2 = 0$. But this is impossible, because $A > 1$. Hence the one-one dependence W'' on W' implies that the discriminant of the quadratic equation is identically zero. The discriminant is a polynomial of degree 8 of (z, W, W'). The 11 coefficients of this polynomial are as follows:

$$\{-4\,AB^3\,(B-1)\,(B+1)\,, \qquad -4\,B^2\,(B-1)\,(B+1)\,\left(2\,A^2+1\right),$$
$$-4\,AB\,(B-1)\,(B+1)\,\left(A^2+2\right), \qquad -4\,A^2\,(B-1)\,(B+1)\,,$$
$$-8\,B^2\,(B-1)\,(B+1)\,(A-1)\,(A+1)\,,\; -16\,AB\,(B-1)\,(B+1)\,(A-1)\,(A+1)\,,$$
$$-8\,A^2\,(B-1)\,(B+1)\,(A-1)\,(A+1)\,, \qquad -4\,B^2\,(A-1)\,(A+1)\,,$$
$$-4\,AB\,\left(B^2+2\right)(A-1)\,(A+1)\,,\; -4\,A^2\,\left(2\,B^2+1\right)(A-1)\,(A+1)\,,$$
$$-4\,A^3B\,(A-1)\,(A+1)\}.$$

It is easily checked that the available restrictions on the parameters A and B does not make possible that all the coefficients vanish. This means that the cone $C_1(A,B)$ is nonspherical for all admissible values of the parameters.

The families $C_2(A,B)$, $C_3(A)$, $C_4(A)$ are dealt with similarly. $\square$

Proposition 2.5. *If C is a nondegenerate nonspherical cone (that is, a cone from the list C_1, C_2, C_3, C_4), then the decomposition of the algebra* $\operatorname{aut} C$ *of germs of infinitesimal automorphisms at the origin has the form* $\operatorname{aut} C = g_{-1} + g_0 + g_1$.

Proof. Expressing the variable $\bar w$ from the equation for the cone $\rho(z,\bar z,w,\bar w)=0$, we have $\bar w = \phi(z,w,\bar z)$. The tangency condition of the vector field $X=(F(z,w),G(z,w))$ to the cone C is written as

$$E(z,w,\bar z) = 2\operatorname{Re}(\rho'_z\,F + \rho'_w\,G) = 0, \quad \text{for} \;\; \bar w = \phi(z,w,\bar z).$$

The presence of a grading field $X=(z,w)$ means that each weight component of any tangent field is also a tangent field. Let $X=(F,G)$ be a field of weight $k+2$ ($k \geqslant 0$), that is, F and G are homogeneous polynomials of degree $k+3$, which we write as

$$F(z,w) = a\,w^{k+3} + c\,w^{k+2}\,z + z^2\,f(z,w),$$
$$G(z,w) = b\,w^{k+3} + d\,w^{k+2}\,z + e\,w^{k+1}\,z^2 + z^3\,g(z,w),$$

where $a,b,c,d,e \in \mathbf{C}$; $f(z,w)$, $g(z,w)$ are homogeneous polynomials of degree $k+1$ and k, respectively. Our aim is to show that

$$a=b=c=d=e=0, \quad f(z,w)=g(z,w)=0.$$

This is proved by direct calculation involving several cases. Namely, each of the four series of cones is considered separately, and in each series we single out a general case and several special ones. Let us show the corresponding calculations for $C_4(A)$. For this series, $A \geqslant 0$, but we will additionally assume that $A \notin \{0,1\}$.

