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On Exceptional Quadrics

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Abstract. It is proved that the graded Lie algebras of infinitesimal holomorphic automorphisms of a nondegenerate quadric of codimension k do not have weight components more than $2k$. It is also proved that, for $k \leqslant 3$, there are no graded components of weight greater than 2. Several questions are formulated.

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1. INTRODUCTION

It was recently discovered [1] that the proof of one of my 1990 statements [2] contains an error. We are talking about the theorem on page 19, which, in particular, implies that the graded Lie algebra $\operatorname{aut}\mathscr{Q}$ of infinitesimal holomorphic automorphisms of a nondegenerate quadric $\mathscr{Q}$ of arbitrary codimension consists of at most five weight components:

$$\operatorname{aut}\mathscr{Q} = g_{-2} + g_{-1} + g_0 + g_1 + g_2.$$

In [1], a counterexample to this assertion, namely a quadric of codimension five in $\mathbf{C}^9$ was given and, among the counterexamples in [3], there is a quadric of codimension four in $\mathbf{C}^{10}$. We call these quadrics (i.e., nondegenerate quadrics whose algebra contains fields of weight greater than two) *exceptional*. All known examples of exceptional quadrics are very complicated. The nature of these rare examples remains unclear at present.

The paper is structured as follows. In Sec. 1, for a vector field, we analyze the membership conditions to the algebra of automorphisms in the context of the Ehrenpreis–Palamodov theorem. This is the technique of a work with systems of linear differential equations with constant coefficients which led to a criterion for the finite-dimensionality of the Lie algebra of infinitesimal holomorphic automorphisms of a quadric [5]. In Theorem 5, a criterion for the exceptionality of a quadric is given. In Sec. 2, we apply the criterion thus obtained to the study of quadrics of the CR-types $(3,3)$ and $(3,4)$. The use of classifications constructed in the works of Palinchak [8] and Anisova [9] enables us show that there are no exceptional quadrics among these types. Section 3 discusses the situation with quadrics of small codimensions. The main result, Theorem 19, is the assertion that there are no exceptional quadrics in codimension three. In Sec. 4 we return to the context of Sec. 1. A certain submodule of a free module over a polynomial ring (characteristic submodule) is assigned to a quadric, and, using this submodule, we present another criterion for the exceptionality of a quadric. We also prove an upper bound for the weights and degrees of fields in the algebra of automorphisms of a quadric using its codimension.

2. EXCEPTIONALITY CRITERION

Let M_ξ be the germ at a point ξ of a smooth real generating submanifold of a complex space $\mathbf{C}^N$ of CR-dimension $n>0$ and codimension $k>0$ ($N=n+k$). Let

$$z=(z_1,\dots,z_n), \quad w=(w_1,\dots,w_k), \quad w_j=u_j+i\,v_j, \quad j=1,\dots,k,$$

be the coordinates in $\mathbf{C}^N$. Let us assign weights to the variables:

$$[z]=[\bar z]=1, \quad [w]=[\bar w]=[u]=2.$$

This enables us to expand power series into weight components. If we assume that the differentiation with respect to z and $\bar z$ has the weight (-1) and the differentiation with respect to w and $\bar w$ has the weight (-2), then the Lie algebra of formal vector fields, and together with all its subalgebras, becomes graded. After a simple quadratic transformation, the equation of M_ξ can be represented as

$$M_\xi = \{v_j = < z,\bar z >_j + o(2), \quad j=1,\dots,k\} = \{v = < z,\bar z > + o(2)\},$$

where $< z, \bar{z} >_j$ are Hermitian forms on $\mathbf{C}^n$, and $o(m)$ are the functions whose Taylor expansion at zero does not contain terms of weight m and lower. The main object of our attention is the tangent quadric

$$\mathcal{Q} = \{v =< z, \bar{z} >\}.$$

Let $\operatorname{aut} M_\xi$ be the Lie algebra of germs of the vector fields in $\mathbf{C}^N$ tangent to M_ξ that have the form

$$X = 2\,\mathrm{Re}\,\left(f(z,w)\,\frac{\partial}{\partial z} + g(z,w)\,\frac{\partial}{\partial w}\right) = (f(z,w), g(z,w)), \tag{2.1}$$

where f and g are germs of vector functions holomorphic in ξ. These are fields that generate local 1-parameter groups of holomorphic transformations of M_ξ. The model surface $\mathcal{Q}$ is weighted homogeneous (given by weighted homogeneous equations) and holomorphically homogeneous (the group of holomorphic automorphisms acts transitively on $\mathcal{Q}$). Therefore, instead of $\operatorname{aut} \mathcal{Q}_\xi$, we can write $\operatorname{aut} \mathcal{Q}$ (there is no dependence of the algebra on the point).

According to our grading,

$$\operatorname{aut} \mathcal{Q} = g_{-2} + g_{-1} + g_0 + g_1 + g_2 + \dots\,.$$

It can readily be seen that the condition that a vector field $X = \sum X_j$ belongs to the Lie algebra $\operatorname{aut} \mathcal{Q}$ is equivalent to the condition that each of its weight components X_j belongs to this algebra. In this situation, the finite-dimensionality condition for the algebra is equivalent to the condition that the algebra is finitely graded. In [5] it was proved that the finite-dimensionality criterion for $\operatorname{aut} \mathcal{Q}$ is the following pair of conditions: (1) the absence of a core, i.e., if $< e, \bar{z} >= 0$ for all z, then $e = 0$; (2) the coordinates of the form $< z, \bar{z} >$ are linearly independent (nondegeneracy condition).

Writing out the condition that the field (2.1) belongs to the algebra $\operatorname{aut} \mathcal{Q}$, we obtain

$$2\,\mathrm{Re}\,(i\,g(z, u + i\,< z, \bar{z} >) + 2\,< f(z, u + i\,< z, \bar{z} >), \bar{z} >) = 0. \tag{2.2}$$

Write $\Delta(\varphi(u)) = \partial_u \varphi(u)(< z, \bar{z} >)$. A simple analysis (see [5]) of relation (2.2) enables us to establish the following assertion.

Proposition 1. *A pair* (f, g) *satisfies the relation* (2.2) *if and only if*

$$f = a(w) + C(w)\,z + A(w)(z,z), \quad g = b(w) + 2\,i\,< z, \bar{a}(w) >\,.$$

(*the dependence on* z *is indicated explicitly, i.e., of degree zero, one, and two, and the dependence on* w *is analytic in a neighborhood of the origin*), *and* a, A, C, b *satisfy the following relations that decompose into two systems of equations*

$$< A(u)(z,z), \bar{z} > = 2\,i\,< z, \Delta\bar{a}(u) >, \quad < z, \Delta^2 \bar{a}(u) >= 0. \tag{2.3}$$

$$\mathrm{Im}\, b(u) = 0, \quad \Delta\, b(u) = 2\,\mathrm{Re}\,< C(u)\,z, \bar{z} >, \tag{2.4}$$

$$\mathrm{Im}\,< \Delta\, C(u)\,z, \bar{z} > = 0, \quad \Delta^3\, b(u) = 0.$$

We can write

$$C(u)\,z = \sum C_j(u)\,z_j, \quad A(u)(z,z) = \sum A_{kj}(u)\,z_k\,z_j, \quad A_{kj} = A_{jk}.$$

Then either of the systems of equations can be written as a system of linear equations with constant coefficients. The first system is for the set of vector functions $(a(u), A_{kj}(u))$, and the other for $(b(u), C_j(u))$. It can readily be proved that, if the Hermitian vector form is nondegenerate, then neither of these systems has any exponential solutions with nonzero exponent. Applying, the Ehrenpreis–Palamodov exponential representation theorem [11], to each of these systems, we conclude that the space of solutions of each of them is a finite-dimensional subspace of the space of polynomials in u. It is clear that the finite-dimensionality implies a uniform estimate for the degree of the solutions.

It should be noted that any vector field X can be decomposed into a sum of an even, X^0, and odd, X^1, components, where

$$X^0 = X_{-2} + X_0 + X_2 + \dots, \quad X^1 = X_{-1} + X_1 + X_3 + \dots$$

in our grading. At the same time, it can be noted that, if $X \in \operatorname{aut} \mathcal{Q}$, then

$$X^0 = (C(w)\,z, b(w)), \quad X^1 = (a(w) + A(w)(z,z), 2\,i\,< z, \bar{a}(w) >).$$

Thus the solution (2.3) is exactly the even component X^0, and the solution (2.4) is the odd component X^1. Note that the fields of even weight form a subalgebra in $\operatorname{aut}\mathcal{Q}$.

As is well known, the subalgebra $g_- = g_{-2} + g_{-1}$ is Tanaka fundamental ([6, 7]) for any nondegenerate quadric. Therefore, if the corresponding component g_j vanishes for some j, then $g_J = 0$ for all $J > j$. In this connection, we give a definition.

Definition 2. A nondegenerate CR-quadric $\mathcal{Q}$ is said to be *exceptional* if $g_3 \neq 0$ for $\mathcal{Q}$.

It is clear that a quadric is not exceptional if and only if

$$\operatorname{aut}\mathcal{Q} = g_{-2} + g_{-1} + g_0 + g_1 + g_2.$$

Meylan's and Gregorovič's examples are examples of exceptional quadrics.

The condition of the absence of weight components greater than two is four conditions on (a, A, b, C):

$$\deg_u a \leqslant 1,\ \deg_u A = 0,\ \deg_u C \leqslant 1,\ \deg_u b \leqslant 2.$$

These conditions ensure that the even component terminates at the second weight and the odd on the first weight. However, since the criterion of exceptionality is a condition on the third component, it follows that we can confine ourselves to considering the first system (2.3) only. Note also that the first relation in (2.3), for a chosen a, is uniquely solvable with respect to A (nondegeneracy condition). Therefore, the parameter A can be excluded.

Denote by $z \cdot \bar{\zeta}$ the standard Hermitian form

$$(z \cdot \bar{\zeta}) = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n.$$

Then an arbitrary Hermitian form can be written as $(Hz \cdot \bar{z})$, where H is a Hermitian matrix, and z is understood as a column. Denote by $H_j z \cdot \bar{\zeta}$ the j-th coordinate of the vector form $< z, \bar{z} >$.

