Modification of Poincaré's construction and its application in CR-geometry of hypersurfaces in $\mathbf{C}^4$

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Abstract. The modified Poincaré construction (a generalization of Poincaré's homological operator) was earlier used to estimate the dimension of the local automorphism group for an arbitrary germ of a real-analytic hypersurface in $\mathbf{C}^3$. In the present paper we prove the following alternative. For every hypersurface in $\mathbf{C}^4$, this dimension is either infinite or does not exceed 24. Moreover, 24 occurs only for a non-degenerate hyperquadric (one of the two). If the hypersurface is 2-nondegenerate (resp. 3-nondegenerate) at a generic point, the bound can be improved to 17 (resp. 20).

Keywords: CR-manifold, automorphisms, model surfaces.

§ 1. Introduction

A leading element of the method of model surfaces is Poincaré's construction, which was used by him in celestial mechanics and CR-geometry (the homological operator of Poincaré or Poincaré–Dulac). Its application in CR-geometry was described in 1907 [1] (see also [2]). This construction is essentially a version of implicit mapping theorem in the class of formal power series.

This construction is commonly used in CR-geometry as follows. Given a nonlinear differential or functional relation $F(x,\phi(x))=0$, we assume that the ring of formal power series in x is endowed with a grading (weight) in such a way that the μth component of the relation is of the form

$$L(x,\phi_\mu(x)) = \text{an expression depending on } \phi_\nu \text{ with } \nu \leqslant \mu-1,$$

where $L(x,y)$ depends linearly on y. Then the dimension of the vector space of solutions of the equation $L(x,\phi(x))=0$ is clearly greater than or equal to the dimension of the set of solutions of the original equation $F(x,\phi(x))=0$.

A modification of this construction (reduction of depth two) was used in [3] to estimate the dimension of the Lie algebra of infinitesimal automorphisms of an arbitrary holomorphically non-degenerate real hypersurface in $\mathbf{C}^3$. Namely, let the μth component of the relation $F(x,\phi(x))=0$ be of the form

$$L_1(x,\phi_\mu(x)) + L_2(x,\phi_{\mu-1}(x)) = \text{an expression depending on } \phi_\nu \text{ with } \nu \leqslant \mu-2,$$

where $L_1(x,y)$ and $L_2(x,y)$ depend linearly on y. Then the dimension of the vector space of solutions of the equation $L_1(x,\phi(x))+L_2(x,\phi(x))=0$ is clearly greater

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than or equal to the dimension of the set of solutions of the original equation $F(x, \phi(x)) = 0$. We similarly define a generalization of this construction (reduction of an arbitrary depth k).

In this paper we give another example of using the reductions of depth two and three. We use this modification of Poincaré's construction to give an upper bound for the dimension of the Lie algebra of infinitesimal automorphisms of an arbitrary holomorphically non-degenerate real hypersurface in $\mathbf{C}^4$ (Theorem 2).

This result confirms the following old conjecture [4]. The dimension of the automorphism group of any real-analytic hypersurface germ is either infinite or not exceeding the dimension of this group for any non-degenerate standard hyperquadric (the latter dimension is equal to 24 in $\mathbf{C}^4$).

Note that the ordinary Poincaré construction (of depth one) suffices to obtain the well-known bound for hypersurfaces in $\mathbf{C}^2$. However, to obtain a bound in the space of dimension $(n+1)$, we need to use the reductions of all depths from 1 to n.

Since the k-reduced Poincaré construction differs from the classical one, we expose the scheme of its application in a general form. Let V be the vector space of all infinite sequences of real numbers and let $x \in V$ be a sequence split into finite intervals, which are denoted by x_j. Thus we have $x = (x_1, x_2, \dots)$, where each x_j is an element of some finite-dimensional real vector space. We fix a positive integer k. Here is a general construction which may naturally be called the recursion scheme of depth k. We assume throughout that the subscripts of all variables in the group x are positive (all occurring variables with non-positive subscripts are omitted).

Theorem 1. *Suppose that we are given an infinite system of polynomial relations of the form*

$$\begin{aligned}\Theta_j(x_1, \dots, x_j) = L_{j1}(x_j) + \dots + L_{jk}(x_{j-k+1})& \\ + \theta_j(x_{j-k}, x_{j-k-1}, \dots, x_1) = 0, &\qquad j = k, k+1, \dots, \end{aligned} \tag{1.1}$$

where $(L_{j1}, \dots, L_{jk})$ *are linear and*

$$L_j(x) = L_{j1}(x_j) + L_{j2}(x_{j-1}) + \dots + L_{jk}(x_{j-k+1}), \qquad L(x) = (L_1(x), L_2(x), \dots).$$

Assume that $\operatorname{Ker} L$ *is contained in a finite-dimensional subspace in* V *of the form* $\widetilde{V}_l = \{(x_1, \dots, x_l, 0, 0, \dots)\}$. *Then the number of parameters on which the general solution of* (1.1) *depends does not exceed the dimension of* $\operatorname{Ker} L$.

Proof. Let W be a complement to $\operatorname{Ker} L$ in V. It can be constructed in the following way. Choose an arbitrary complement $\widetilde{W}$ to $\operatorname{Ker} L$ in the finite-dimensional space $\widetilde{V}_l$ and take its sum with the subspace $V_l = \{(0, \dots, 0, x_{l+1}, x_{l+2}, \dots)\}$. Thus the only solution of $L(x) = 0$ in W is $x = 0$. Successively considering the equations $\Theta_j(x) = L_j(x) + \theta_j(x) = 0$, we see that this system also has at most one solution in W. Any vector $x' \in V$ is of the form $x + a$, where $x \in W$, $a \in \operatorname{Ker} L$. Consider the system with a fixed a. We obtain $\Theta_j(x+a) = L_j(x) + \theta_j(x+a) = 0$. It follows that this system also has at most one solution. Thus the set of solutions of (1.1) is parametrized by some subset of $\operatorname{Ker} L$. □

Let Γ be a real-analytic hypersurface in $\mathbf{C}^4$ and let Γ_ξ be its germ at a point ξ. We write $\operatorname{aut}\Gamma_\xi$ for the Lie algebra of the germs at ξ of real-analytic vector fields that are tangent to Γ_ξ. If Γ is Levi non-degenerate outside a proper real-analytic subset, then the standard approach (reduction of depth one) yields the standard bound: $\dim \operatorname{aut} \Gamma_\xi$ does not exceed the dimension of the automorphism algebra of the non-degenerate tangent hyperquadric. The latter dimension is equal to 24 independently of the signature.

A bound can be obtained only when Γ is holomorphically non-degenerate. The holomorphic non-degeneracy of a hypersurface in $\mathbf{C}^4$ is equivalent to its l-non-degeneracy outside a proper analytic subset, where $l \leqslant 3$ (see [5]). Thus, to obtain a general bound, we need to consider two different cases: 1) Γ is uniformly 2-nondegenerate in a neighbourhood of ξ; 2) Γ is uniformly 3-nondegenerate in a neighbourhood of ξ.

There are many examples of uniformly 2-nondegenerate hypersurfaces in $\mathbf{C}^4$. For example, they are contained in the familiar paper of Fels and Kaup [6]. In particular, these authors describe tube hypersurfaces over real cones and their holomorphic automorphisms. In $\mathbf{C}^4$, the automorphism groups of such cones are of dimension 15. We know two examples of uniformly 3-nondegenerate hypersurfaces in $\mathbf{C}^4$. They were given by Kaup [6] and Santi [7] respectively. Both are holomorphically homogeneous and the dimension of the automorphism algebra in Santi's example is equal to 8.