From the equation $\operatorname{Re}(z^2 + i\,A\,z\,w) + \operatorname{Im}(z\,\bar w) = 0$ we get

$$\bar w = \frac{-i(iAzw + i\bar z w + \bar z^2 + z^2)}{A\bar z + z}.$$

Since $E(0,w,0) = A^{k+3}\,(1-A^2)\,w^{k+4}\,\bar a = 0$ we conclude that $a=0$. Next, $E'_z(z,w,0) = -Aw^3(A^2(Aw)^k\bar b - w^k b) = 0$, and hence $b=0$. Similarly, on the next iteration, from $E(0,w,0)=0$ we have $c=d=0$, and from $E(z,w,0)=0$ we get $f(z,w) = -i\,A\,g(z,w)$. Another iteration gives $g(z,w)=0$, which completes our argument.

The argument in the remaining cases both for this series and for the remaining series is similar. This proves the theorem.

So, for computation of the algebra of the nonspherical nondegenerate cones $C_1(A,B)$, $C_2(A,B)$, $C_3(A)$, $C_4(A)$ it suffices to directly calculate the three components g_{-1}, g_0, g_1. This calculation, which is quite trivial, produces the following results.

The dimension of g_{-1} is 0, except the cases $C_1(A,1)$, $C_2(A,1)$, $C_2(1,B)$, $C_2(A,A)$, where it is equal to 1.

The dimension of g_0 is 1, except the cases $C_1(A,A)$, $C_1(A,0)$, $C_2(A,A)$, $C_2(A,0)$, $C_3(A)$ for $\operatorname{Im} A = 0$, $C_4(0)$, where it is equal to 2.

The dimension of g_1 is zero, except the cases $C_1(A,1)$, $C_2(A,1)$, $C_2(1,B)$, $C_4(1)$, where it is equal to 1.

Note that the fields from g_{-1} are the fields that generate shifts. All the exceptional cases, in which such fields exist, are the cases where the quadratic form is of real rank 3. In this case, the cone is the product of a nondegenerate 2-dimensional cone and the real line. The singular points of this cone (the edge) is the real line defining the shift direction. In all the remaining cases, the edge is only the origin, and no shifts are possible.

The algebras of automorphisms for the spherical cones $C_2(1,0)$, $C_5(A)$ and C_6 will be computed in the next section (see Examples 3.4, 3.5, and 3.6), but here we will use these results.

As a result, we arrive at the following theorem.

Theorem 2.6. *Let ρ be a nonzero real quadratic form in $\mathbf{C}^2 \approx \mathbf{R}^4$, $C = \{\rho = 0\}$ be a quadratic cone in $\mathbf{C}^2$ with edge at the origin (ρ and $d\,\rho$ vanish at the origin), $\operatorname{aut} C$ be the algebra of germs of holomorphic infinitesimal automorphisms of C at the origin, and let d be its dimension. Then:*

$d = \infty \Leftrightarrow C$ *is degenerate; if C is nondegenerate, then $d \leqslant 5$.*

$d = 5 \Leftrightarrow C = C_5(0)$, *or* $C = C_6$.

$d = 4 \Leftrightarrow C = C_2(1,0)$.

$d = 3 \Leftrightarrow C \in \{C_1(A,1),\ C_2(A,1)(A \neq 0),\ C_2(1,B)\,(B \neq 0$ *and* $1),\ C_2(A,A)\ (A \neq 1),\ C_4(1)\}$.

$d = 2 \Leftrightarrow C \in \{C_1(A,A),\ C_1(A,0),\ C_2(A,1)\,(A \neq 1),\ C_3(A)\ (\operatorname{Im} A = 0),\ C_4(0),\ C_5(1)\}$.

$d = 1$ *in the case of all other nondegenerate cones.*

The groups corresponding to nonspherical nondegenerate cones consist of affine transformations; the groups corresponding to spherical cones are algebraic and will be written explicitly below (see Section 3). Note that the weight decomposition of the algebra of automorphisms for all cones has the form $\operatorname{aut} C = g_{-1}+g_0+g_1$, except $C_5(0)$ and C_6, for which it has the form $\operatorname{aut} C = g_0 + g_1 + g_2$.