Let $B(\bar{z}) = (B_1 \cdot \bar{z}, \dots, B_k \cdot \bar{z})$ be a family of k linear forms. Consider the equation

$$< x, \bar{z} >= B(\bar{z}), \quad x \in \mathbf{C}^n, \quad B(\bar{z}) = (B_1, \dots, B_k) \in \mathbf{C}^{kn}, \tag{2.5}$$

i.e., assume that for a chosen right-hand side, we are looking for a vector $x \in \mathbf{C}^n$ such that the equation becomes an identity with respect to z. Such an equation can be written in the form $H_j x = B_j,\ j = 1, \dots, k$. Now consider the mapping

$$\nu : \mathbf{C}^n \to \mathbf{C}^{kn}, \quad \nu(z) = (H_1 z, \dots, H_k z).$$

Denote its image by $\mathscr{L}'$, and denote by $\mathscr{L}''$ the direct complement of the image to the entire space. Let π be the projection of $\mathbf{C}^{kn}$ onto $\mathscr{L}''$ along $\mathscr{L}'$. Since the forms $(H_1, \dots, H_k)$ have no common kernel, it follows that ν is an isomorphism of $\mathbf{C}^n$ and $\mathscr{L}'$, and the inverse mapping ν^{-1} is defined on $\mathscr{L}'$. The reasoning of this paragraph can be summarized as follows.

Lemma 3. (a) *Equation* (2.5) *is solvable with respect to* x *if and only if* $\pi(B) = 0$.
(b) *If* $\pi(B) = 0$*, then the unique solution of* (2.5) *has the form* $x = \nu^{-1}(B)$.

Applying Lemma 3 to the first relation (2.3), we obtain the following assertion.

Lemma 4. (a) *The equation* $< A(u)(z, z), \bar{z} >= 2\,i\ < z, \Delta\bar{a}(u) >$ *is equivalent to the pair of relations*

$$\pi(< z, \Delta\bar{a}(u) >) = 0, \quad A(z, z) = 2\,i\,\nu^{-1}(< z, \Delta\bar{a}(u) >). \tag{2.6}$$

(b) *If* $\deg_u a = d$*, then* $\deg_u A \leqslant d - 1$.

Before stating the resulting theorem, we give a definition. A nondegenerate quadric is said to be *rigid* if $g_+ = 0$ ($g_j = 0$ if $j \geqslant 1$). Due to the fundamentality of the algebra, this is equivalent to the condition $g_1 = 0$. In turn, this is equivalent to the fact that the equation $\pi(< z, \Delta\bar{a}(u) >) = 0$ has no nonzero solutions linear in u.

A quadric can be treated as a point of the space of families of k Hermitian forms of an n-dimensional variable. These families form a real linear space $\mathscr{H}$ of dimension $k\,n^2$. Denote the set of nondegenerate quadrics by $\mathscr{H}'$. It is clear that the degenerate quadrics form an algebraic subset $\mathscr{H}$ (given by polynomial conditions on the coordinates). Thus, $\mathscr{H}'$, which is the complement of $\mathscr{H}$, is semi-algebraic (given by polynomial equalities and polynomial inequalities). Denote by $\mathscr{H}_1'$ the set of nondegenerate quadrics for which g_+ is of length at least l ($g_l \neq 0$). It is clear that the nonrigid quadrics form $\mathscr{H}_1$, and the exceptional quadrics form $\mathscr{H}_3$. It immediately follows from our description of the components of the algebra that $\mathscr{H}_l$ is a semi-algebraic set for all l,

$$\mathscr{H}' \supseteq \mathscr{H}_1 \supseteq \mathscr{H}_2 \supseteq \mathscr{H}_3 \supseteq \dots$$

Let us present two criteria: for the exceptionality and nonrigidity.

Theorem 5. *A nondegenerate quadric* $\mathscr{Q}$
(a) *is exceptional if and only if there is a nonzero quadratic vector form* $a(u,u) = (a_1(u,u), \dots, a_n(u,u))$ *satisfying two relations*

$$\pi(< z, \Delta\bar{a}(u,u) >) = 0, \quad < z, \Delta^2\bar{a}(u,u) >= 0.$$

(b) *is nonrigid if and only if there is a nonzero linear vector form* $a(u) = (a_1(u), \dots, a_n(u))$ *satisfying the relation*

$$\pi(< z, \Delta\bar{a}(u) >) = 0.$$

3. CR-QUADRICS OF THE TYPES $(3,3)$ AND $(3,4)$

Let us state two conditions that can be satisfied by a Hermitian vector form $< z, \bar{z} >$ of a quadric $\mathscr{Q}$.
(I) The set $\{z \in \mathbf{C}^n : \text{rank}(H_1 z, \dots, H_k z) = n\}$ is nonempty (and hence open and dense).
(II) The image of the mapping from $\mathbf{C}^{2n}$ to $\mathbf{C}^k$ given by $(p,q) \to < p, q >$ contains interior points.

Theorem 6. *If a nondegenerate quadric satisfies conditions* (I) *and* (II)*, then it is not exceptional.*

Proof. We have $< z, \Delta^2\bar{a} >= 0$. It follows from condition (I) that $\Delta^2 a = d^2 a(< z, \bar{z} >, < z, \bar{z} >) = 0$. Let us complexify the resulting equality (i.e., let z and $\zeta = \bar{z}$ be independent variables). It now follows from condition (II) that, in a neighborhood of some value z, the differentials $du =< z, \bar{z} >$ can be treated as independent ones. This means that $d^2 a = 0$, and hence $\deg_u a \leqslant 1$. This completes the proof of the theorem. Note that the proof of the assertion is based on the second relation of the criterion only and does not use the first one.

Choose an $n > 0$. By the condition of linear independence of the coordinate Hermitian forms, nondegenerate quadrics of codimension k are possible only in the interval $1 \leqslant k \leqslant n^2$. Moreover, in the interval $2 \leqslant k \leqslant n^2 - 2$, for a generic quadric, we have $g_+ = 0$ (no fields of positive weight). A criterion for this "rigidity" is the condition $g_1 = 0$. N. Palinchak [8] classified all quadrics for $n = k = 3$ with $g_1 \neq 0$ up to holomorphic equivalence. This is a set of eight quadrics. A similar list with nine quadrics for $n = 3,\ k = 4$ was compiled by E. Anisova [9]. By the fundamentality, it follows from the fact that $g_1 = 0$ that the subsequent components also vanish. Therefore, the exceptional quadrics of these types, if they exist, should be included in the Palinchak and Anisova lists.

Theorem 7. (a) *There are no exceptional quadrics of type* $(3,3)$.
(b) *There are no exceptional quadrics of type* $(n,4)$ *for* $n \leqslant 3$.

Proof. We directly verify that all quadrics in both lists satisfy condition (II). As for condition (I), it is satisfied by all quadrics of the Anisova list and by all quadrics of the Palinchak list with one exception. This is a quadric which was denoted there by Q_5. The forms that define it are

$$2\,\text{Re} z_1\,\bar{z}_3, \quad 2\,\text{Re} z_2\,\bar{z}_3, \quad 2\,\text{Im} z_1\,\bar{z}_2.$$

The space $B^\perp = \{z \in \mathbf{C}^3 :< z, \bar{B} >= 0\}$ is defined by two independent relations

$$z_2\,\bar{B}_3 + z_3\,\bar{B}_2 = z_1\,\bar{B}_2 - z_2\,\bar{B}_1 = 0.$$

Therefore, the solution has the form

$$B^\perp = \{z_1 = \lambda\,\bar{B}_1,\ z_2 = \lambda\,\bar{B}_2,\ z_3 = -\lambda\,\bar{B}_3\}.$$

Thus, if $a(u) = (a_1(u), a_2(u), a_3(u))$ satisfies the relation $< \Delta^2 a, \bar{z} >= 0$, then $\Delta^2 a_j(u)$ is divisible by $\bar{z}_j$. Therefore, the restriction $\Delta^2 a_1$ to the plane $\bar{z}_1 = 0$ is zero. Here the restrictions of the coordinate Hermitian forms are $(z_1\,\bar{z}_3,\ z_2\,\bar{z}_3 + z_3\,\bar{z}_2, -i\,z_1\,\bar{z}_2)$. If we consider the corresponding mapping of $\mathbf{C}^5$ onto $\mathbf{C}^3$, then we see that it is of rank three. This enables us to claim that $d^2 a_1 = 0$, and the degree of a_1 does not exceed one. A similar reasoning gives the same bound for the degree for a_2 and a_3. Thus, for all quadrics except for Q_5 in the list of Palinchak, the nonexceptionality follows from Theorem 6, and for this quadric Q_5 it follows from our considerations. This proves part (a) and the absence of exceptional quadrics of type (3,4). A quadric of type (1,4) cannot be nondegenerate. There is only one nondegenerate quadric of type (2,4). This is the last quadric given by the basis of the space of Hermitian forms on $\mathbf{C}^2$. It is easy to check that it is not exceptional (see also Proposition 20). This proves (b). This completes the proof of the theorem.

4. QUADRICS OF SMALL CODIMENSIONS

The minimal codimension for which an example of an exceptional quadric is known is $k = 4$. For the codimensions k equal to one and two, the situation is as follows. The fact that there are no exceptional quadrics in codimension one directly follows from [4], and in [1] a proof for $k = 2$ was given.

In this paper, we prove that there are no exceptional quadrics in the codimension $k = 3$ either. Thus, the codimension four is the minimal codimension in which the existence of exceptional quadrics is possible.

Let $f = \sum_0^\infty f_j,\ g = \sum_0^\infty g_j$ be expansions of f and g into sums of the weight components. Then X_m, i.e., the m-th weight field component (2.1), has the form

$$X_m = 2\,\mathrm{Re}\left(f_{m+1}(z,w)\,\frac{\partial}{\partial z} + g_{m+2}(z,w)\,\frac{\partial}{\partial w}\right), \qquad \text{where}$$
$$2\,\mathrm{Re}\,(i\,g_{m+2}(z, u + i < z,\bar z >) + 2 < f_{m+1}(z, u + i < z,\bar z >), \bar z >) = 0. \tag{4.1}$$

Let us consider the lowest components of the algebra, assuming that $\mathcal{Q}$ is nondegenerate. Here we write the components f and g that arise in this way as multilinear forms, and we shall also assume that they are symmetric inside every group of variables, z or w. Here are the results.

Weight (–2). $f_{-1} = 0,\ g_0 = q,\quad X_{-2} = (0, q)$. We see from the condition (4.1) that $q \in \mathbf{R}^k$.