We successively consider the cases of 3-nondegeneracy and 2-nondegeneracy. Note that a single use of homological operators of depth greater than one is insufficient for obtaining the desired bound. We use a two-step procedure to study 2-nondegenerate hypersurfaces. After performing a recursive procedure with some weight, we change the weight and analyze the kernel of the old homological operator from the point of view of the new weight recursion. This yields a new homological operator. Analysis of a special case (of big kernels) requires another change of weight and a new recursion.

As a result, we obtain two bounds for the dimension: 17 (resp. 20) in the 2-nondegenerate (resp. 3-nondegenerate) case. Our technique actually yields the bound 18 in the former case, but using a recent result of Sykes and Zelenko [8] enables us to improve it to 17.

§ 2. 3-nondegenerate hypersurfaces

We denote the coordinates in $\mathbf{C}^4$ by $(z, \zeta, \eta, w = u + iv)$. Suppose that Γ is *uniformly* 3-*nondegenerate* in a neighbourhood of $\xi \in \Gamma$. Since our aim is to obtain bounds for the dimension of the automorphism group, we can restrict ourselves to the so-called rigid hypersurfaces. Indeed, if the automorphism algebra contains a field transversal to the complex tangent, its local straightening enables us to assume that the group contains all shifts along the u-axis, therefore, the local equation of Γ takes the form

$$v = F(z, \overline{z}, \zeta, \overline{\zeta}, \eta, \overline{\eta}),$$

where the right-hand side is independent of u. By 3-nondegeneracy, the rank of the Levi form at a generic point is equal to one. Hence we can write the local equation of Γ in the form $v = |z|^2 + F_3 + F_4 + \cdots$, where $F_j(z, \overline{z}, \zeta, \overline{\zeta}, \eta, \overline{\eta})$ is a homogeneous polynomial of degree j. Simple transformations enable us to remove all pluriharmonic terms from the right-hand side, as well as all terms which are linear in z and $\overline{z}$ except for $|z|^2$.

It is necessary for uniform 3-nondegeneracy that the rank of the complex Hessian of $F(z,\overline{z},\zeta,\overline{\zeta},\eta,\overline{\eta})$ be at most one everywhere. Since $F_{z\overline{z}}$ is equal to one at the origin, we can represent this condition in the form of vanishing of three minors of order two. Namely,

$$\begin{aligned}\delta_1(F) &= F_{z\overline{z}}F_{\zeta\overline{\zeta}} - |F_{z\overline{\zeta}}|^2 = 0,\\ \delta_2(F) &= F_{z\overline{z}}F_{\zeta\overline{\eta}} - F_{\zeta\overline{z}}F_{z\overline{\eta}} = 0,\\ \delta_3(F) &= F_{z\overline{z}}F_{\eta\overline{\eta}} - |F_{z\overline{\eta}}|^2 = 0.\end{aligned} \tag{2.1}$$

Lemma 1. *If* Γ *is uniformly* 3*-nondegenerate*, *then a polynomial change of variables enables us to write the equation of* Γ *in the form*

$$v = |z|^2 + F_3 + F_4 + F_5 + F_6 + O(7), \tag{2.2}$$

where

$$\begin{aligned}&F_3 = 2\operatorname{Re}(z^2\overline{\zeta}), \qquad F_4 = 2\operatorname{Re}(z^3\overline{\eta}) + 4|z|^2|\zeta|^2,\\ &F_5 = 2\operatorname{Re}(r_1\overline{z}^4\zeta + r_2\overline{z}^4\eta + r_3 z\overline{\zeta}^2\overline{\eta}^2 + r_4 z\overline{\eta}^4 + 4z^2\zeta\overline{\zeta}^2 + 6z^2\overline{z}\,\zeta\overline{\eta}),\\ &F_6 = 2\operatorname{Re}(8\overline{r}_1 z^3\overline{z}\zeta\overline{\zeta} + 8\overline{r}_2 z^3\overline{z}\,\zeta\overline{\eta} + 2r_3 z\eta^2\overline{\zeta}\zeta^2 + 2r_4 z\eta^4\overline{\zeta} + s_1 z\overline{z}^4\zeta + s_2\overline{z}^5\zeta\\ &\qquad + s_3 z\overline{z}^4\eta + s_4\overline{z}^5\eta + s_5\overline{z}^4\zeta^2 + s_6\overline{z}^4\zeta\eta + s_7\overline{z}^4\eta^2 + s_8\overline{z}\,\eta^5 + 12\overline{z}^3\zeta\overline{\zeta}\,\eta\\ &\qquad + 12z\overline{z}^2\zeta^2\overline{\eta}) + 16|z|^2|\zeta|^4 + 9|z|^4|\eta|^2.\end{aligned}$$

Proof. We give a general formula for F_3 after our simplification. Distinguishing the component of degree one in (2.1), we obtain $F_3 = 2\operatorname{Re}(a_1\zeta + a_2\eta)\overline{z}^2$. Since F_3 is not identically equal to zero, $(a_1,a_2)\neq 0$ and we can take $a_1\zeta + a_2\eta$ for a new variable ζ. Then a simple transformation enables us to remove all summands of the form $2\operatorname{Re}(A(z,\zeta,\eta)\overline{z}^2)$ with a holomorphic coefficient A from all subsequent components of F.

We give a general formula for F_4 after our simplification. Distinguishing the component of degree two in (2.1), we obtain $F_4 = 2\operatorname{Re}\overline{z}^3(a_3\eta + a_4\zeta) + 4|z|^2|\zeta|^2$. The uniform 3-nondegeneracy implies that $a_3\neq 0$ and we can take $a_3\eta + a_4\zeta$ for a new variable η. Then a simple transformation enables us to remove all summands of the form $2\operatorname{Re}(B(z,\zeta,\eta)\overline{z}^3)$ with a holomorphic coefficient B from all subsequent components of F. Distinguishing the component of degree three, we obtain the formula for F_5. Then the component of degree four yields the formula for F_6. □

Consider a map of the germ at the origin of a hypersurface Γ of the form (2.2) to another such hypersurface $\widetilde{\Gamma}$. Denote the coordinates of the germ of this map at the origin by

$$\Phi = \big(z\to f(z,\zeta,\eta,w),\ \zeta\to g(z,\zeta,\eta,w),\ \eta\to h(z,\zeta,\eta,w),\ w\to e(z,\zeta,\eta,w)\big).$$

We assume that the hypersurfaces are fixed. Introduce a grading in the space of formal power series in $(z,\overline{z},\zeta,\overline{\zeta},\eta,\overline{\eta},u)$, as well as in $(z,\overline{z},\zeta,\overline{\zeta},\eta,\overline{\eta},w,\overline{w})$, by prescribing the following weights of the variables:

$$[z] = [\overline{z}] = [\zeta] = [\overline{\zeta}] = [\eta] = [\overline{\eta}] = 1, \qquad [w] = [\overline{w}] = [u] = 2.$$

The quadruple $(f_{\mu-1}, g_{\mu-2}, h_{\mu-3}, e_\mu)$ of weighted components is denoted by ϕ_μ (the μth weighted component of Φ). The following relation expresses the fact that Φ maps Γ to $\widetilde{\Gamma}$:

$$\begin{aligned}\Theta(z,\overline{z},\zeta,\overline{\zeta},u) &= -2\operatorname{Im} e(z,\zeta,\eta,w) + 2|f|^2 + 4\operatorname{Re}(f^2\overline{g}\,) + 4\operatorname{Re}(f^3\overline{h}\,) \\ &\quad + 8|f|^2|g|^2 + 2F_4(f,\overline{f},g,\overline{g},h,\overline{h}\,) + 2F_5(f,\overline{f},g,\overline{g},h,\overline{h}\,) \\ &\quad + 2F_6(f,\overline{f},g,\overline{g},h,\overline{h}\,) + \cdots = 0 \\ \text{for}\quad w &= u + i\bigl(|z|^2 + 2\operatorname{Re}(z^2\overline{\zeta}\,) + 2\operatorname{Re}(z^3\overline{\eta}\,) + 4|z|^2|\zeta|^2 + \cdots\bigr).\end{aligned} \tag{2.3}$$