3. RESOLUTION OF SINGULARITIES AND FIELDS WITH SINGULAR COEFFICIENTS

Example 3.1. In [9], a hypersurface in $\mathbf{C}^2$ of the form $\Gamma = \{\operatorname{Im} W = |Z|^4\}$ was considered and the algebra $\operatorname{aut}\Gamma_0$ of the germ of this hypersurface at the origin was computed. The result of this computation can be obtained differently. Namely, here one can use the fact that Γ is the image of the spherical hypersurface $Q = \{\operatorname{Im} w = |z|^2\}$ under the mapping $\phi = (Z = \sqrt{z},\ W = w)$. The intersection of the singular set of this mapping with Q is the real line $(u,0)$, which is transformed to the line $(U,0)$ on Γ. Away from these singular lines, the mapping establishes a biholomorphic equivalence of germs of spherical hypersurfaces. The differential of this mapping establishes an isomorphism of their algebras. The algebra Q is well known. In case of natural grading, this algebra consists of five components $g_{-2} + g_{-1} + g_0 + g_1 + g_2$ of the form

$$\begin{gathered} X_{-2} = (0,q),\ \ q \in \mathbf{R},\quad X_{-1} = (p,\ 2\,i\,\bar p\,z),\ p \in \mathbf{C},\quad X_0 = (\lambda\,z,\ 2\operatorname{Re}\lambda\,w),\ \lambda \in \mathbf{C}, \\ X_1 = (a\,w + 2\,i\,\bar a\,z^2,\ 2\,i\,\bar a\,z\,w),\quad a \in \mathbf{C}, X_2 = (r\,z\,w,\ r\,w^2),\ r \in \mathbf{R}. \end{gathered} \tag{16}$$

On these vector fields, the differential ϕ of the form $d\,\phi(dz,dw) = (\frac{dz}{2\sqrt{z}}, dw)$ acts. Let $Y_j = d\,\phi(X_j)$. Then, in the coordinates (Z,W):, we have

$$\begin{gathered} Y_{-2} = (0,q),\quad Y_{-1} = (\frac{p}{2\,Z},\ 2\,i\,\bar p\,Z^2),\quad Y_0 = (\frac{\lambda}{2}\,Z,\ 2\,(\operatorname{Re}\lambda)\,W), \\ Y_1 = (a\,\frac{W}{Z} + 2\,i\,\bar a\,Z^3,\ 2\,i\,\bar a\,Z^2\,W),\quad Y_2 = (\frac{r}{2}\,Z\,W,\ r\,W^2). \end{gathered}$$

The above result clearly implies that if we are interested only in fields with coefficients holomorphic near the origin, then this is the 4-dimensional subalgebra formed by Y_{-2}, Y_0, Y_2. (This very algebra was computed in [9].)

This example demonstrates two phenomena. First, the image of a hypersurface which is Levi-nondegenerate everywhere is a hypersurface with Levi-degenerations. Second, the algebra consisting of fields with holomorphic coefficients is transformed to the algebra of fields with meromorphic coefficients. In order to distinguish the algebras in terms of the class of coefficients, we introduce the following notation. Assume, for definiteness, that we are concerned with fields in the neighborhood of the origin in $\mathbf{C}^2$. Let $V_{\mathcal{O}}$ be the class of fields $X = (f,g)$ whose coefficients f and g are germs of the functions which are holomorphic at the origin, and let $V_{\mathcal{M}}$ be the class of fields whose coefficients are germs of the functions meromorphic at the origin.

In addition to holomorphic and meromorphic functions, we consider another class of functions.

Let f be a multivalued function analytic in a domain Ω (a connected neighborhood of the origin) minus a proper complex-analytic set σ containing the origin. In other words, f extends analytically along any path in $\Omega \setminus \sigma$. Two such functions (f_1,Ω_1,σ_1) and (f_2,Ω_2,σ_2) are said to be equivalent if $(\Omega_1 \cap \Omega_2) \setminus (\sigma_1 \cup \sigma_2)$ contains a point at which some germ of f_1 coincides with some germ of f_2. Taking the quotient of such functions by this equivalence relation, we get the germ of $f$unctions analytic at the origin. Let $V_{\mathcal{A}}$ be the set of fields with such coefficients.