Weight (–1). $f_0 = p,\ g_1 = l(z)$, where $p \in \mathbf{C}^n$ and $l(z)$ is a linear form. We see from the condition (4.1) that $l(z) = 2\,i < z, \bar p >$. Thus, $X_{-1} = (p,\ 2\,i < z, \bar p >)$.

Weight 0. $f_1 = C\,z,\ g_2 = \alpha(z,z) + \rho\,w$. We see from the condition (4.1) that $\alpha(z,z) = 0,\ \mathrm{Im}\,(\rho\,u) = 0,\quad 2\,\mathrm{Re} < C\,z, \bar z > = \rho < z, \bar z >$. Thus, $X_0 = (Cz,\ \rho\,w)$ with the conditions found above.

Weight 1. $f_2 = a\,w + A(z,z),\ g_3 = \alpha(z,z,z) + \beta(z)\,w$. We see from conditions (4.1) that

$$\alpha(z,z,z) = 0,\ < A(z,z), \bar z > = 2\,i < z, \bar a < z, \bar z >>,\ \beta(z)\,u = 2\,i < z, \bar a\,u > .$$

We obtain $X_1 = (aw + A(z,z),\ 2\,i < z, \bar a\,w >)$.

Weight 2. $f_3 = B(w)z + b(z,z,z),\ g_4 = \alpha(z,z,z,z) + \beta(z,z)\,w + r(w,w)$. We see from condition (4.1) that $\alpha(z,z,z,z) = 0,\ \beta(z,z)\,u = 0,\ b(z,z,z) = 0$, and also

$$\mathrm{Re} < B(u)z, \bar z > = r(< z, \bar z >, u),\ \mathrm{Im} < B(< z, \bar z >)z, \bar z > = 0.$$

We obtain $X_2 = (B(w)\,z,\ r(w,w))$ with the conditions found above.

Weight 3. $f_4 = d(w,w) + D(w)(z,z) + e(z,z,z,z),\ g_5 = \alpha(z,z,z,z,z) + \beta(z,z,z)\,w + \gamma(z)(w,w)$. We see from condition (4.1) that $\alpha(z,z,z,z,z) = 0,\ \beta(z,z,z)(u) = 0,\ e(z,z,z,z) = 0$, and also

$$\gamma(z)(u,u) = 2\,i < z, \bar d(u,u) >,$$
$$< D(u)(z,z), \bar z > = 4\,i < z, \bar d(< z, \bar z >, u) >, \quad < D(< z, \bar z >)(z,z), \bar z > = 0.$$

We obtain $X_3 = (d(w,w) + D(w)(z,z),\ 2\,i < z, \bar d(w,w) >)$ with the conditions found above.

Thus, we have the following assertion.

Proposition 8. (a) *If $\mathcal{Q}$ is a nondegenerate quadric, then the lower weight components of* aut $\mathcal{Q}$ *are of the form*

$$X_{-2} = (0, q), \quad q \in \mathbf{R}^k,$$
$$X_{-1} = (p,\ 2\,i < z, \bar p >), \quad p \in \mathbf{C}^n,$$
$$X_0 = (Cz,\ \rho\,w),\ C \in gl(n, \mathbf{C}), \qquad \rho \in gl(k, \mathbf{R}), \quad 2\,\mathrm{Re} < C\,z, \bar z > = \rho < z, \bar z >,$$
$$X_1 = (aw + A(z,z),\ 2\,i < z, \bar a\,w >), \qquad < A(z,z), \bar z > = 2\,i < z, \bar a < z, \bar z >>,$$
$$X_2 = (B(w)\,z,\ r(w,w)), \quad \mathrm{Re} < B(u)z, \bar z > = r(< z, \bar z >, u),\ \mathrm{Im} < B(< z, \bar z >)z, \bar z > = 0,$$
$$X_3 = (d(w,w) + D(w)(z,z),\ 2\,i < z, \bar d(w,w) >),$$

$$< D(u)(z,z), \bar z > = 4\,i < z, \bar d(< z, \bar z >, u) >,\ < D(< z, \bar z >)(z,z), \bar z > = 0. \tag{4.2}$$

(b) *$\mathcal{Q}$ is not exceptional if and only if the only solution to* (4.2) *is $D(u)(z,z) = 0$ and $d(u,u) = 0$.*

Lemma 9. (a) *If $\mathcal{Q}$ is a nondegenerate quadric, then*

$$D(< z, \bar{p} >)(z, z) = 0.$$

(b) *$\mathcal{Q}$ is not exceptional if and only if the only solution to*

$$< D(< z, \bar{p} >)(z, z), \bar{z} >= 0, \quad < D(u)(z, z), \bar{z} >= 4\, i\, < z, \bar{d}(< z, \bar{z} >, u) > \tag{4.3}$$

is $D(u)(z, z) = 0$ and $d(u, u) = 0$.

Proof. Let us prove (a). Evaluating the commutator directly, we obtain $[X_3,\, X_{-1}] = (F(z, w), G(z, w))$, where

$$F = -2\, d(2i < z, \bar{p} >, w) - D(2i < z, \bar{p} >)(z, z) - 2D(w)(z, p),$$
$$G = (2\, i\, < d(w, w) + D(w)(z, z), \bar{p} > -2\, i\, < p, \bar{d}(w, w) > -4\, i\, < z, \bar{d}(2i < z, \bar{p} >, w) >).$$

The commutator of a field of weight 3 with a field of weight (-1) is an element of g_2. We know that the z-coordinate of such a field does not contain any cubic form in z and, therefore, $D(< z, \bar{p} >)(z, z) = 0$. This immediately implies (b). This completes the proof of the lemma.

Let $D(u)(z, z) = D_1(z, z)\, u_1 + \cdots + D_k(z, z)\, u_k$, where $D_j(z, z) = (D_j^1(z, z), \ldots, D_j^n(z, z))$ is a vector-valued quadratic form. Then the first relation (4.3) becomes

$$(H_1 z \cdot \bar{p})\, D_1(z, z) + \cdots + (H_k z \cdot \bar{p})\, D_k(z, z) = 0. \tag{4.4}$$

Here the ν-th coordinate (4.4) looks as follows:

$$(H_1 z \cdot \bar{p})\, D_1^\nu(z, z) + \cdots + (H_k z \cdot \bar{p})\, D_k^\nu(z, z) = 0. \tag{4.5}$$

Removing the convolution with the independent parameter $\bar{p}$, we obtain the vector relation of the form

$$D_1^\nu(z, z)\, H_1 z + \cdots + D_k^\nu(z, z)\, H_k z = 0. \tag{4.6}$$

To a quadratic vector form $d(u, u)$, there uniquely corresponds a symmetric bilinear vector form

$$d(u, U) = \sum_{\alpha\beta} d_{\alpha\beta}\, u_\alpha\, U_\beta = \sum_{\beta} \delta_\beta(u)\, U_\beta, \quad \delta_\beta(u) = \sum_{\alpha} d_{\alpha\beta}\, u_\alpha, \quad d_{\alpha\beta} \in \mathbf{C}^n.$$

If $D(u)(z, z) = 0$, then the condition on the form $d(u, u)$ can be written out as follows:

$$< z, \bar{\delta}_j(< z, \bar{z} >) >= 0, \quad j = 1, \ldots, k.$$

Since all coefficients of the forms d satisfy the same relation, we write this relation in a form which is free from indices:

$$< z, \bar{\delta}(< z, \bar{z} >) >= 0, \quad \delta(u) = u_1\, \alpha_1 + \cdots + u_k\, \alpha_k. \tag{4.7}$$

Thus, to describe g_3 under the condition $D(z, z) = 0$, one must describe all families $(\delta_1(u), \ldots, \delta_k(u))$ of linear $\mathbf{C}^n$-valued forms such that each of them satisfies relation (4.7), and the entire collection satisfies the symmetry condition for the bilinear form $d(u, U)$, namely, $d_{\alpha\beta} = d_{\beta\alpha}$.

Before proceeding to the case of codimension three, which is of interest for us, let us consider the cases $k = 1$ and $k = 2$ in our context.

Case k=1. If $k = 1$, then the first relation (4.3) becomes $D(< z, \bar{p} >)(z, z) = 0$; whence we immediately obtain $D = 0$. After this, the second relation becomes $< d(< z, \bar{z} >, u), \bar{z} >=< \alpha, \bar{z} >< z, \bar{z} > u = 0$. Whence we immediately obtain $d = 0$ and $g_3 = 0$.

Case k=2. Write

$$Pz = H_1 z,\ Qz = H_2 z, \quad Pz = (p_1(z), ..., p_n(z)),\ Qz = (q_1(z), ..., q_n(z)).$$

Then (4.6) becomes

$$l(z, z)\, Pz + m(z, z)\, Qz = 0,\ l(z, z) = D_1^\nu(z, z),\ m(z, z) = D_2^\nu(z, z).$$

There is a μ such that $p_\mu(z) = \alpha(z)$ is not proportional to $q_\mu = \beta(z)$. Then we obtain

$$l(z,z) = \lambda(z)\,\beta(z),\ m(z,z) = -\lambda(z)\,\alpha(z).$$

If $\lambda(z) \neq 0$, then we obtain $\beta(z)\,H_1 z = \alpha(z)\,H_2\,z$, which implies that $H_1 z = \alpha(z)\,A$, $H_2 z = \beta(z)\,A$.

Let us state a lemma needed here and in what follows.

Lemma 10. (a) *If a Hermitian matrix H is of rank 1, i.e., $Hz = l(z)A$ $(l \neq 0, A \neq 0)$, then $l(z) = \rho\,(z \cdot \bar{A})$, $\rho \neq 0$, and the corresponding Hermitian form looks as follows*:

$$(Hz \cdot \bar{z}) = \rho\,|(z \cdot \bar{A})|^2,\ \rho \in \mathbf{R}.$$

(b) *If a Hermitian matrix H is of rank 2, then the corresponding Hermitian form looks as follows*:

$$(Hz \cdot \bar{z}) = \rho\,|(z \cdot \bar{A})|^2 + \tau\,|(z \cdot \bar{B})|^2.$$

Proof. The rank of a Hermitian form is invariant under nondegenerate linear changes. Let us bring the form under consideration to a diagonal form. The number of nonzero coefficients in the first case is equal to one and in the other to two. Returning to the original coordinates, we obtain both the assertions of the lemma.