Among all maps holomorphic at the origin, we distinguish the class of maps of the following form:

$$\mathcal{V}_5 = \bigl\{\Phi = \mathrm{Id} + \phi_5 + \cdots = \bigl(z + O(4),\, \zeta + O(3),\, \eta + O(2),\, w + O(5)\bigr)\bigr\}. \tag{2.4}$$

We estimate the dimension of the family of such maps from Γ to $\widetilde{\Gamma}$ by the multiple recursion scheme of depth $k = 3$ (see Theorem 1). To do this, we write Θ_μ for the μth weighted component in (2.3) and consider the terms in Θ_μ depending only on $(\phi_\mu, \phi_{\mu-1}, \phi_{\mu-2})$, that is, on

$$(e_\mu, e_{\mu-1}, e_{\mu-2}, f_{\mu-1}, f_{\mu-2}, f_{\mu-3}, g_{\mu-2}, g_{\mu-3}, g_{\mu-4}, h_{\mu-3}, h_{\mu-4}, h_{\mu-5}).$$

We put $\Delta_1\psi(u) = iF_3\psi'(u)$, $\Delta_2\psi(u) = iF_4\psi'(u)$. Successively distinguishing the terms of this form in all summands of the expression Θ (beginning with $-2\operatorname{Im} e$ and ending with F_6), we obtain the following result.

Lemma 2. *For all $\mu \geqslant 5$, the μth component of Θ is of the form*

$$\Theta_\mu = L_1(\phi_\mu) + L_2(\phi_{\mu-1}) + L_3(\phi_{\mu-2}) + \theta_\mu(\phi_{\nu<\mu-2}),$$

where $w = u + i|z|^2$,

$$\begin{aligned}
L_1(\phi) &= 2\operatorname{Re}(ie + 2\overline{z}f + 2\overline{z}^{\,2}g + 2\overline{z}^{\,3}h), \\
L_2(\phi) &= \Delta_1(L_1(\phi)) + l_2(\phi), \\
L_3(\phi) &= \Delta_1(L_2(\phi)) + \Delta_2(L_1(\phi)) + \Delta_1^2(L_1(\phi)) + l_3(\phi), \\
l_2(\phi) &= 2\operatorname{Re}\{4z\overline{\zeta}f + 8z\overline{z}\,\overline{\zeta}\eta g + (4\overline{z}^{\,4}r_2 + 4\overline{z}\,\zeta^2\eta r_3 + 8\overline{z}\,\eta^3 r_4 + 12\overline{\zeta}\overline{z}^{\,2}z)h\}, \\
l_3(\phi) &= 2\operatorname{Re}\{(8\overline{\zeta}\overline{z}\,\zeta + 6\overline{\eta}z^2)f + (4\overline{z}^{\,4}r_1 + 4\overline{z}\,\zeta^2\eta r_3 + 12\overline{\eta}\,\overline{z}\,z^2 + 8\overline{\zeta}^{\,2}z + 16\overline{\zeta}\overline{z}\,\zeta)g \\
&\qquad + (16\overline{\zeta}\overline{z}^{\,3}zr_2 + 8\overline{\zeta}\zeta^2\eta zr_3 + 16\overline{\zeta}\eta^3 zr_4 + 2\overline{z}^{\,5}s_4 + 2\overline{z}^{\,4}\zeta s_6 + 4\overline{z}^{\,4}\eta s_7 \\
&\qquad + 2\overline{z}^{\,4}zs_3 + 10\overline{z}\,\eta^4 s_8 + 24\overline{\zeta}^{\,2}cz^2z + 24\overline{\zeta}\overline{z}^{\,3}\zeta + 18\overline{\eta}\,\overline{z}^{\,2}z^2)h\}.
\end{aligned}$$

Note that the expression $L(\phi) = L_1(\phi) + L_2(\phi) + L_3(\phi)$ is linear with respect to ϕ and independent of μ.

Let V_5 be the vector space of the germs at the origin of formal power series of the form

$$\Phi = \phi_5 + \phi_6 + \cdots = (f_4 + f_5 + \cdots,\; g_3 + g_4 + \cdots,\; h_2 + h_3 + \cdots,\; e_5 + e_6 + \cdots).$$

By Theorem 1, the dimension of the family of all maps from Γ to $\widetilde{\Gamma}$ in $\mathcal{V}_5$ does not exceed the dimension of the kernel of L on V_5.

We proceed to estimate the dimension of the kernel of L. This kernel is the space of solutions of the relation

$$L(\phi)=0, \quad \text{where} \quad \phi\in V_5. \tag{2.5}$$

We put

$$f(0,0,0,u)=a(u),\quad g(0,0,0,u)=b(u),\quad h(0,0,0,u)=c(u),\quad e(0,0,0,u)=d(u).$$

Letting $\overline{z}=\overline{\zeta}=\overline{\eta}=0$ in (2.5), we obtain a relation which yields immediately that

$$\begin{aligned} e(z,\zeta,\eta,u) &= d(u)+2iz\overline{a}\,(u)+2iz^2\overline{b}\,(u)+2iz^3\overline{c}\,(u)\\ &\qquad + 4iz^4(\overline{r}_2\overline{c}\,(u)+\overline{r}_1\overline{b}\,(u))+2i\overline{s}_4z^5\overline{c}\,(u), \end{aligned} \tag{2.6}$$

where $d(u)$ is real.

We put

$$\begin{aligned} &f'_z(0,0,0,u)=k_1(u), && g'_z(0,0,0,u)=k_2(u), && h'_z(0,0,0,u)=k_3(u),\\ &f'_\zeta(0,0,0,u)=m_1(u), && g'_\zeta(0,0,0,u)=m_2(u), && h'_\zeta(0,0,0,u)=m_3(u),\\ &f'_\eta(0,0,0,u)=n_1(u), && g'_\eta(0,0,0,u)=n_2(u), && h'_\eta(0,0,0,u)=n_3(u). \end{aligned}$$

Substitute the resulting value of e into L (preserving the notation L). Letting $\overline{z}=\overline{\zeta}=\overline{\eta}=0$ in $L_{\overline{z}}$, $L_{\overline{\zeta}}$, $L_{\overline{\eta}}$, we obtain