Next, let M_0 be a germ of a real-analytic submanifold, which is not contained in any complex analytic germ at the origin. We denote by $\mathrm{aut}_{\mathcal{O}} M_0$, $\mathrm{aut}_{\mathcal{M}} M_0$, and $\mathrm{aut}_{\mathcal{A}} M_0$, respectively, the set of fields from $V_{\mathcal{O}}$, $V_{\mathcal{M}}$, or $V_{\mathcal{A}}$ which are tangent to M_0. Here, we assume that, in the meromorphic and analytic cases, the tangency condition should be tested only outside the singularities of the coefficients. It is clear that

$$\mathrm{aut}_{\mathcal{O}} M_0 \subseteq \mathrm{aut}_{\mathcal{M}} M_0 \subseteq \mathrm{aut}_{\mathcal{A}} M_0.$$

In the same way, we can also change from consideration of the pseudogroup of holomorphic automorphisms of the germ M_0 to the pseudogroups of meromorphic and multivalued analytic automorphisms, which we denote by

$$\mathrm{Aut}_{\mathcal{O}} M_0 \subseteq \mathrm{Aut}_{\mathcal{M}} M_0 \subseteq \mathrm{Aut}_{\mathcal{A}} M_0.$$

Note that if ϕ is a germ at a point of an $\mathcal{A}$-mapping, then the inverse of it (a germ of a multivalued function obtained by local inversion of ϕ outside the zeros of the Jacobian and singular points) is also a germ from the class $\mathcal{A}$. In particular, this pertains to germs of meromorphic mappings.

For a germ of the hypersurface Γ_0 from Example 3.1 , we have

$$\dim \mathrm{aut}_{\mathcal{O}} \Gamma_0 = 4, \quad \dim \mathrm{aut}_{\mathcal{M}} \Gamma_0 = 8.$$

Note that the operation of commutation of vector fields induces the structure of Lie algebras on $\mathrm{aut}_{\mathcal{M}} M_0$ and $\mathrm{aut}_{\mathcal{A}} M_0$ (similarly to $\mathrm{aut}_{\mathcal{O}} M_0$).

Theorem 3.2. *If a manifold M is holomorphically homogeneous and* $\dim \mathrm{aut}_{\mathcal{O}} M_0 < \infty$, *then all three algebras coincide, that is,*

$$\mathrm{aut}_{\mathcal{O}} M_0 = \mathrm{aut}_{\mathcal{M}} M_0 = \mathrm{aut}_{\mathcal{A}} M_0.$$

Proof. Let $[X_1]_0, \dots, [X_s]_0$ be a basis set of germs of the fields generating $\mathrm{aut}_{\mathcal{O}} M_0$. And let $X_1, \dots, X_s$ be the family of fields representing these germs in some neighborhood V of the origin. By holomorphic homogeneity, the germs of these fields $[X_1]_p, \dots, [X_s]_p$ also form a basis for $\mathrm{aut}_{\mathcal{A}} M_p$ at any point $p \in V$. Let Y be a field analytic in $\Omega \setminus \sigma$, that is, in some neighborhood of the origin minus the singular set, such that the germ $[Y]_0$ at the origin lies in $\mathrm{aut}_{\mathcal{A}} M_0$. Let $p \in \Omega \setminus \sigma$ and $[Y]_p$ be some germ of the field Y at $p \in (\Omega \cap V) \setminus \sigma$. So, $[Y]_p$ is a linear combination of $[X_1]_p, \dots, [X_s]_p$. Correspondingly, Y is a linear combination of $X_1, \dots, X_s$ near p. And this linear combination is holomorphic near the origin. $\square$

Example 3.1 shows that the above result ceases to hold without homogeneity assumption. The finite-dimensionality assumption cannot also be rejected in the theorem — here, as an example, one can consider the hyperplane $\{\mathrm{Im}\, w = 0\}$ or the straight line $\{w = 0\}$. In this case, homogeneity is present, but the algebras are different.