In accordance with the lemma,

$$(Pz \cdot \bar{z}) = \tau_1|(z \cdot \bar{A})|^2, \quad (Qz \cdot \bar{z}) = \tau_2|(z \cdot \bar{A})|^2,$$

which contradicts the non degeneracy condition and, therefore, we have $D = 0$.

Let us write out (4.7); we obtain

$$(Pz \cdot \bar{z}) < z, \bar{\alpha}_1 > +(Qz \cdot \bar{z}) < z, \bar{\alpha}_2 >= 0. \tag{4.8}$$

A pair of vectors (α_1, α_2) can be of rank zero, one, or two. The rank zero means that $\delta = 0$. Let it be one; then the vectors are proportional, write $\alpha_2 = \lambda\,\alpha_1,\ \alpha_1 \neq 0$. From (4.8), we obtain

$$((Pz \cdot \bar{z}) + \bar{\lambda}\,(Qz \cdot \bar{z}))\ < z, \bar{\alpha}_1 >= 0.$$

This implies the linear dependence of P and Q, which contradicts the nondegeneracy condition for the pair (P, Q).

Now let (α_1, α_2) be linearly independent $(n \geqslant 2)$. Let us choose a basis in $\mathbf{C}^n$ of the form $(\alpha_1, \alpha_2, \dots)$. Let $p_j(z)$ be the j-th coordinate of Pz in this basis, and let $q_j(z)$ be the j-th coordinate of Qz. Then (4.8) becomes

$$p_1(z)\,Pz + p_2(z)\,Qz = 0, \quad q_1(z)\,Pz + q_2(z)\,Qz = 0. \tag{4.9}$$

Write out the first coordinate of the first equality and the second coordinate of the other. We have

$$(p_1(z))^2 + p_2(z)\,q_1(z) = 0, \quad (q_2(z))^2 + p_2(z)\,q_1(z) = 0.$$

If $p_1 = 0$, then $p_2 = 0$, as follows from the first equality (otherwise $q_1 = 0$, and α_1 falls into the common kernel). Now it follows from the second equality that $q_2 = 0$, and α_2 falls into the common kernel. A contradiction. Thus, p_1 and q_2 are nonzero; however, then (4.9) and the nondegeneracy imply that p_2 and q_1 are nonzero either. Then, applying Lemma 11 (see below) to the first equality in (4.9), we see that only version (b.2) is possible, i.e., $Pz = p_2(z)\,A$, $Qz = -p_1(z)\,A$. According to Lemma 10, the forms $(Pz \cdot \bar{z})$ and $(Qz \cdot \bar{z})$ are proportional to $|(z \cdot \bar{A})|^2$, which contradicts the nondegeneracy condition.

Thus, for $k = 2$, we have $g_3 = 0$.

In connection with the study of (4.3) for small k, we formulate several obvious auxiliary assertions.

Lemma 11. *Let $m(z)\,P\,z = l(z)\,Q\,z$, where P and Q are square matrices and m and l are scalar linear forms*; *then one of the following possibilities must occur.*

(a) *$m(z)\,P\,z = l(z)\,Q\,z = 0$, i.e., every pair has at least one factor which is equal to zero (4 versions).*

(b.1) *Linear dependence. $P\,z = \lambda\,Q\,z \neq 0,\ \lambda\,m(z) = l(z) \neq 0$.*

(b.2) *Linear independence. $P\,z = l(z)\,A,\ \ Q\,z = m(z)\,A,\ \ A \neq 0$, and the forms l and m are not proportional.*

Lemma 12. *Let $m(z)\,A = l(z)\,B$, where A and B are vectors and m and l are scalar linear forms; then one of the following possibilities holds.*

(a) *$m(z)\,A = l(z)\,B = 0$, i.e., at least one of the factors in every pair is zero (4 versions).*
(b) *Linear dependence. $B = \lambda\, A \neq 0,\ \ m(z) = \lambda\, l(z) \neq 0$.*

Let $I(l_1(z), l_2(z))$ be the ideal in the ring of polynomials in z generated by two the linear forms l_1 and l_2.

Lemma 13. (a) *$l(z) \in I(l_1(z), l_2(z))$ if and only if $l(z) = \lambda_1\, l_1(z) + \lambda_2\, l_2(z)$.*
(b) *The ideal $I(l_1(z), l_2(z))$ is simple, i.e., if $p(z)\, q(z) \in I(l_1(z), l_2(z))$, then either $p(z) \in I(l_1(z), l_2(z))$ or $q(z) \in I(l_1(z), l_2(z))$.*

Let us pass to the consideration of codimension $\mathbf{k = 3}$.

Let
$$< z, \bar{z} >= ((Pz \cdot \bar{z}),\, (Qz \cdot \bar{z}), (Rz \cdot \bar{z})).$$
Relation (4.6) becomes
$$l(z, z)\, Pz + m(z, z)\, Qz + n(z, z)\, Rz = 0. \tag{4.10}$$

Lemma 14. *Relation* (4.10) *is impossible for linearly dependent (l, m, n).*

Proof. Let $\mathrm{rank}(l, m, n) = 1$, i.e.,
$$l(z, z) = \lambda\, \varphi(z, z),\ m(z, z) = \mu\, \varphi(z, z),\ n(z, z) = \nu\, \varphi(z, z), \varphi(z, z) \neq 0.$$
Then $\lambda\, Pz + \mu\, Qz + \nu\, Rz = 0$. A contradiction.

Let $\mathrm{rank}(l, m, n) = 2$, i.e. (to be definite), $n(z, z) = \lambda\, l(z, z) + \mu\, m(z, z)$, where l and m are not proportional. We have
$$l(z, z)(Pz + \lambda\, Rz) + m(z, z)(Qz + \mu\, Rz) = 0.$$
Let l and m be not coprime, i.e.,
$$l(z, z) = \alpha(z)\, \gamma(z),\ m(z, z) = \beta(z)\, \gamma(z),$$
where α and β are not proportional, $\gamma \neq 0$. We obtain
$$\alpha(z)(Pz + \lambda\, Rz) + \beta(z)(Qz + \mu\, Rz) = 0.$$
This implies (Lemma 11) that
$$(Pz + \lambda\, Rz) = \beta(z)\, A,\ (Qz + \mu\, Rz) = -\alpha(z)\, A,\ A \in \mathbf{C}^n.$$

If λ and μ are real, then the matrices on the left-hand sides of these equalities are Hermitian and, by Lemma 10, $\alpha(z)$ and $\beta(z)$ are proportional to $(z \cdot \bar{A})$. A contradiction.

If at least one of the numbers λ and μ is not real, for example, λ, then, separating the imaginary part in $(Pz \cdot \bar{z} + \lambda\, Rz \cdot \bar{z}) = \beta(z)\, A \cdot \bar{z}$, we obtain
$$(\mathrm{Im}\lambda)\, Rz \cdot \bar{z} = \mathrm{Im}(\beta(z)\, A \cdot \bar{z}).$$
Whence, by Lemma 10, we see that $\beta(z)$ is proportional to $z \cdot \bar{A}$; this implies that all three Hermitian forms $(Pz \cdot \bar{z},\ Qz \cdot \bar{z},\ Rz \cdot \bar{z})$ are proportional to $|z \cdot \bar{A}|^2$. A contradiction.

Let l and m be coprime. Then $(Pz + \lambda\, Rz) = (Qz + \mu\, Rz) = 0$. A contradiction. This completes the proof of the lemma.

Further, we assume that $(l(z, z), m(z, z), n(z, z))$ are linearly independent. Consider the ideals generated by pairs of quadratic forms:
$$I_1 = (m, n),\ I_2 = (l, n),\ I_3 = (l, m)$$
and the sets of their zeros V_1, V_2, V_3. The linear independence of the forms means that
$$l(z, z) \notin I_1, \quad m(z, z) \notin I_2, \quad n(z, z) \notin I_3. \tag{4.11}$$
Consider the zero sets of these ideals,
$$V_1 = \{m(z, z) = n(z, z) = 0\},\ V_2 = \{l(z, z) = n(z, z) = 0\},\ V_3 = \{l(z, z) = m(z, z) = 0\}.$$

Lemma 15. *Let the dimension of the space* $\mathbf{C}^n$ *be not less than four, and let* l *be nonidentically zero on* V_1*, let* m *be nonidentically zero on* V_2*, and let* n *be nonidentically zero on* V_3*. Then the relation* (4.10) *is impossible.*

Proof. It follows from relation (4.10) that $Pz = 0$ on the nonempty set

$$V_1' = \{z \in \mathbf{C}^n : m(z,z) = n(z,z) = 0,\ l(z,z) \neq 0\}.$$

Therefore, $Pz = 0$ on the set V_1 as well. By a similar reasoning for Qz and Rz, we see that all three linear operators vanish on

$$\{z \in \mathbf{C}^n : m(z,z) = n(z,z) = l(z,z) = 0\},$$

which is of dimension at least $(n-3)$ and, therefore, contains a nonzero z. A contradiction. This completes the proof of the lemma.

Consider the situation in which at least one of the three conditions of the previous lemma is violated. Let, for example, $l = 0$ on V_1. Taking into account that (l, m, n) are linearly independent and $l \notin I_1$, it immediately follows that I_1 is not radical. This means that, among the linear combinations of m and n, there is a reducible quadratic form. Together with the condition that l vanishes on V_1, this means that, from the linearly independent forms (l, m, n), we can pass, using a nondegenerate linear transformation, to their linear combinations of the form

$$l'(z,z) = \lambda(z)\,\mu_2(z),\ m'(z,z) = \mu_1(z)\,\mu_2(z),\ n'(z,z) = n(z,z).$$

Relation (4.10) acquires the form

$$l'(z,z)\,P'z + m'(z,z)\,Q'z + n'(z,z)\,R'z = \lambda(z)\,\mu(z)\,P'z + \mu_1(z)\,\mu_2(z)\,Q'z + n(z,z)\,R'z = 0, \tag{4.12}$$

where $(P'z, Q'z, R'z)$ are obtained from (Pz, Qz, Rz) by a nondegenerate linear transformation (below we omit the primes).

Lemma 16. *If the dimension of the space* $\mathbf{C}^n$ *is at least 4 and one of three conditions of Lemma* 15 *is violated, then* (4.10) *does not hold.*

Proof. Let $n(z,z)$ be divisible by $\mu_2(z)$, i.e., $n(z,z) = \mu_2(z)\,\nu(z)$. Then (4.12) becomes

$$\lambda(z)\,Pz + \mu_1(z)\,Qz + \nu(z)\,Rz = 0.$$

Then, just as in the proof of Lemma 15, we show that all three operators vanish on the subspace

$$\lambda(z) = \mu_1(z) = \nu(z) = 0$$

of positive dimension. This means that the original triple has a common kernel of the operators. A contradiction.