$$\begin{aligned} &2z^2\overline{k}_2(u)+2z^3\overline{k}_3(u)+10\eta^4s_8h(z,\zeta,\eta,u)+24\overline{c}\,(u)\zeta^2z^2+2z^5\overline{s}_4\overline{k}_3(u)\\ &\quad+8h(z,\zeta,\eta,u)\eta^3\overline{r}_4+4z^4\overline{r}_2\overline{k}_3(u)+2\overline{c}\,(u)z^4\overline{s}_3+4z^4\overline{r}_1\overline{k}_2(u)-2a(u)\\ &\quad+2z\overline{k}_1(u)+8\zeta^2\overline{b}\,(u)+4\zeta\overline{a}\,(u)-2d'(u)z+2f(z,\zeta,\eta,u)+12\zeta z^2\overline{c}\,(u)\\ &\quad-4iz^2a'(u)-(4i)z^4\overline{c}\,'(u)-4iz^3\overline{b}\,'(u)+4h(z,\zeta,\eta,u)\zeta^2\eta r_3+16\overline{c}\,(u)\zeta z^3\overline{r}_2\\ &\quad-8iz^5\overline{r}_2\overline{c}\,'(u)-8i\overline{r}_1z^5\overline{b}\,'(u)-4i\overline{s}_4z^6\overline{c}\,'(u)+4\zeta^2\eta r_3g(z,\zeta,\eta,u)=0,\\ &16h(z,\zeta,\eta,u)\eta^3zr_4-4iz^5\overline{c}\,'(u)-2iz^7\overline{s}_4c'(u)-4iz^6\overline{r}_1\overline{b}\,'(u)+2z^2\overline{m}_2(u)\\ &\quad+2z^3\overline{m}_3(u)+2z\overline{m}_1(u)-2z^2e'(u)+4zf(z,\zeta,\eta,u)+8h(z,\zeta,\eta,u)\zeta^2\eta zr_3\\ &\quad+2\overline{c}\,(u)z^4\overline{s}_6+24\overline{c}\,(u)\zeta z^3-4iz^4\overline{b}\,'(u)-4iz^3a'(u)-8iz^6\overline{r}_2\overline{c}\,'(u)\\ &\quad+4z^4\overline{r}_2\overline{m}_3(u)+2z^5\overline{s}_4\overline{m}_3(u)+8\zeta z\overline{a}\,(u)+16\zeta z\overline{b}\,(u)+4z^4\overline{r}_1\overline{m}_2(u)=0,\\ &4z^4\overline{r}_1\overline{n}_2(u)+4z^4\overline{s}_7\overline{c}\,(u)+2z^5\overline{s}_4\overline{n}_3(u)+6z^2f(z,\zeta,\eta,u)+4z^4\overline{r}_2\overline{n}_3(u)\\ &\quad-4iz^5\overline{b}\,'(u)-4iz^4a'(u)-4iz^7\overline{r}_2\overline{c}\,(u)-2z^3e'(u)-4iz^7\overline{r}_1\overline{b}\,'(u)\\ &\quad-2iz^8\overline{s}_4\overline{c}\,'(u)-4iz^6\overline{c}\,'(u)+2z\overline{n}_1(u)+2z^2\overline{n}_2(u)+2z^3\overline{n}_3(u)=0. \end{aligned} \tag{2.7}$$

It follows from the third relation in (2.7) that $n_1(u)=0$ and

$$\begin{aligned} 3f(z,\zeta,\eta,u) &= -\overline{n}_2(u)+z\big(e'(u)-\overline{n}_3(u)\big)\\ &\quad+2z^2\big(-\overline{r}_1\overline{n}_2(u)-\overline{s}_7\overline{c}\,(u)-\overline{r}_2\overline{n}_3(u)+i\overline{a}\,'(u)\big)+z^3\big(2i\overline{b}\,'(u)-\overline{s}_4\overline{n}_3(u)\big)\\ &\quad+2iz^4\overline{c}\,'(u)+2iz^5\big(\overline{r}_2\overline{c}\,'(u)+\overline{r}_1\overline{b}\,'(u)\big)+iz^6\overline{s}_4\overline{c}\,'(u). \end{aligned} \tag{2.8}$$

Substituting the resulting value of f into the first and second relations in (2.7) and distinguishing the leading component with respect to η, we obtain

$$\begin{gathered} 2r_3\zeta^2 g(z,\zeta,\eta,u) + (2r_3\zeta^2 + 5s_8\eta^2)h(z,\zeta,\eta,u) = 0, \\ (r_3\zeta^2 + 2r_4\eta^2)h(z,\zeta,\eta,u) = 0. \end{gathered} \tag{2.9}$$

If (2.9) holds, then the other conditions guaranteeing (2.7) take the form of the following system:

$$\begin{gathered} a = b = c = n_1 = n_2 = k_3 = 0, \\ d' = -2\operatorname{Re} n_3, \qquad k_2 = \frac{2}{3}\, r_2 n_3, \qquad m_2 = \frac{1}{3}(n_3 - \overline{n}_3), \qquad m_3 = \frac{4}{3}\, r_2 n_3, \\ r_1 r_2 n_3 = 0, \qquad s_4 n_3 = 0, \qquad (r_1 - s_4 + 4r_2^2)n_3 = r_1 \overline{n}_3. \end{gathered} \tag{2.10}$$

In this case we have

$$e(z,\zeta,\eta,u) = d(u), \qquad f(z,\zeta,\eta,u) = \frac{d'(u) - \overline{n}_3(u)}{3} z - \frac{2}{3}\,\overline{r}_2\overline{n}_3(u) z^2 - \frac{\overline{s}_4\overline{n}_3(u)}{3} z^4.$$

Consider two cases.

1. Suppose that $r_3 \neq 0$. Then it follows directly from (2.9) that $g = h = 0$. The remaining relations enable us to conclude that $f = 0$ and d is a real constant. Thus, in this case we have $\operatorname{Ker} L = V^0 = \{(0,0,0,d_0)\}$ and $d_0 \in \mathbf{R}$.

2. Suppose that $r_3 = 0$. If $(r_4, s_8) \neq 0$, then it follows from (2.9) that $h = 0$. Considering again (2.7), we see that $n_3 = n_2 = d' = 0$. Hence $f = 0$ and e is a real constant. We denote $g''_{zz}(0,0,0,u)$ by $k_{22}(u)$. Calculating $L''_{\overline{z}\,\overline{z}}$ with $\overline{z} = \overline{\zeta} = \overline{\eta} = 0$, we obtain

$$g(z,\zeta,\eta,u) = 2iz^5\overline{r}_1\overline{k}_2'(u) - z^4\overline{r}_1\overline{k}_{22}(u) + iz^3\overline{k}_2'(u) - \frac{1}{2}z^2\overline{k}_{22}(u) - 4\zeta^2\overline{k}_2(u).$$

Substituting this value of g into L and equating the coefficients of $z^2\overline{\zeta}^2$ and $z^2\overline{z}\,|\zeta|^2$ to zero, we obtain $k_2 = k_{22} = 0$, that is, $g = 0$. Hence $\operatorname{Ker} L = V^0$.

Suppose that $r_4 = s_8 = 0$. Put

$$h''_{zz}(0,0,0,u) = k_{32}(u), \qquad h'''_{zzz}(0,0,0,u) = k_{33}(u).$$

Calculating $L''_{\overline{z}\,\overline{z}}$ with $\overline{z} = \overline{\zeta} = \overline{\eta} = 0$ as above, we obtain an expression for g. Calculating $L'''_{\overline{z}\,\overline{z}\,\overline{z}}$ with $\overline{z} = \overline{\zeta} = \overline{\eta} = 0$, we obtain an expression for h. Then an analysis of the non-leading coefficients of L yields that $n_3 = d' = k_2 = k_{22} = k_{32} = k_{33} = 0$. It follows that $f = g = h = 0$ and e is a real constant. Hence $\operatorname{Ker} L = V^0$.

This proves the following Lemma 3.