It is also clear that the above argument does not depend on the specifics of the two-dimensional space.

Example 3.3. Changing the variables $(z = Z\,W,\ w = W)$ (a σ-process) in the equation of the spherical hypersurface $Q = \{\mathrm{Im}\, w = |z|^2\}$, we have $\Gamma = \{\mathrm{Im}\, W = |Z\,W|^2\}$. This is the image of Q under the mapping $\phi = (Z = z/w,\ W = w)$. The differential has the form $d\phi(dz, dw) = (dz/w - z\,dw/w^2, dw)$. Let $Y_j = d\,\phi(X_j)$ be the image of the fields (16) that form $\mathrm{aut}_{\mathcal{O}} Q_0$. In the coordinates (Z, W), we have

$$Y_{-2} = (-q\,\frac{Z}{W},\ q), \quad Y_{-1} = (\frac{p}{W} - 2\,i\,\bar{p}\,Z^2,\ 2\,i\,\bar{p}\,Z\,W), \quad Y_0 = (-\bar{\lambda}\,Z,\ 2\,(\mathrm{Re}\,\lambda)\,W),$$
$$Y_1 = (a,\ 2\,i\,\bar{a}\,Z\,W^2), \qquad Y_2 = (0,\ r\,W^2).$$

So, $\mathrm{aut}_{\mathcal{O}} \Gamma_0$ is a 5-dimensional subalgebra formed by (Y_0, Y_1, Y_2) in the 8-dimensional algebra $\mathrm{aut}_{\mathcal{M}} \Gamma_0$. Unlike the hypersurfaces from Example 3.1, this is a hypersurface of infinite Bloom-Graham-type at the origin. According to [10], the maximal dimension for this hypersurface is 5.

Example 3.4. Changing the variables $(z = Z,\ w = Z\,W)$ in the equation of the sphere $Q = \{\mathrm{Im}\, w = |z|^2\}$, we have $\Gamma = \{\mathrm{Im}(Z\,W) = |Z|^2\}$. This is the image of Q under the mapping $\phi = (Z = z,\ W = w/z)$. The differential has the form $d\phi(dz, dw) = (dz,\ -w\,dz/z^2 + dw/z)$. We have

$$Y_{-2} = \left(0,\ \frac{q}{Z}\right), \quad Y_{-1} = \left(p,\ 2\,i\,\bar{p} - p\,\frac{W}{Z}\right), \quad Y_0 = \left(\lambda\,Z,\ \bar{\lambda}\,W\right),$$
$$Y_1 = \left(a\,Z\,W + 2\,i\,\bar{a}\,Z^2,\ -a\,W^2\right), \quad Y_2 = \left(r\,Z^2\,W,\ 0\right).$$

The hypersurfaces Γ is a cone which can be easily transformed to the cone C_6 from the list in the previous section. So, we have

$$\dim \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = 5, \quad \dim \mathrm{aut}_{\mathcal{M}}\,\Gamma_0 = 8.$$

These fields are holomorphic for $p = q = 0$. The weight decomposition of the subalgebra of holomorphic fields has the form $\mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = g_0 + g_1 + g_2$.