Let now $n(z,z)$ be non divisible by $\mu_2(z)$; we have

$$\mu_2(z)(\lambda(z)\,Pz + \mu_1(z)\,Qz) + n(z,z)\,Rz = 0.$$

Whence it follows that

$$Rz = \mu_2(z)\,A, \quad (\lambda(z)\,Pz + \mu_1(z)\,Qz) + n(z,z)\,A = 0.$$

Hence we obtain $n(z,z) = \alpha(z)\,\lambda(z) + \beta(z)\,\mu_1(z)$. Then we have

$$\lambda(z)(Pz + \alpha(z)\,A) + \mu_1(z)(Qz + \beta(z)\,A) = 0.$$

This implies in turn that

$$(Pz + \alpha(z)\,A) = \mu_1(z)\,B, \quad (Qz + \beta(z)\,A) = -\lambda(z)\,B.$$

Finally,

$$Pz = -\alpha(z)\,A + \mu_1(z)\,B, \quad Qz = -\beta(z)\,A - \lambda(z)\,B, \quad Rz = \mu_2(z)\,A.$$

That is (Pz, Qz, Rz) lie in the linear span of two constant vectors (A, B). From (Pz, Qz, Rz), using a nondegenerate linear transformation, one can pass to the original Hermitian matrices that appear in relation

(4.10). Here the vectors (A, B) correspond to two constant vectors (A', B'). In this case, as follows from Lemma 10, all three Hermitian forms are linear combinations of $|z \cdot \bar{A}'|^2$ and $|z \cdot \bar{B}'|^2$, and hence are linearly dependent. A contradiction.

Applying Lemma 14, Lemma 15, Lemma 16, and Proposition 8, we obtain the following assertion.

Proposition 17. (a) *Let $k = 3$. If $D(u)(z, z)$ satisfies relation* (4.4)*, then* $D(u)(z, z) = 0$.
(b) *In this case, g_3 consists of fields of the form*

$$X = (d(w, w),\, 2\,i\, < z, \bar{d}(w, w) >), \qquad \textit{where} \qquad < z, \bar{d}(< z, \bar{z} >, u) >= 0.$$

For $k = 3$, the form δ is

$$\delta(u) = \alpha_1\, u_1 + \alpha_2\, u_2 + \alpha_3\, u_3.$$

Let us write out (4.7); we obtain

$$(Pz \cdot \bar{z})\ < z, \bar{\alpha}_1 > +(Qz \cdot \bar{z})\ < z, \bar{\alpha}_2 > +(Rz \cdot \bar{z})\ < z, \bar{\alpha}_3 >= 0. \tag{4.13}$$

The rank of the triple of vectors $(\alpha_1, \alpha_2, \alpha_3)$ can be zero, one, two, or three. The rank zero means that $\delta = 0$.

Case 1 (*rank one*). Let the rank be equal to one, i.e., $\alpha_j = \lambda_j\, \alpha,\ \alpha \in \mathbf{C}^n \setminus \{0\},\ (\lambda_1, \lambda_2, \lambda_3) \neq 0$. It follows from (4.13) that

$$(\bar{\lambda}_1\, (Pz \cdot \bar{z}) + \bar{\lambda}_2\, (Qz \cdot \bar{z}) + \bar{\lambda}_3\, (Rz \cdot \bar{z}))\ < z, \bar{\alpha} >= 0.$$

This implies the linear dependence of (P, Q, R), which contradicts the nondegeneracy condition for this set.

Case 2 (*rank two*). Let the rank be equal to two. We may assume that $\alpha_3 = \bar{\lambda}\, \alpha_1 + \bar{\mu}\, \alpha_2$, where the vectors α_1, α_2 are linearly independent. It follows from (4.13) that

$$((Pz \cdot \bar{z}) + \lambda\, (Rz \cdot \bar{z}))\ < z, \bar{\alpha}_1 > +((Qz \cdot \bar{z}) + \mu\, (Rz \cdot \bar{z}))\ < z, \bar{\alpha}_2 >= 0.$$

In $\mathbf{C}^n$, choose a basis of the form $(\alpha_1, \alpha_2, \dots)$. Writing out relation (4.13), we obtain

$$\begin{aligned} p_1(z)\, ((Pz \cdot \bar{z}) + \lambda\, (Rz \cdot \bar{z})) + p_2(z)\, ((Qz \cdot \bar{z}) + \mu\, (Rz \cdot \bar{z})) &= 0, \\ q_1(z)\, ((Pz \cdot \bar{z}) + \lambda\, (Rz \cdot \bar{z})) + q_2(z)\, ((Qz \cdot \bar{z}) + \mu\, (Rz \cdot \bar{z})) &= 0, \\ r_1(z)\, ((Pz \cdot \bar{z}) + \lambda\, (Rz \cdot \bar{z})) + r_2(z)\, ((Qz \cdot \bar{z}) + \mu\, (Rz \cdot \bar{z})) &= 0. \end{aligned} \tag{4.14}$$

It follows from the condition that (P, Q, R) is nondegenerate that, in every pair of the form $(p_1, p_2), (q_1, q_2), (r_1, r_2)$, either both forms are equal to zero or are not proportional. Let $r_1 = r_2 = 0$. Then, writing out the first coordinate of the first relation and the second of the other and applying Lemma 5, we see that p_1 is proportional to p_2 and q_1 is proportional to q_2. Thus, all these forms are equal to zero, which means that both vectors α_1 and α_2 are contained in the kernel of $< z, \bar{z} >$. This contradiction means that (r_1, r_2) are not proportional. Then, taking into account Lemma 11 and the nondegeneracy condition, from the third relation (4.14) we obtain

$$P\, z + \lambda\, R\, z = r_2(z)\, A, \quad Q\, z + \mu\, R\, z = -r_1(z)\, A, \quad A \neq 0. \tag{4.15}$$

Substituting (4.15) into the first and second relations (4.14) and taking into account the fact that $A \neq 0$, we can write

$$p_1(z)\, r_2(z) - p_2(z)\, r_1(z) = 0, \quad q_1(z)\, r_2(z) - q_2(z)\, r_1(z) = 0.$$

Taking into account the fact that r_1 and r_2 are not proportional, we see that

$$p_1(z) = \nu\, r_1(z),\ p_2(z) = \nu\, r_2(z),\ q_1(z) = \kappa\, r_1(z),\ q_2(z) = \kappa\, r_2(z).$$

Writing out the first and second coordinates of (4.15), we obtain

$$\begin{aligned} &\nu\, r_1(z) + \lambda\, r_1(z) = a_1\, r_2(z), \quad \nu\, r_2(z) + \lambda\, r_2(z) = a_2\, r_2(z), \\ &\kappa\, r_1(z) + \mu\, r_1(z) = -a_1\, r_1(z), \quad \kappa\, r_2(z) + \mu\, r_2(z) = -a_2\, r_1(z). \end{aligned}$$

This implies

$$\nu = -\lambda,\ \kappa = -\mu,\ a_1 = a_2 = 0.$$

Let us represent (4.15) in the form

$$(Pz\cdot\bar z)+\lambda\,(Rz\cdot\bar z)=r_2(z)\,(A\cdot\bar z),\;\;(Qz\cdot\bar z)+\mu\,(Rz\cdot\bar z)=-r_1(z)\,(A\cdot\bar z). \tag{4.16}$$

If the coefficients λ and μ are real, then Lemma 10 implies that both forms r_1 and r_2 are proportional to $(z\cdot\bar A)$. A contradiction. Further, let only one of them be real, say, μ, while $L=\operatorname{Im}\lambda\neq 0$. Then the second relation (4.16) implies that $r_1(z)=\tau\,(z\cdot\bar A)$. Separating the imaginary part of the first relation (4.16) we see that

$$(Rz\cdot\bar z)=\frac{1}{L}\operatorname{Im}\left(r_2(z)\,(A\cdot\bar z)\right).$$

This is equivalent to the relation

$$r_j(z)=\frac{1}{2\,i\,L}(r_2(z)\,a_j-(z\cdot\bar A)\,\bar r_2^j).$$

For $j=2$ we obtain

$$r_2(z)=\frac{\bar r_2^2}{2\,i\,L}\,(z\cdot\bar A).$$

Whence it follows that r_1 and r_2 are proportional. A contradiction.

Now let $L=\operatorname{Im}\lambda\neq 0$ and $M=\operatorname{Im}\mu\neq 0$. Separating the imaginary parts of the first and second relations (4.16), we obtain

$$(Rz\cdot\bar z)=\frac{1}{L}\operatorname{Im}\left(r_2(z)\,(A\cdot\bar z)\right)=-\frac{1}{M}\operatorname{Im}\left(r_1(z)\,(A\cdot\bar z)\right).$$

This implies that

$$(\bar z\cdot A)\,(L\,r_1(z)+M\,r_2(z))=(z\cdot\bar A)\,(L\,\overline{r_1(z)}+M\,\overline{r_2(z)}),$$

which in turn implies

$$r_2(z)=-\frac{L}{M}\,r_1(z)+\nu\,(z\cdot\bar A).$$

Now, separating the real parts of relations (4.16) and substituting the values for $(Rz\cdot\bar z)$ and $r_2(z)$ obtained above into these parts, we obtain expressions for $(Pz\cdot\bar z)$ and $(Qz\cdot\bar z)$. As a result, we see that all three Hermitian forms are proportional to $\operatorname{Im}\left(r_1(z)\,(A\cdot\bar z)\right)$. A contradiction.