Lemma 3. *If $L(\phi) = 0$, then $\phi = (0,0,0,d_0)$, where d_0 is a real constant. In particular, the kernel in V_5 is trivial.*

We now explain the structure of lower-order jets of any map from Γ to $\widetilde{\Gamma}$ that leaves the origin fixed. Distinguishing the components of weights one and two in (2.3), we immediately obtain

$$e_1 = 0, \qquad e_2 = |\lambda|^2 w, \qquad f_1 = \lambda z, \qquad \lambda \in \mathbf{C}^*.$$

We write

$$e_3 = |\lambda|^2(d_3 + d_1 w), \qquad f_2 = \lambda(aw + a_2), \qquad g_1 = b_1,$$

where d_3, d_1, a_2, b_1 are homogeneous holomorphic forms of the corresponding degrees in (z, ζ, η) and a is a constant. The third weighted component in (2.3) is of the form

$$\begin{aligned} &|\lambda|^2 \operatorname{Im}\bigl[i2\operatorname{Re}\bigl(z^2\overline{\zeta} + d_1(z,\zeta,\eta)(u + i|z|^2)\bigr)\bigr] \\ &\quad = |\lambda|^2 2\operatorname{Re}[(a(u + i|z|^2) + a_2(z,\zeta,\eta))\overline{z}\,] + 2\operatorname{Re}[\lambda^2 z^2 \overline{b}_1(\overline{z}, \overline{\zeta}, \overline{\eta}\,)]. \end{aligned}$$

Distinguishing the terms which are linear in u, we obtain $d_1 = 2i\overline{a}\, z$. Then consider the terms of bidegree $(2,1)$ and separate the components which are linear in $\overline{z}$, in $\overline{\zeta}$, and in $\overline{\eta}$. We obtain

$$a_2(z,\zeta,\eta) = \left(2i\overline{a} - \frac{\lambda}{\overline{\lambda}}\,\overline{b}_1^{\,1}\right) z^2, \qquad b_1^2 = \frac{\lambda}{\overline{\lambda}}, \quad b_1^3 = 0,$$

where $b_1 = b_1^1 z + b_1^2\zeta + b_1^3\eta$. Here and in what follows, the superscripts indicate the relation between the coefficients and the variables. Writing $b_1^1 = \alpha\lambda/\overline{\lambda}$, we obtain

$$e_3 = 2i|\lambda|^2\overline{a}\, zw, \qquad f_2 = \lambda\bigl(aw + (2i\overline{a} - \overline{\alpha}\,)z^2\bigr), \qquad g_1 = \frac{\lambda}{\overline{\lambda}}(\alpha z + \zeta).$$

Furthermore, we write

$$\begin{aligned} e_4 = |\lambda|^2(d_4 + d_2 w + d_0 w^2), &\qquad f_3 = \lambda(a_3 + a_1 w), \\ g_2 = \frac{\lambda}{\overline{\lambda}}(b_2 + b_0 w), &\qquad h_1 = \frac{\lambda}{\overline{\lambda}^2} c_1, \end{aligned}$$

where the coefficients are holomorphic homogeneous forms of the corresponding degrees in (z, ζ, η). Consider the component of weight four in (2.3), removing the common factor $|\lambda|^2$:

$$\begin{aligned} &\operatorname{Im}[d_4(z,\zeta,\eta) + d_2(z,\zeta,\eta)(u + i|z|^2) + d_0(u^2 + 2i|z|^2 u - |z|^4) \\ &\qquad + 2\overline{a}z(z^2\overline{\zeta} + \overline{z}^{\,2}\zeta) + i(z^3\overline{\eta} + \overline{z}^{\,3}\eta) + 4|z|^2|\zeta|^2] \\ &\quad = 2\operatorname{Re}[(a_3(z,\zeta,\eta) + a_1(z,\zeta,\eta)(u + i|z|^2))\overline{z}\,] + |a|^2(u^2 + |z|^4) \\ &\qquad - 2\operatorname{Re}[(a(u + i|z|^2)(2ia + \alpha)\overline{z}^{\,2})] + |2ia + \alpha|^2|z|^4 + 2\operatorname{Re}\bigl[a\bigl(2i\operatorname{Re}(z^2\overline{\zeta}\,)\bigr)\bigr] \\ &\qquad + 2\operatorname{Re}\bigl[2\bigl(a(u + i|z|^2) + (2i\overline{a} - \overline{\alpha}\,)z^2\bigr)z(\overline{\alpha}\,\overline{z} + \overline{\zeta}\,)\bigr] \\ &\qquad + 2\operatorname{Re}\bigl[\bigl(b_2(z,\zeta,\eta) + b_0(u + i|z|^2)\bigr)\overline{z}^{\,2}\bigr] + 2\operatorname{Re}[c_1(z,\zeta,\eta)\overline{z}^{\,3}] + 4|z|^2|\alpha z + \zeta|^2. \end{aligned} \tag{2.11}$$

Separating the coefficient in front of u^2 in (2.11), we obtain $\operatorname{Im} d_0 = |a|^2$. Put $d_0 = \gamma + i|a|^2$. Separating the coefficient in front of u in (2.11), we obtain

$$\begin{aligned} \operatorname{Im}[d_2(z,\zeta,\eta)] + |a|^2|z|^2 &= 2\operatorname{Re}[a_1(z,\zeta,\eta)\overline{z}\,] - 2\operatorname{Re}[a(2ia + \alpha)\overline{z}^{\,2}] \\ &\qquad + 2\operatorname{Re}[az(\overline{\alpha}\,\overline{z} + \overline{\zeta}\,)] + 2\operatorname{Re}[b_0\overline{z}^{\,2}]. \end{aligned}$$

It follows that

$$d_2 = (2i\overline{a}^2 - \overline{a}\,\overline{\alpha} + \overline{\beta}\,)z^2, \qquad a_1 = \big((|a|^2 - 2\,\mathrm{Re}(a\overline{\alpha}\,)) + i\delta\big)z + \overline{a}\,\zeta,$$

where $\beta = b_0$, $\delta = \mathrm{Im}\,a_1^1$. The component of bidegree $(4,0)$ yields at once that $d_4 = 0$. In bidegree $(3,1)$ we obtain

$$\begin{aligned} id_2(z,\zeta,\eta)|z|^2 - \overline{a}\,z^3\overline{\zeta} + z^3\overline{\zeta} &= a_3(z,\zeta,\eta)\overline{z} \\ &\quad + i\overline{a}(-2i\overline{a} + \overline{\alpha}\,)z^3\overline{z} - i\overline{\beta}\,z^3\overline{z} + z^3\overline{c_1(z,\zeta,\eta)}. \end{aligned}$$

Hence

$$a_3 = (-4\overline{a}^2 - 2i\overline{a}\,\overline{\alpha} + 2\overline{\beta} - \overline{\nu}\,)z^3, \qquad c_1 = \nu z - a\zeta + \eta,$$

where $\nu = c_1^1$. In bidegree $(2,2)$ we have

$$\begin{aligned} &2\,\mathrm{Re}[-i\overline{a}\,z\zeta\overline{z}^{\,2} + ia_1 z\overline{z}^{\,2} + 2iaz^2\overline{z}\,(\overline{\alpha}\,\overline{z} + \overline{\zeta}\,) + b_2\overline{z}^{\,2} + 4\alpha z^2\overline{z}\,\overline{\zeta}\,] \\ &\quad + (|a|^2 + 4|\alpha|^2 + |2ia + \alpha|^2)|z|^4 = 0. \end{aligned}$$

We further obtain

$$\begin{aligned} b_2 &= \left(\delta - 2|\alpha|^2 - \frac{|a|^2}{2} - \frac{1}{2}\,|2ia + \alpha|^2 - 2\,\mathrm{Re}(ia\overline{\alpha} + i\kappa)\right)z^2 + (ia + 2\alpha)z\zeta, \\ c_1 &= \nu z - a\zeta + \eta, \end{aligned}$$

where $\kappa = \mathrm{Im}\,b_2^1$. This proves the following lemma.