Example 3.5. Changing the variables ($z = Z,\ w = i\,(A\,Z^2 + W^2)$ in the equation of the sphere $Q = \{\mathrm{Im}\, w = |z|^2\}$, we have $\Gamma = \{\mathrm{Im}(i\,(A\,Z^2 + W^2)) = |Z|^2\}$. This is the image of Q under the mapping $\phi = (Z = z,\ W = \sqrt{-i\,w - A\,z^2})$. The differential has the form

$$d\phi(dz, dw) = \left(dz,\ \frac{-2\,A\,z\,dz - i\,dw}{2\sqrt{-i\,w - A\,z^2}}\right).$$

We have

$$Y_{-2} = \left(0,\ -\frac{i}{2}\,\frac{q}{W}\right),\ Y_{-1} = \left(p,\ \frac{(\bar p - A\,p)\,Z}{W}\right),\ Y_0 = \left(\lambda\,Z,\ \frac{A\,(\bar\lambda - \lambda)\,Z^2 + 2\,(\mathrm{Re}\lambda)\,W^2}{2\,W}\right),$$

$$Y_1 = \left(i\,a\,(A\,Z^2 + W^2) + 2\,i\,\bar a\,Z^2,\ -i\,\frac{A\,(a + \bar a)\,Z^3 + (a - \bar a)\,W^2}{W}\right),$$

$$Y_2 = \left(i\,r\,Z\,(A\,Z^2 + W^2),\ -i\,r\,\frac{A^2\,Z^4 - W^4}{W}\right).$$

The hypersurface Γ is the cone $C_5(A)$ from the list in the previous section. Irrespective of A, all the above fields are contained in the algebra $\mathrm{aut}_{\mathcal{M}}\,\Gamma_0$, which is isomorphic to the algebra of Q; its dimension is 8. The subalgebra $\mathrm{aut}_{\mathcal{O}}\,\Gamma_0$ is a function of A:

$$\text{if } A = 0, \text{ then } \dim \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = 5, \text{ and } \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = g_0 + g_1 + g_2;$$
$$\text{if } A = 1, \text{ then } \dim \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = 3, \text{ and } \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = g_{-1} + g_0 + g_1;$$
$$\text{if } A \notin \{0, 1\} \text{ then } \dim \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = 2, \text{ and } \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = g_0 + g_1.$$

Example 3.6. Changing the variables ($z = Z^2,\ w = c\,W$), $c = \pm\frac{i}{\sqrt{2}}$ in the equation of the sphere $Q = \{\mathrm{Im}\, w = |z|^2\}$, we have $\Gamma = \{(2\,\mathrm{Re}\, W)^2 = |Z|^4\}$. This is the image of Q under the mapping $\phi = (Z = \sqrt{z},\ W = w/c)$. The differential has the form

$$(dZ, dW) = d\phi(dz, dw) = \left(\frac{dz}{2\,Z},\ \frac{dw}{c}\right).$$

As a result,

$$Y_{-2} = \left(0,\ \frac{q}{c}\right),\ Y_{-1} = \left(\frac{p}{2\,Z},\ \frac{2\,i\,\bar p}{c}\,Z^2\right),\ Y_0 = \left(\frac{\lambda\,Z}{2},\ (2\,\mathrm{Re}\lambda)\,W\right),$$

$$Y_1 = \left(\frac{c\,a\,W + 2\,i\,\bar a\,Z^4}{2\,Z}, 2\,i\,\bar a\,Z^2\,W\right), \qquad Y_2 = c\,r\,\left(\frac{Z\,W}{2}, W^2\right).$$

Swapping Z and W in the equation of the hypersurface Γ, we get the cone $C_2(1, 0)$ from the list from the previous section. Note that

$$\dim \mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = 4, \quad \dim \mathrm{aut}_{\mathcal{M}}\,\Gamma_0 = 8.$$

The weight decomposition of the subalgebra of holomorphic fields has the form $\mathrm{aut}_{\mathcal{O}}\,\Gamma_0 = g_{-1} + g_0 + g_1$, which is the same as for the nonspherical cones.