Case 3 (rank three). Assume that $n\geqslant 3$. As is well known, there is no counterexample of CR-dimension $n\leqslant 2$. Choose a basis in $\mathbf{C}^n$ of the form $(\alpha_1,\alpha_2,\alpha_3,\dots)$. Relation (4.13) becomes

$$\begin{aligned}
p_1(z)\,(Pz\cdot\bar z)+p_2(z)\,(Qz\cdot\bar z)+p_3(z)\,(Rz\cdot\bar z)&=0,\\
q_1(z)\,(Pz\cdot\bar z)+q_2(z)\,(Qz\cdot\bar z)+q_3(z)\,(Rz\cdot\bar z)&=0,\\
r_1(z)\,(Pz\cdot\bar z)+r_2(z)\,(Qz\cdot\bar z)+r_3(z)\,(Rz\cdot\bar z)&=0.
\end{aligned}\tag{4.17}$$

Or, in the vector form,

$$\begin{aligned}
p_1(z)\,Pz+p_2(z)\,Qz+p_3(z)\,Rz&=0,\\
q_1(z)\,Pz+q_2(z)\,Qz+q_3(z)\,Rz&=0,\\
r_1(z)\,Pz+r_2(z)\,Qz+r_3(z)\,Rz&=0.
\end{aligned}\tag{4.18}$$

Let us write the first coordinate of the first relation (4.18), the second coordinate of the second relation, and the third coordinate of the third relation. We obtain

$$\begin{aligned}
p_1^2+p_2\,q_1+p_3\,r_1&=0,\\
q_1\,p_2+q_2^2+q_3\,r_2&=0,\\
r_1\,p_3+r_2\,q_3+r_3^2&=0.
\end{aligned}$$

Whence, using Lemma 13, we have

$$\begin{aligned}
p_1(z)&=\alpha_2\,p_2(z)+\alpha_3\,p_3(z),\\
q_2(z)&=\beta_1\,q_1(z)+\beta_3\,q_3(z),\\
r_3(z)&=\gamma_1\,r_1(z)+\gamma_2\,r_2(z).
\end{aligned}$$

Then (4.18) becomes

$$p_2(z)\,(Q+\alpha_2\,P)\,z+p_3(z)\,(R+\alpha_3\,P)\,z=0,$$
$$q_1(z)\,(P+\beta_1\,Q)\,z+q_3(z)\,(R+\beta_3\,Q)\,z=0,$$
$$r_1(z)\,(P+\gamma_1\,R)\,z+r_2(z)\,(Q+\gamma_2\,R)\,z=0. \tag{4.19}$$

It follows from the linear independence of (P,Q,R) that in every pair

$$(p_2,p_3), \qquad (q_1,q_3), \qquad (r_1,r_2)$$

either both the forms vanish simultaneously or they are nonproportional.

Case 3.0. All three pairs cannot be zero, because it would mean that all the first three basis vectors fall into a common kernel. Thus, one of the three pairs is nonzero and nonproportional. Let it be (r_1,r_2).

Case 3.1. Let both remaining pairs be zero. Write out the first coordinate of the third relation in (4.19); we obtain $\gamma_1\,r_1^2+\gamma_2\,r_1\,r_2=0$. Whence it follows that $\gamma_1=\gamma_2=0$. Then, using Lemma 3 and the nondegeneracy, we obtain $P\,z=r_2(z)\,A,\ Q\,z=-r_1(z)\,A,\ A\neq 0$ and, from Lemma 6, that both r_1 and r_2 are proportional to $(z\cdot\bar{A})$. A contradiction.

Case 3.2. Let only one pair vanish, say, $p_1=p_2=0$, and the forms (q_1,q_3), as well as (r_1,r_2), be pairwise nonproportional. Let us write out the first coordinate of the second relation (4.19); we obtain $\beta_1\,q_1^2+q_3\,r_1+\beta_3\,q_3\,q_1=0$. Whence it immediately follows that $r_1=\nu\,q_1$. Let us write out the first coordinate of the third relation (4.19); we obtain $\gamma_1\,r_1^2+r_2\,(q_1+\gamma_2\,r_1)=0$. Whence, since r_2 is not proportional to r_1, it follows that $\gamma_1=0,\ \gamma_2=-1/\nu$, i.e., $r_3=-r_2/\nu$. The relation $r_1(z)\,P\,z+r_2(z)\,(R-Q/\nu)\,z=0$ implies that $r_j=q_j/\nu$. Whence we see that $R=Q/\nu$. A contradiction.

Case 3.3. Let all three pairs be pairwise nonproportionate. Then, from (4.19), Lemma 11, and the nondegeneracy, we obtain

$$(Q+\alpha_2\,P)\,z=-p_3(z)\,A,\quad (R+\alpha_3\,P)\,z=p_2(z)\,A,$$
$$(P+\beta_1\,Q)\,z=-q_3(z)\,B,\quad (R+\beta_3\,Q)\,z=q_1(z)\,B,$$
$$(P+\gamma_1\,R)\,z=-r_2(z)\,C,\quad (Q+\gamma_2\,R)\,z=r_1(z)\,C, \tag{4.20}$$

where A,B,C are three nonzero vectors. Writing out the third coordinate for the first equality, the second coordinate for the second equality, the third coordinate for the third equality, the first coordinate for the fourth equality, the second coordinate for the fifth equality, and the first coordinate for the sixth equality, we obtain the following family of relations:

$$q_3(z)=-(\alpha_2+a_3)\,p_3(z),\quad (\alpha_2+a_3)(\beta_1+b_3)=1, \tag{4.21}$$
$$r_2(z)=(a_2-\alpha_3)\,p_2(z),\quad (\alpha_3-a_2)(\gamma_1+c_2)=1,$$
$$r_1(z)=(b_1-\beta_3)\,q_1(z),\quad (b_1-\beta_3)(c_1-\gamma_2)=1.$$

Thus, all nine linear forms $(p_1,p_2,p_3,q_1,q_2,q_3,r_1,r_2,r_3)$ are linear combinations of three of them, (q_1,p_2,p_3):

$$p_1=\alpha_2\,p_2+\alpha_3\,p_3,\ \ q_2=\beta_1\,q_1-\beta_3\,(a_3+\alpha_2)p_3,\ \ q_3=-(\alpha_2+a_3)\,p_3,$$
$$r_1=(b_1-\beta_3)\,q_1,\ \ r_2=(a_2-\alpha_3)\,p_2,\ \ r_3=\gamma_1\,(b_1-\beta_3)\,q_1+\gamma_2\,(a_2-\alpha_3)\,p_2.$$

We also see that

$$\beta_1=\frac{1}{\alpha_2+a_3}-b_3,\ \ \gamma_1=\frac{1}{\alpha_3-a_2}-c_2,\ \ \gamma_2=\frac{1}{\beta_3-b_1}+c_1,$$

where the denominators do not vanish.

In this case, all eighteen relations obtained from the first three coordinates (4.20) are a system $L(q_1,p_2,p_3)=0$ of linear relations among (q_1,p_2,p_3). The coefficients of this system rationally depend on $(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3)$ and $(\alpha_2,\alpha_3,\beta_3)$, and the denominators do not vanish by (4.21).

Let us number these equations successively, i.e., the first coordinate of the first group, the second coordinate of the first group, the third coordinate of the first group, the first coordinate of the second group, and so on until the third coordinate of the sixth group. We obtain

$$l_j^1\,q_1+l_j^2\,p_2+l_j^3\,p_3=0,\quad j=1,\ldots,18.$$

Taking into account that the rank of the pair of forms (p_2, p_3) is equal to two, we see that the rank of the resulting matrix of coefficients of size 3×18 is equal to one. We obtain a system of 34 equations

$$e_{2j-3} = l_j^2\, l_1^1 - l_j^1\, l_1^2, \quad e_{2j-2} = l_j^3\, l_1^1 - l_j^1\, l_1^3, \qquad j = 2, \ldots, 18.$$

Let E_m be the numerator of the expression e_m. We obtain a system of 34 polynomial relations for 12 variables: $E_m = 0$, $m = 1, \ldots, 34$. We have used above a part of the relations (4.20). We immediately see that

$$E_3 = E_4 = E_7 = E_8 = E_{15} = E_{17} = E_{18} = E_{26} = 0.$$

Denote the system of the remaining 26 equations by

$$\mathscr{L}(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, \alpha_2, \alpha_3, \beta_3) = 0. \tag{4.22}$$

Lemma 18. (a) *The solutions of system* (4.22) *form the union of three irreducible components*

$$\begin{aligned} &a_1 = a_2 = a_3 = \alpha_2 = \alpha_3 = 0,\\ &a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = \alpha_3 = 0,\\ &a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = \alpha_2\, \beta_3 + \alpha_3 = 0. \end{aligned}$$

(b) *None of these solutions is compatible with the condition of Case* (3.3).

Proof. Part (a) is verified by a direct analysis of the system, which is simplified due to the fact that the equations $a_3 = a_2$ and $(c_2 - c_1)(a_2 - \alpha_3) = 0$ are part of the system. Since (4.21) implies that $(a_2 - \alpha_3) \neq 0$, it follows that $c_2 = c_1$. Part (a) is also easily verified with using the Maple system.

Further, if a solution belongs to the first or second component, then $\alpha_3 - a_2 = 0$, which contradicts (4.21). Let a solution belong to the third component. Then

$$\beta_1 = \frac{1}{\alpha_2},\ \beta_3 = -\frac{\alpha_3}{\alpha_2},\ \gamma_1 = \frac{1}{\alpha_3},\ \gamma_2 = -\frac{\alpha_2}{\alpha_3}.$$

Here the dependent forms are expressed in terms of (q_1, p_2, p_3) as follows:

$$q_2 = -\alpha_2\, p_2,\ q_3 = -\alpha_2\, p_3,\ r_1 = -\alpha_2\, \alpha_3\, p_2 - \alpha_3^2\, p_3,\ r_2 = -\alpha_3\, p_2,\ r_3 = -\alpha_3\, p_3.$$

Let us use the first two equalities in (4.20) to express Q and R

$$Q\, z = -\alpha_2\, P\, z - p_3(z)\, A, \quad R\, z = -\alpha_3\, P\, z + p_2(z)\, A. \tag{4.23}$$

Substituting the result into the four remaining equalities; we obtain

$$B = -\frac{1}{\alpha_2}\, A,\ q_1 = -\alpha_2^2\, p_2 - \alpha_2\, \alpha_3\, p_3,\ C = \frac{1}{\alpha_3^2}\, A,\ \frac{\alpha_2}{\alpha_3}\, p_2 = 0.$$

The last equality cannot be satisfied, because $\alpha_2 \neq 0$. This contradiction completes our discussion.

Theorem 19. *If $\mathscr{Q}$ is a nondegenerate model quadric of codimension $k = 3$, then the decomposition* aut $\mathscr{Q}$ *into the weight components has the form*

$$g_{-2} + g_{-1} + g_0 + g_1 + g_2.$$

Proof. As is well known, for every nondegenerate quadric, the subalgebra $g_- = g_{-2} + g_{-1}$ is Tanaka fundamental ([6, 7]). We have shown above that $g_3 = 0$. Whence the triviality of all components with greater weights immediately follows. This completes the proof of the theorem.