Lemma 4. a) *Every origin-preserving locally invertible map from* Γ *to* $\widetilde{\Gamma}$ *can be represented as the composite of maps of the form*

$$\begin{aligned} z &\to \lambda\big(z + aw + (2i\overline{a} - \overline{\alpha}\,)z^2 + (-4\overline{a}^2 + 2i\overline{a}\,\overline{\alpha} + 2\overline{\beta} - \overline{\nu}\,)z^3 \\ &\qquad + \big((|a|^2 - 2\,\mathrm{Re}(a\overline{\alpha}\,) + i\delta)z + \overline{a}\zeta\big)w\big), \\ \zeta &\to \frac{\lambda}{\overline{\lambda}}\big(\zeta + \alpha z + \tau z^2 + (ia + 2\alpha)z\zeta + \beta w\big), \\ \eta &\to \frac{\lambda}{\overline{\lambda}^2}(\eta + \nu z - a\zeta), \\ w &\to |\lambda|^2\big(w + 2i\overline{a}\,zw + (2i\overline{a}^{\,2} - \overline{a}\,\overline{\alpha} + \overline{\beta}\,)z^2 w + (\gamma + i|a|^2)w^2\big), \\ \tau &= \left(\delta - 2|\alpha|^2 - \frac{|a|^2}{2} - \frac{1}{2}\,|2ia + \alpha|^2 - 2\,\mathrm{Re}(ia\overline{\alpha} + i\kappa)\right), \end{aligned}$$

and the map

$$z \to z + O(4), \quad \zeta \to \zeta + O(3), \quad \eta \to \eta + O(2), \quad w \to w + O(5).$$

b) *Moreover,*

$$\lambda \in \mathbf{C}^*, \quad a, \alpha, \beta, \nu \in \mathbf{C}, \quad \gamma, \delta, \kappa \in \mathbf{R}.$$

This yields 13 *real parameters.*

c) *Such a map is uniquely determined by its* 3*-jet at the origin.*

We are now ready to prove the following statement.

Statement 1. *If* Γ *is a real-analytic hypersurface in* $\mathbf{C}^4$ *and it is* 3*-nondegenerate at a generic point, then the dimension of the pseudogroup of local holomorphic automorphisms at every point does not exceed* 20.

Proof. The dimension of the group at an arbitrary point does not exceed the sum of the dimension of the hypersurface and the dimension of the stabilizer at a generic point. The dimension of the hypersurface is equal to 7. The dimension of the stabilizer does not exceed 13 by Theorem 1 and Lemmas 1–4. This proves the statement since $7+13=20$. □

§ 3. General 2-nondegenerate hypersurfaces

We denote the coordinates in $\mathbf{C}^4$ by $(z=(z_1,z_2),\ \zeta,\ w=u+iv)$ and proceed to consider the 2-nondegenerate case. As above, we can restrict ourselves to rigid hypersurfaces. Let Γ be uniformly 2-nondegenerate in a neighbourhood of $\xi\in\Gamma$. The Levi form of a uniformly 2-nondegenerate hypersurface has everywhere minimal degeneration. Namely, its rank is equal to 2. Thus we can write the local equation of Γ in the form

$$v=\langle z,\overline{z}\,\rangle+F_3(z,\overline{z},\zeta,\overline{\zeta}\,)+F_4(z,\overline{z},\zeta,\overline{\zeta}\,)+\cdots, \tag{3.1}$$

where F_j is a homogeneous real polynomial of degree j and $\langle z,\overline{z}\,\rangle$ is a non-degenerate Hermitian form of the variable $z\in\mathbf{C}^2$. Simple triangular-polynomial changes of the variables z and w enable us to assume that the right-hand side

$$F=\langle z,\overline{z}\,\rangle+F_3+F_4+\cdots$$

of the equation of Γ contains no pluriharmonic summands (that is, summands of degrees $(m,0)$ and $(0,m)$) and no summands which are linear in z and $\overline{z}$ except for $\langle z,\overline{z}\rangle$. Here are the remaining summands in F_3 and F_4:

$$\begin{aligned}
F_3&=2\operatorname{Re}\bigl(K(z,z)\overline{\zeta}+A_1(z)|\zeta|^2+A_2\zeta^2\overline{\zeta}\,\bigr),\\
F_4&=2\operatorname{Re}\bigl((P(z,z,\overline{z}\,)+Q(z,z,z))\overline{\zeta}+R(z,z)\overline{\zeta}^{\,2}\bigr)+S(z,\overline{z}\,)|\zeta|^2+T(z,z,\overline{z},\overline{z}\,)\\
&\qquad+2\operatorname{Re}\bigl(B_1(z,z)|\zeta|^2+B_2(z)\zeta^2\overline{\zeta}+B_3(z)\zeta\overline{\zeta}^{\,2}\bigr).
\end{aligned}\tag{3.2}$$

It follows from the condition of 2-nondegeneracy of Γ that the form $K(z,z)$ is not identically equal to zero. In what follows we shall need to reduce the pair of forms $(\langle z,\overline{z}\rangle, K(z,z))$ on $\mathbf{C}^2$ by complex-linear changes of coordinates to an expression with minimum number of parameters. The following classification holds (see also [9]).

Lemma 5. *Suppose that* $\langle z,\overline{z}\,\rangle$ *is nondegenerate and* $K(z,z)$ *is not identically equal to zero. Then there is a complex-linear transformation sending this pair to a pair in the following list*:

1*) $(|z_1|^2+|z_2|^2,\ kz_1^2+mz_2^2)$, $k,m>0$, $k\neq m$;
2*) $(|z_1|^2+|z_2|^2,\ k(z_1^2+z_2^2))$, $k>0$;
3*) $(|z_1|^2+|z_2|^2,\ kz_1^2)$, $k>0$;

4*) $(|z_1|^2 - |z_2|^2,\ kz_1^2 + mz_2^2)$, $k, m > 0$, $k \neq m$;
5*) $(|z_1|^2 - |z_2|^2,\ k(z_1^2 + z_2^2))$, $k > 0$;
6*) $(|z_1|^2 - |z_2|^2,\ kz_1^2)$, $k > 0$;
7*) $(2\,\mathrm{Re}(z_1\overline{z}_2),\ z_1^2 + mz_2^2)$, $m \notin \mathbf{R}$;
8*) $(2\,\mathrm{Re}(z_1\overline{z}_2),\ z_1^2 + mz_2^2)$, $m \in \mathbf{R}^*$;
9*) $(2\,\mathrm{Re}(z_1\overline{z}_2),\ z_1^2)$.

Proof. Suppose that $\langle z, \overline{z}\rangle$ is positive definite and ν is an eigenvector of the operator given by the matrix

$$\begin{bmatrix} k & l \\ l & m \end{bmatrix}.$$

We take

$$\frac{\nu}{\sqrt{\langle \nu, \overline{\nu}\,\rangle}}$$

for the first vector of a new basis and find the second vector to make this basis orthonormal. This yields the pairs 1*)–3*) for various values of the rank of K. The parameters k and m can be made positive by rotations in the planes z_1 and z_2.

Suppose that $\langle z, \overline{z}\,\rangle$ is of signature $(1, 1)$. If the operator has an eigenvector ν with $\langle \nu, \overline{\nu}\,\rangle \neq 0$, then the same argument applies and yields the pairs 4*)–6*).