The group of holomorphic automorphisms of the spherical hypersurface Q is well known — this is a subgroup of the group of projective automorphisms of the 2-dimensional projective space, and which is locally isomorphic to $SU(2, 1)$. This makes it possible to describe the groups of automorphisms of the hypersurfaces Γ which are obtained from Q by some transformations. Indeed, if ϕ sends Q onto Γ, then any automorphism $\Gamma : C \to C$ can be written as $H = \phi \circ h \circ (\phi)^{-1}$, where $h : Q \to Q$ is a holomorphic automorphism of Q. We say that the pseudogroup of automorphisms $\tilde G(\Gamma)$ is *i*nduced by the mapping ϕ from $\mathrm{Aut}\, Q$.

Here, one can start not only from holomorphic automorphisms of Q, but also from the group of automorphisms of an arbitrary class of an arbitrary germ. In addition, it is worth noting that, in accordance with the formula, the class of resulting automorphisms is defined by the class of original automorphisms and the class of mappings ϕ and ϕ^{-1}.

Remark 3.7. If the original algebra of automorphisms consists of fields with meromorphic coefficients, and the mapping ϕ is inverse of a meromorphic one, then the induced algebra also consists of fields with meromorphic coefficients. This result was illustrated by the above examples. At the same time, if the original group consists of holomorphic transformations, one may assert only analyticity with respect to the induced one. This will be illustrated below.

Let us return to our examples. The group of automorphisms of Q can be described as follows. Let G_- be the subgroup generated by the subalgebra $g_{-2}+g_{-1}$, the subgroup G_0 be be generated by g_0, the subgroup G_+ be generated by g_1+g_2. Then

$$G_- = \left\{z \to z+p,\ w \to w + 2\,i\,\bar p\,z + (q+i|p|^2)\right\}, \quad G_0 = \left\{z \to \lambda\,z,\ w \to |\lambda|^2\,w\right\},$$
$$G_+ = \left\{z \to \frac{z+a\,w}{1-(2\,i\,\bar a\,z + (r+i|a|^2)\,w)},\ w \to \frac{w}{1-(2\,i\,\bar a\,z+(r+i|a|^2)\,w)}\right\},$$
$$p, a \in \mathbf{C},\ \lambda \in \mathbf{C}^*,\ q, r \in \mathbf{R}.$$

And if we have an explicit expression for ϕ and ϕ^{-1}, then we can immediately write down the group of automorphisms of Γ of the corresponding class. For example, for Example 3.1 we get

$$\begin{gathered} \tilde G_- = \left\{Z \to \sqrt{Z^2+p},\ W \to W + 2\,i\,\bar p\,Z^2 + (q+i|p|^2)\right\}, \\ \tilde G_0 = \left\{Z \to \sqrt{\lambda}\,Z,\ W \to |\lambda|^2 W\right\}, \\ \tilde G_+ = \left\{Z \to \sqrt{\frac{Z^2+a\,W}{1-(2\,i\,\bar a\,Z^2+(r+i|a|^2)\,W}},\ W \to \frac{W}{1-(2\,i\,\bar a\,Z^2+(r+i|a|^2)\,W)}\right\}. \end{gathered} \tag{17}$$

So, to the group of holomorphic automorphisms of Q there corresponds the group of analytic (multivalued) automorphisms of Γ. In other words, (17) is $\operatorname{Aut}_{\mathcal{A}}\Gamma$. Applying this construction to the mapping from Example 3.3, we get the group of meromorphic automorphisms $\operatorname{Aut}_{\mathcal{M}}\Gamma$. In each of these groups one can easily define the subgroup of holomorphic automorphisms.

Example 3.8. Let us apply this approach to the germs from Section 1. Let $M_0 = \{w = \bar z\}$ be a germ of a totally real 2-dimensional plane at the origin. Consider the inverse mapping ϕ of $(z = Z^\beta, \; ; w = W/Z^\alpha)$, that is, $\phi(z,w) = (z^{\frac{1}{\beta}},\ z^{\frac{\alpha}{\beta}}\,w)$. We have $\tilde M_0 = \{W = Z^\alpha\,\bar Z^\beta\}$. If $\beta > 0$, then the tangent at the origin is $W = 0$, that is, an RC-singular point. In addition, $\tilde M_0$ is a monomial surface considered in Section 1. The group and the algebra of automorphisms of the plane M_0 are infinite-dimensional.