The above proofs of nonexceptionality of the quadrics of small codimension use the second relation (2.3) only. To quadrics of small codimensions, the quadrics of very high codimensions are opposed, in a sense. The largest codimension for a fixed n is $k = n^2$. This is the only quadric given by the basis set in the space Hermitian forms. Introduce the following notation for the variables of the group w:

$$\begin{aligned} &w_{\alpha\alpha} = u_{\alpha\alpha} + i\, v_{\alpha\alpha}, \quad 1 \leqslant \alpha \leqslant n.\\ &w^R_{\alpha\beta} = u^R_{\alpha\beta} + i\, v^R_{\alpha\beta}, \quad w^I_{\alpha\beta} = u^I_{\alpha\beta} + i\, v^I_{\alpha\beta}, \quad 1 \leqslant \beta < \alpha \leqslant n. \end{aligned}$$

Then the equations of this quadric become

$$v_{\alpha\alpha} = z_\alpha\,\bar z_\alpha, \quad 1 \leqslant \alpha \leqslant n.$$
$$v^R_{\alpha\beta} = 2\,\mathrm{Re}\, z_\alpha\,\bar z_\beta, \quad v^I_{\alpha\beta} = 2\,\mathrm{Im}\, z_\alpha\,\bar z_\beta, \quad 1 \leqslant \beta < \alpha \leqslant n.$$

Proposition 20. *If $\mathcal{Q}$ is a nondegenerate quadric of codimension $k = n^2$ (the last quadric), then $g_3 = 0$. In other words, such a quadric is not exceptional.*

Proof. We select some coordinates in the first vector relation (2.3). Let us write out the $(\alpha\alpha)$-coordinate,

$$A_\alpha(u)(z,z)\,\bar z_\alpha = 2\,i\,z_\alpha\,\Delta\bar a_\alpha(u).$$

The divisibility by $\bar z_\alpha$ implies that a_α depends only on $u_{\alpha\alpha}$, and $A_\alpha(u)(z,z) = 2\,i\,z_\alpha^2\,\bar a'_\alpha(u_{\alpha\alpha})$. Writing out the $(\alpha 1)^R$-coordinate (2.3) for $\alpha > 1$, we obtain

$$\bar a'_\alpha(u_{\alpha\alpha})\,z_\alpha^2\,\bar z_1 - \bar a'_1(u_{11})\,z_1^2\,\bar z_\alpha = z_\alpha\,\bar a'_1|z_1|^2 + z_1\,\bar a'_\alpha\,|z_\alpha|^2.$$

This gives $a = \mathrm{const},\ A = 0$. Whence it immediately follows that $g_1 = 0$, and, due to fundamentality, $g_+ = 0$ as well. This completes the proof of the proposition.

Note that here, in contrast to small codimensions, we have used the first relation (2.3) only, and a small part of it, and not used the other relation (2.3) at all.

5. *RAQ*-QUADRICS

In [10], a very interesting class of quadrics such that $n = k$ was considered (the codimension is equal to the CR-dimension), the so-called RAQ-quadrics. These quadrics are in one-to-one correspondence with the finite-dimensional real commutative (associative) algebras with identity.

Let $\mathscr{A}$ be such an algebra of dimension n. If X and Y are elements of $\mathscr{A}$, then by $X \cdot Y$ we denote their product as elements of algebra. Let $\mathscr{A}^c = \mathscr{A} \bigotimes \mathbf{C}$ be the complexification of $\mathscr{A}$. If $Z \in \mathscr{A}^c$, then by $\bar Z$ we denote the complex conjugation canonically defined in $\mathscr{A}^c$. The quadric corresponding to the algebra $\mathscr{A}$ has the form

$$Q = \{(Z,W) \in (\mathscr{A}^c)^2 :\ \mathrm{Im}\, W = Z \cdot \bar Z\}. \tag{5.1}$$

If we ask a following question: When does a quadric of type (n,n) have this form? The answer is given by two conditions on $< z, \bar z >$:
– real property: in some coordinates, the forms on real vectors takes real values,
– associativity $<< p,q >, r >=< p, < q,r >>$ for all $p,\ q,\ r \in \mathbf{R}^n$.
That is why they are called RAQ-quadrics – Real Associative Quadrics.

In [10], it was shown that the nondegeneracy condition for an RAQ-quadric is equivalent to the presence of unit in $\mathscr{A}$.

Theorem 21. *There are no exceptional RAQ-quadrics.*

Proof. To prove this, we use the sufficient nonexceptionality condition of Theorem 6. Let us verify the validity of conditions (I) and (II).

Condition (I). Let $(E_1,\dots,E_n)$ be the basic elements of the algebra $\mathscr{A}$, where E_1 is the unit of the algebra. Then the coordinate operators have the form $H_j\,Z = E_j \cdot Z$,and thus $H_j\,E_1 = E_j$, and the set $(E_1,\dots,E_n)$ has rank n.

Condition (II). The image of the mapping $(Z', Z) \to < Z', \bar Z >= Z' \cdot \bar Z$ coincides with $\mathscr{A}^c$, because (Z', E_1) goes to Z'. This completes the proof of the theorem.

Corollary 22. *If $\mathcal{Q}$ is a nondegenerate RAQ-quadric, then*
(a) $\mathrm{aut}\,\mathcal{Q} = g_{-2} + g_{-1} + g_0 + g_1 + g_2$.
(b) *The subgroup G_+, the subgroup of nonlinear automorphisms $\mathcal{Q}$ preserving the origin, is described by the Poincaré formula (see [10]), namely,*

$$Z^* = (Z + a\cdot W)\cdot(1 - 2\,i\,\bar a\cdot Z - (r + i\,a\cdot\bar a)\cdot W)^{-1},$$
$$W^* = W\cdot(1 - 2\,i\,\bar a\cdot Z - (r + i\,a\cdot\bar a)\cdot W)^{-1}, \tag{5.2}$$

where $a \in \mathscr{A}^c,\ r \in \mathscr{A}$.

6. A BOUND FOR THE LENGTH OF THE SUBALGEBRA $\mathbf{g}_+$

In [12], a theorem (Theorem 12.3.11) is given that enables one to estimate the number of a jet j on which the mapping of an analytic l-nondegenerate germ of codimension k (with the minimality condition) depends, namely, $j \leqslant (k+1)\, l$. Since a nodegenerate quadric $\mathscr{Q}$ is 1-nondegenerate and has finite type at every point (in particular, it is minimal), it follows that, for the degree d of the coefficients of fields in aut $\mathscr{Q}$, we obtain the following bound:

$$d \leqslant (k+1).$$

In this part of the paper, we obtain this bound and other similar results, using methods different from those used in [12] (Segre sets). Namely, we use the technique of the Fourier transform in the space of distributions (generalized functions), which underlies the proof of the Ehrenpreis–Palamodov theorem [11]. This theorem was used in [5] to obtain a criterion for the finite-dimensionality of the algebra of a quadric $\mathscr{Q}$.

The Ehrenpreis–Palamodov theorem is a very general result which gives a description of the kernel of an arbitrary linear differential operator with constant coefficients (a system of equations). The number q of unknown functions, the number p of independent variables they depend on, and also the maximal order r of differential relations are arbitrary. By a solution of such a system in the present context one means a vector of distributions that satisfies all the relations included in the system of equations.

By the characteristic set for such a system one means an algebraic subset χ of the space of independent variables $\mathbf{C}^p$, which is formed by the exponents $\lambda \in \mathbf{C}^p$ for which there exists a nonzero vector $v \in \mathbf{C}^q$ such that

$$v\, \exp(\lambda_1\, x_1 + \cdots + \lambda_p\, x_p)$$

is a solution to the system. The theorem states that every generalized solution of the system can be written in the form

$$\int (\mathscr{P}(x) \exp(\lambda_1\, x_1 + \cdots + \lambda_p\, x_p))\; d\,\mu(\lambda),$$

where $\mu(\lambda)$ is a charge supported by χ and $\mathscr{P}(x)$ is a polynomial vector.

Let $F(s) = (F_1(s), \ldots, F_n(s))$ be a vector distribution which is the Fourier transform of the vector function $a(u) = (a_1(u), \ldots, a_n(u))$. Then, applying the Fourier transform to the second equation (2.3), we see that the following relation holds for every $z \in \mathbf{C}^n$:

$$(s_1\, < z, \bar z >_1 + \cdots + s_k\, < z, \bar z >_k)^2\, < F(s), \bar z >= 0. \tag{6.1}$$

Let $C[s] = C[s_1, \ldots, s_k]$ be the polynomial ring in the k-dimensional variable s with complex coefficients, and let $C[u] = C[u_1, \ldots, u_k]$ be the same polynomial ring in the k-dimensional variable u. Let $(C[s])^k$ be a free module of dimension k over $C[s]$, and let $(C[u])^n$ be a free module of dimension n over $C[u]$. Let

$$\{e_j = (0, \ldots, 0, 1 \text{ (at the jth place)}, 0, \ldots, 0),\ j = 1, \ldots, k\}$$

be the set of generators of the module $(C[s])^k$.

Introduce the module $M = M(\mathscr{Q})$, the submodule of the module $(C[s])^k$ generated by all elements of the form

$$(s_1\, < z, \bar z >_1 + \cdots + s_k\, < z, \bar z >_k)^2\, (< \varphi(s), \bar z >_1\; e_1 + \cdots + < \varphi(s), \bar z >_k\; e_k), \tag{6.2}$$

where $z \in \mathbf{C}^n$, $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s)) \in (C[s])^n$. We call $M(\mathscr{Q})$ the *characteristic* submodule of the quadric $\mathscr{Q}$.

Let us present a set of assertions which, under the condition that the quadric Q is nondegenerate, follow for the equation

$$\Delta^2\, < a(u), \bar z > =\; 0 \tag{6.3}$$

directly from the general scheme [11] (see also [13], Chap. 10).

(a) The space of solutions of (6.3) is a linear subspace $\mathscr{L}_1$ of the space of polynomials of degree not exceeding some degree $d_{\max}$.

(b) The space of solutions of (6.1) is a linear subspace $\mathscr{L}_2$ of the space of distributions which is generated by δ_0 (the delta function with the support at zero) and its derivatives of order at most $d_{\max}$.