Let (e_1, e_2) be a basis of $\mathbf{C}^2$ which diagonalizes $K(z, z)$, that is, $K(z, z) = kz_1^2 + mz_2^2$, and $\langle e_1, \overline{e}_1\rangle = \langle e_2, \overline{e}_2\rangle = 0$. Writing $z = z_1e_1 + z_2e_2$, we have

$$\langle z, \overline{z}\,\rangle = 2\,\mathrm{Re}(\langle e_1, \overline{e}_2\rangle z_1\overline{z}_2).$$

After a dilation with respect to z_1, the Hermitian form can be written as $\langle z, \overline{z}\,\rangle = 2\,\mathrm{Re}(z_1\overline{z}_2)$. Using the transformation

$$z_1 \to \lambda z_1, \quad z_2 \to \frac{z_2}{\overline{\lambda}}, \qquad \lambda \in \mathbf{C}^*,$$

which does not change the Hermitian form, we obtain the pairs 7*)–9*). □

Lemma 6. *If* Γ *is given by* (3.1) *and the Levi form of* Γ *is identically degenerate, then* F_3 *and* F_4 *can be written in the form* (3.2), *where* $A_1 = A_2 = B_1 = B_2 = B_3 = 0$ *and* S *is of the following form* (*depending on the number of the pair in Lemma* 5):
1*) $S = 4(k^2|z_1|^2 + m^2|z_2|^2)$;
2*) $S = 4k^2(|z_1|^2 + |z_2|^2)$;
3*) $S = 4k^2|z_1|^2$;
4*) $S = 4(k^2|z_1|^2 - m^2|z_2|^2)$;
5*) $S = 4k^2(|z_1|^2 - |z_2|^2)$;
6*) $S = 4k^2|z_1|^2$;
7*) $S = 4(\overline{m}\, z_1\overline{z}_2 + mz_2\overline{z}_1)$;
8*) $S = 4m(z_1\overline{z}_2 + z_2\overline{z}_1)$;
9*) $S = 0$.

Proof. Calculating the determinant of the complex Hessian matrix with respect to the variables (z_1, z_2, ζ) and separating the components of degree one, we see that $A_1 = A_2 = 0$. Further separation of the component of degree two yields that $B_1 = B_2 = B_3 = 0$ and S is of the form stated. □

We can now write

$$F_3 = 2\,\mathrm{Re}[K(z,z)\overline{\zeta}\,],$$
$$F_4 = 2\,\mathrm{Re}\big[(P(z,z,\overline{z}\,) + Q(z,z,z))\overline{\zeta} + R(z,z)\overline{\zeta}^{\,2}\big] + S(z,\overline{z}\,)|\zeta|^2 + T(z,z,\overline{z},\overline{z}\,).$$

Hence the equation of the hypersurface takes the form

$$\begin{aligned} v = \langle z,\overline{z}\,\rangle + 2\,\mathrm{Re}[K(z,z)\overline{\zeta}\,] + 2\,\mathrm{Re}\big[(P(z,z,\overline{z}\,) + Q(z,z,z))\overline{\zeta} + R(z,z)\overline{\zeta}^{\,2}\big] \\ + S(z,\overline{z}\,)|\zeta|^2 + T(z,z,\overline{z},\overline{z}\,) + O(5). \end{aligned} \tag{3.3}$$

We introduce a grading on the space of power series in $(z,\overline{z},\zeta,\overline{\zeta},u)$, as well as in $(z,\overline{z},\zeta,\overline{\zeta},w,\overline{w}\,)$, by prescribing the following weights of the variables

$$[z] = [\overline{z}\,] = [\zeta] = [\overline{\zeta}\,] = 1, \qquad [w] = [\overline{w}\,] = [u] = 2.$$

Let Γ and $\widetilde{\Gamma}$ be hypersurfaces with equations

$$\begin{aligned} v &= \langle z,\overline{z}\,\rangle + 2\,\mathrm{Re}(K(z,z)\overline{\zeta}\,) + O(4), \\ v &= \langle z,\overline{z}\,\rangle + 2\,\mathrm{Re}(\widetilde{K}(z,z)\overline{\zeta}\,) + O(4), \end{aligned} \tag{3.4}$$

and let

$$\phi = \big(z \to f = f_1 + f_2 + O(3),\, \zeta \to g = g_1 + O(2),\, w \to h = h_1 + h_2 + h_3 + O(4)\big)$$

be a locally invertible origin-preserving holomorphic map of Γ to $\widetilde{\Gamma}$. Here the components of the coordinates of the map are the components of a given weight and $O(j)$ stands for a sum of terms of weight at least j.

The following relation is an analytic expression of the condition that this map sends Γ to $\widetilde{\Gamma}$:

$$\begin{aligned} \Theta = -2\,\mathrm{Im}\,h + 2\langle f,\overline{f}\,\rangle + 4\,\mathrm{Re}(\widetilde{K}(f,f)\overline{g}\,) + \widetilde{F}_4 + \cdots = 0 \\ \text{for } w = u + i(\langle z,\overline{z}\,\rangle + 2\,\mathrm{Re}(K(z,z)\overline{\zeta}\,) + F_4 + O(5)). \end{aligned} \tag{3.5}$$

Separating the component of weight 1, we obtain $h_1 = A(z) + B\zeta = 0$.

Write $h_2 = \Phi_2(z,\zeta) + \rho w$ and $f_1 = Cz + d\zeta$, where Φ_2 is a form of degree two in (z,ζ). Separating the component of weight 2 in (3.5), we obtain $\Phi_2(z,\zeta) = 0$ and $\langle Cz,\overline{Cz}\,\rangle = \rho\langle z,\overline{z}\rangle$, that is, $f_1 = Cz$, $h_2 = \rho w$. Note that the matrix C is non-degenerate and $\rho \neq 0$ since the map is invertible.

We now write

$$h_3 = \rho\big(\Phi_3 + (A(z) + B\zeta)w\big), \qquad f_2 = C\big(aw + b(z,z) + c(z)\zeta + d\zeta^2\big),$$
$$g_1 = \langle z,\overline{\alpha}\,\rangle + \beta\zeta,$$

where Φ_3 is a form of degree three in (z,ζ). Separating the component of weight 3 in (3.5), we obtain

$$h_3 = 2i\rho\langle z,\overline{a}\,\rangle w, \qquad f_2 = C\big(aw + 2i\langle z,\overline{a}\,\rangle z - K(z,z)\mu\big),$$
$$g_1 = \langle z,\overline{\alpha}\,\rangle + \beta\zeta,$$

where $\beta \neq 0$ since the map is invertible.

Our calculations show that the following point of view is convenient for finding the weighted j-jet:

$$\phi = \sum \phi_j, \qquad \phi_j = (f_{j-1}, g_{j-2}, h_j).$$

Thus the weighted j-jet of the map is regarded as a tuple of jets of the coordinates consisting of the jth weighted jet of h, the $(j-1)$th jet of f and the $(j-2)$th jet of g. The calculation above yields a description of the action of holomorphic maps on the 3-jet of the equation of a hypersurface of the form (3.4).

Lemma 7. a) *The pseudogroup of locally invertible origin-preserving holomorphic maps acts in the following way on the set of weighted* 3*-jets of* 2*-nondegenerate hypersurfaces of the form* (3.4):

$$\begin{aligned} &z \to C\big(z + aw + 2i\langle z, \overline{a}\,\rangle z - K(z,z)\alpha\big) + O(3),\\ &\zeta \to \langle z, \overline{\alpha}\,\rangle + \beta\zeta + O(2),\\ &w \to \rho(w + 2i\langle z, \overline{a}\,\rangle w) + O(4), \end{aligned}$$

where $C \in \mathrm{GL}(2,\mathbf{C})$, $\rho \in \mathbf{R}^*$, $a, \alpha \in \mathbf{C}^2$, $\beta \in \mathbf{C}^*$, *and*

$$\langle z, \overline{z}\,\rangle = \rho\langle C^{-1}z, \overline{C^{-1}z}\,\rangle, \qquad \widetilde{K}(z,z) = \frac{\rho}{\overline{\beta}} K(C^{-1}z, C^{-1}z). \tag{3.6}$$

b) *Up to this action, every invertible origin-preserving holomorphic map from* Γ *to* $\widetilde{\Gamma}$ *is of the form*

$$z \to z + O(3), \qquad \zeta \to \zeta + O(2), \qquad w \to w + O(4). \tag{3.7}$$

To study the maps of a hypersurface Γ to itself, we consider the subgroup of all linear automorphisms of the following form in the automorphism group of Γ:

$$\begin{gathered} G_0 = \{(z \to Cz,\ \zeta \to \beta\zeta,\ w \to \rho w)\} \text{ provided that}\\ \langle Cz, \overline{Cz}\,\rangle = \rho\langle z, \overline{z}\,\rangle, \qquad K(Cz, Cz) = \frac{\rho}{\overline{\beta}} K(z,z). \end{gathered} \tag{3.8}$$

We now calculate the dimension of this group for each of the six pairs of forms listed in Lemma 6.