Indeed, the condition that the field $X = (f(z,w), g(z,w))$ is a germ from $\operatorname{aut} M_0$ has the form

$$g(z,w) = T(f)(z,w) = \bar f(w,z),$$

that is, $\operatorname{aut} M_0$ consists of fields of the form $X = (f, T(f))$. This algebra via the action $X = (f, T(f)) \to Y = d\phi(f, T(f)) = (\tilde f(Z,W), \tilde g(Z,W))$ induces on $\tilde M_0$ an infinite-dimensional subalgebra in the algebra of fields with meromorphic coefficients. This algebra consists of fields of the form

$$\begin{aligned} \tilde f(Z,W) &= \frac{1}{\beta\,Z^{\beta-1}}\,f(Z^\beta, \frac{W}{Z^\alpha}), \\ \tilde g(Z,W) &= \frac{1}{\beta}(\frac{\alpha\,W}{Z^\beta}\,f(Z^\beta, \frac{W}{Z^\alpha}) + Z^\alpha\,\bar f(\frac{W}{Z^\alpha}, Z^\beta)). \end{aligned}$$

Theorem 1.9. can now be interpreted as follows: the subalgebra of fields with holomorphic coefficients in the algebra of fields with meromorphic coefficients is finite-dimensional if and only if $\beta > \alpha$.

4. PROBLEMS FOR FURTHER STUDY

The problems discussed above have been hardly touched in CR geometry even in the framework of 2-dimensional complex spaces. Of course, similar problems can also be posed in the multivariate setting. This leaves us with more questions than answers, but still we formulate some problems naturally related to the above results.

Problem 4.1. Find the structure of the automorphisms of the model RC-singular 2-dimensional surface defined by the equation $Q = \{w = A\, z^{\alpha}\, \bar{z}^{m-\alpha} + B\, z^{\beta}\, \bar{z}^{m-\beta}\}$, where $m \geqslant \alpha > \beta > 0$. In particular, the problem of automorphisms is unsolved even in the following simplest case:

$$Q = \{w = \bar{z}^2 + A\, z\, \bar{z}\},\ A \neq 0.$$

Problem 4.2. Is it true that the dimension of the automorphisms of the perturbation of a nondegenerate quadratic cone by terms of degree $\geqslant 3$ is majorized by the dimension of the automorphisms of the cone itself? In other words, is it true that

$$\dim \operatorname{aut} \Gamma \leqslant \dim \operatorname{aut} C, \quad \text{where} \quad C = \{\rho_2(z,\bar{z},w,\bar{w}) = 0\}, \quad \Gamma = \{\rho_2(z,\bar{z},w,\bar{w}) + O(3) = 0\}?$$

Problem 4.3. What can be said about the automorphisms of the cones $C = \{\rho_m(z,\bar{z},w,\bar{w}) = 0\}$ of higher degrees, where ρ_m is an irreducible homogeneous real form of degree $m \geqslant 3$?

Problem 4.4. The following problem is base on the examples from Section 3. Which singularities can be removed by holomorphic (irreversible) changes of variables? This problem pertains, in particular, to singularities of all types considered above, that is, to singularities on the smooth structure level (Section 2), to singularities on the CR-structure level (Section 1) and to the singularities of the Levi form (Examples 3.1 and 3.3 in Section 3). In essence, this question is a proposal for construction of the theory of resolution of singularities similar to the well-know algebraic theory, but with emphasis to CR-geometry.

A more general statement of the problem is as follows. Assume that we are given a mapping of a singular manifold to a nonsingular one. Describe the class of this resolution mapping (holomorphic, meromorphic, analytic, ...) ?

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