(c) The spaces $\mathscr{L}_1$ and $\mathscr{L}_2$ are isomorphic, and an isomorphism is established by the Fourier transform.

(d) The characteristic submodule $M(\mathscr{Q})$ is of finite codimension in $(C[s])^k$ (as a linear subspace).

(e) The degree and the dimension of the space $\mathscr{L}_1$ of solutions of (6.3) coincide with the degree and dimension of the quotient space $M' = (C[s])^k \,/\, M(\mathscr{Q})$.

Since the polynomial ring $C[s]$ is Noetherian, it follows that the module M is finitely generated. According to (6.2), this means that M has a finite system of generators of the form

$$\{(l_\nu(s))^2\,(\lambda_\nu^1\,e_1+\cdots+\lambda_\nu^k\,e_k)\},$$

where $l_\nu(s)$ are linear forms and $\lambda\in\mathbf{C}^k$.

Let M_j be the projection of M onto the jth coordinate $(C[s])^k$. This is the ideal in $C[s]$ generated by the forms $\{(l_\nu(s))^2\,\lambda_\nu^j\}$, where $\lambda_\nu^j\neq 0$. Let r_j be the rank of the corresponding family of linear forms $\{l_\nu(s)\}$.

Proposition 23. (a) *For all j, the rank is $r_j=k$.*
(b) *A complement to M, the space M', is the subspace of the space of families of polynomials in s whose total degree does not exceed k. The space $\mathscr{L}_2$ is a subspace of the space of linear combinations of the delta function and its derivatives of order at most k. The space $\mathscr{L}_1$ is a subspace of the space of families of polynomials in u whose total degree does not exceed k.*

Proof. Let there exist a j such that $r_j<k$. Choose a linearly independent family of maximal rank $(L_1(s),\dots,L_{k'}(s))$, $k'<k$, from the family $\{l_\nu(s)\}$. Let us make, in the space of variable s, a nondegenerate linear change $s\to\tilde s$ such that $\tilde s_j=L_j(s)$ for $j=1,\dots,k'$. In the new variables, the generators of M_j are $(\tilde s_1)^2,\dots,(\tilde s_{k'})^2$. The complement of M_j to $C[\tilde s]$ is infinite-dimensional. Indeed, it contains the entire infinite-dimensional ring of polynomials in the last variable $C[\tilde s_k]$. This immediately implies that the complement to M is infinite-dimensional. This contradiction proves part (a).

Thus, among the generators of the ideal, there are the squares of all coordinates. Therefore, the complement contains only polynomials whose degree with respect to any variable does not exceed one. In particular, this means that the total degree does not exceed k. Returning to the old variables, we keep this estimate. Thus, the projection of the complement onto each coordinate does not contain polynomials of degree greater than k. This completes the proof of the assertion.

Lemma 4 immediately implies that, if $A(u)(z,z)$ is a solution to (2.3), then

$$\deg_u A(u)(z,z)\leqslant k-1.$$

Arguments similar to those given above show that, if $(C(u),b(u))$ is a solution to (2.4), then

$$\deg_u C(u)\leqslant k,\quad \deg_u b(u)\leqslant 2\,k.$$

From these bounds, one can obtain general bounds for the degrees of the coefficients of an arbitrary element of the algebra $\operatorname{aut}\mathscr{Q}$. As a result, for the odd component, we obtain the bound $(k+1)$ for the degree, and, for the even component, the bound $2k$. However, one can proceed in different way. This will give a more accurate information, and also does not use other bounds, except for the bound for the degree a obtained above. One can first estimate the *weight* of the odd component, and then use the fundamental property of the algebra.

Theorem 24. *If $\mathscr{Q}$ is a nondegenerate model quadric codimension k, then*
(a) *the weight of fields of odd weight in $\operatorname{aut}\mathscr{Q}$ does not exceed $2\,k-1$;*
(b) *the weight of an arbitrary field does not exceed $2\,k$, i.e., $\operatorname{aut}\mathscr{Q}=g_{-2}+g_{-1}+g_0+g_1+\cdots+g_{2k}$;* (c) *the degrees of the coefficients of fields in $\operatorname{aut}\mathscr{Q}$ do not exceed $(k+1)$.*

Proof. By the bounds obtained for a and A, we immediately see that the weight of the odd component

$$2\,\mathrm{Re}\left((a(w)+A(w)(z,z))\,\frac{\partial}{\partial z}+2\,i\,<z,\bar a(w)>\,\frac{\partial}{\partial w}\right)$$

does not exceed $2k-1$. This proves part (a). It follows from the fundamentality that there are no nonzero of a weight exceeding $2k$. This proves part (b). As was noted above, the degrees of the coefficients for the fields of odd weight do not exceed $(k+1)$. We obtain a bound for the degree of the coefficients for fields of even weight from the bound for the weight. If the weight of a field of even weight

$$2\,\mathrm{Re}\left(C(w)\,z\,\frac{\partial}{\partial z}+b(w)\,\frac{\partial}{\partial w}\right)$$

does not exceed $2k$, then the degrees of both $C(u)z$ and $b(u)$ do not exceed $(k+1)$ either. This completes the proof of the theorem.

As is well known [15], the group of local automorphisms of a nondegenerate quadrics is a subgroup of the group of birational automorphisms of the ambient space $\mathbf{C}^{n+k}$ with a uniform bound for the degree. The bound obtained in the previous theorem for the degree of infinitesimal automorphisms enables us to give a bound for the degree of automorphisms.

Corollary 25. *Let* $\mathcal{Q}$ *be a nondegenerate quadric of type* (n,k)*, and let* $\operatorname{aut}\mathcal{Q}$ *be a local group of its automorphisms that are holomorphic in a neighborhood of zero. Then* $\operatorname{aut}\mathcal{Q}$ *consists of birational transformations* $\mathbf{C}^{n+k}$ *whose degree does not exceed* $(3n+3k+2)(k+1)$.

7. OPEN QUESTIONS

If (n,k) is a CR-type (n is a CR-dimension and k is a codimension), then nondegenerate quadrics are possible in the range $1 \leqslant k \leqslant n^2$.

Question 26. What are the CR-types in this range for which exceptional quadrics exist?

As we have seen, in addition to the cases $k=1,2,3$ mentioned above for which there are no exceptional quadrics, one can add $k=n^2$ to the list of these codimensions (Proposition 20).

As was shown above (Theorem 7), there are no exceptional quadrics of codimension 4 for CR-dimension $n \leqslant 3$. An example of an exceptional quadric of codimension $k=4$ given in [3] has the CR-dimension $n=6$. As a special case of Question 26, the following question can be posed. What is the minimum CR-dimension admitting exceptional quadrics of codimension 4? (Versions of the answer are 4, 5, 6.)

It was shown above (Theorem 24) that the weights of the positive components of the Lie algebra of a nondegenerate quadric of codimension k do not exceed $2k$. However, this bound is not supported by an example, and the question of an sharp bound remains open.

Question 27. Are there non-degenerate quadrics for which the length of g_+ is equal to $2k$? If no, then what is the exact bound?

For the (n,k)-cases in which exceptional quadrics occur, what is the subset of exceptional quadrics in the collection of all nondegenerate quadrics of this type? As noted above, the exceptional quadrics are given by a finite set of polynomial relations of the type "equal to zero" and "not equal to zero" (a semi-algebraic set). Moreover, a natural point of view is to conduct this research in the moduli space [14], i.e., after a factorization of the space $\mathscr{H}$ of families of k Hermitian forms by a known linear action rather than in the space $\mathscr{H}$ itself. An answer to the following question is of interest.

Question 28. What is the codimension of the moduli subspace of the exceptional quadrics in the moduli space of all nondegenerate quadrics?

REFERENCES

[1] F. Meylan, "A Counterexample to the 2-Jet Determination Chern-Moser Theorem in Higher Codimension", *arxiv:2003.11783v1 [math.cv]*, 2020.

[2] V. K. Beloshapka, "A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space", *Math. Notes*, **47**:3 (1990), 239–242.

[3] J. Gregorovič and F. Meylan, "Construction of Counterexamples to the 2-Jet Determination Chern-Moser Theorem in Higher Codimension", *arXiv:2010.10220v1 [math.CV]*, 2020.

[4] S. S. Chern and J. K. Moser, "Real Hypersurfaces in Complex Manifolds", *Acta Math.*, **133** (1974), 219–271.

[5] V. K. Beloshapka, "Finite-dimensionality of the group of automorphisms of a real-analytic surface", *Math. USSR-Izv.*, **52**:2 (1988), 437–442.

[6] M. Sabzevari and A. Spiro, "On the Geometric Order of Totally Nondegenerate CR-Manifolds", *arXiv: 1807.03076v1 [mathCV]*, 2018.

[7] J. Gregorovič, "On the Beloshapka's Rigidity Conjecture for Real Submanifolds in Complex Space", *arXiv: 1807.03502v1 [mathCV]*, 2018.

[8] N. F. Palinchak, "Real quadrics of codimension 3 in $\mathbf{C}^6$ and their non-linear automorphisms", *Izv. Math.*, **59**:3 (1995), 597–617.

[9] E. G. Anisova, "Quadrics of codimension 4 in $\mathbf{C}^7$ and their automorphisms", *Math. Notes*, **64**:6 (1998), 697–703.

[10] V. Ežov and G. Schmalz, "A Matrix Poincaré Formula for Holomorphic Automorphisms of Quadric of Higher Codimension. Real Associative Quadrics", *J. Geom. Anal.*, **8**:1 (1997), 27–41.

[11] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer-Verlag, New York-Berlin, 1970.

[12] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "CR Automorphisms of Real Analytic CR Manifolds in Complex Space", *Comm. Anal. Geom.*, **6** (1998), 291–315.

[13] B. Sturmfels, "Solving Systems of Polynomial Equations", *CBMC, Regional Conference Series in Mathematics*, **97** (2002), 152.

[14] V. K. Beloshapka, "Moduli Space of Model Real Submanifolds", *Russ. J. Math. Phys.*, **13**:3 (2006), 245–252.

[15] A. E. Tumanov, "Finite-Dimensionality of the Group of CR Automorphisms of a Standard CR Manifold and Proper Holomorphic Mappings of Siegel Domains", *Izv. Akad. Nauk SSSR Ser. Mat.*, **52**:3 (1988), 651–659.