Lemma 8. *Let* G_0^j *be the group* G_0 *for the* j*th pair in Lemma* 6*. Then*

$$\dim G_0^1 = 2, \quad \dim G_0^2 = 3, \quad \dim G_0^3 = 3, \quad \dim G_0^4 = 2, \quad \dim G_0^5 = 3,$$
$$\dim G_0^6 = 3, \quad \dim G_0^7 = 3, \quad \dim G_0^8 = 3, \quad \dim G_0^9 = 3.$$

Hence we always have $\dim G_0 \leqslant 3$.

Proof. For the pairs 1^*)–3^*) we have $\langle z, \overline{z}\,\rangle = |z_1|^2 + |z_1|^2$. Hence $C = \lambda U$, $\rho = |\lambda|^2$, where $U \in SU(2)$ and $\lambda \in \mathbf{C}^*$. Writing U in the form

$$\begin{bmatrix} p & q \\ -\overline{q} & \overline{p} \end{bmatrix}, \quad \text{where} \quad |p|^2 + |q|^2 = 1,$$

and substituting this into the second relation, we obtain

$$(kp^2 + m\overline{q}^{\,2},\ -4i\,\mathrm{Im}\,pq,\ kq^2 + m\overline{p}^{\,2}) = \frac{|\lambda|^2}{\overline{\beta}}\,(k, 0, m).$$

Hence $\mathrm{Im}\,pq = 0$, that is, $q = \sigma\overline{p}$, where σ is real. Moreover, $|p|^2(1+\sigma^2) = 1$. Writing $p = \exp(i\phi)/\sqrt{1+\sigma^2}$, we have

$$\exp(4i\phi) = \frac{k\sigma^2 + m}{k + m\sigma^2}\,\frac{k}{m}.$$

It follows that $\phi = 0$ for U in a small neighbourhood of the identity. Then we have either $k = m$, or $\sigma = 0$. The answer for the third pair is obtained in a similar way. The free parameters are λ for the pair 1^*), (λ, σ) for 2^*), and (λ, ϕ) for 3^*).

The pairs 4^*)–6^*) can be considered in the same way taking into account that U is a pseudo-unitary matrix of the form

$$\begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, \quad \text{where} \quad |p|^2 - |q|^2 = 1.$$

We have exactly the same free parameters.

For the Hermitian form of the pairs 7^*)–9^*), pseudo-unitary matrices with unit determinant close to the identity matrix are of the form

$$\begin{bmatrix} p & i\sigma p \\ \dfrac{ir}{(1+r\sigma)p} & \dfrac{1}{(1+r\sigma)p} \end{bmatrix}, \quad \text{where} \quad r, \sigma \in \mathbf{R}, \quad p > 0.$$

This yields the values of the dimensions. The free parameters are (λ, p) for the pair 7^*), (λ, r) for 8^*), and (λ, p) for 9^*). □

We now fix hypersurfaces Γ and $\widetilde{\Gamma}$ of the from (3.4) and give an estimate for the number of parameters encoding the maps of the form (3.7) from Γ to $\widetilde{\Gamma}$ in accordance with the recursion scheme of depth $k = 2$ (Theorem 1). To do this, we describe the μth component of the relation (3.5). We explicitly present the summands depending on ϕ_μ and $\phi_{\mu-1}$ and discard those depending on ϕ_ν with $\nu \leqslant \mu - 2$. We put $f = (f^1, f^2)$ and $\Delta\psi(u) = 2i\,\mathrm{Re}(K(z,z)\overline{\zeta}\,)\psi'(u)$.

Lemma 9. *The μth weighted component Θ_μ of the expression* (3.5) *is of the form*

$$\Theta_\mu = L_1(\phi_\mu) + L_2(\phi_{\mu-1}) + \theta_\mu(\phi_{\nu<\mu-1})$$

with

$$\begin{aligned} L_1(\phi) &= 2\,\mathrm{Re}\big(ih + 2\langle f, \overline{z}\,\rangle + 2\overline{K}(\overline{z}, \overline{z}\,)g\big), \\ L_2(\phi) &= \Delta L_1(\phi) + 2\,\mathrm{Re}\big\{4K(f, z)\overline{\zeta} + 2\big(\overline{P}(\overline{z}, \overline{z}, z) + \overline{Q}(\overline{z}, \overline{z}, \overline{z}\,) \\ &\qquad + 2\overline{R}(\overline{z}, \overline{z}\,)\zeta + S(z, \overline{z}\,)\overline{\zeta}\,\big)g\big\}, \\ &\textit{where} \quad w = u + i\langle z, \overline{z}\,\rangle. \end{aligned}$$

Note that the expression $L(\phi) = L_1(\phi) + L_2(\phi)$ is linear with respect to ϕ and independent of μ. Let V_4 be the vector space of the germs (at the origin) of formal power series of the form

$$\Phi = \phi_4 + \phi_5 + \cdots = (f_3 + f_4 + \cdots\,,\ g_2 + g_3 + \cdots\,,\ h_4 + h_5 + \cdots).$$

By Theorem 1, the number of parameters encoding the maps (3.7) from Γ to $\widetilde{\Gamma}$ does not exceed the dimension of $\operatorname{Ker} L$ on the space V_4. Together with the estimate for the number of parameters in the 3-jet, this yields a general bound for the number of parameters for the maps and, in particular, a bound for the dimension of the local automorphism group of Γ. Thus, to obtain a bound for the dimension of automorphisms of a 2-nondegenerate hypersurface, we only need to estimate the dimension of $\operatorname{Ker} L$ on V_4.

The operator L contains many arbitrary constants. To simplify the estimation of the dimension of its kernel, we use the same approach (recursion of depth two) for the equation $L(f, g, h) = 0$ after changing the weights of the main variables. The new weights are given by

$$[z] = [\overline{z}\,] = 2, \qquad [\zeta] = [\overline{\zeta}\,] = 1, \qquad [w] = [u] = 4.$$

If we put $\phi_\mu = (f_{\mu-2}, g_{\mu-4}, h_\mu)$ in terms of the new weighted decomposition of $\phi = (f, g, h)$, then the μth weighted component of $L(\phi) = 0$ is of the form

$$\begin{aligned} L_\mu = 2\operatorname{Re}[ih_\mu + i\Delta(h_{\mu-1})] + 2\operatorname{Re}[2\langle f_{\mu-2}, \overline{z}\,\rangle + 2\langle \Delta(f_{\mu-3}), \overline{z}\,\rangle + 4K(f_{\mu-3}, z)\overline{\zeta}\,] \\ + 2\operatorname{Re}\big[2\overline{K}(\overline{z}, \overline{z}\,)g_{\mu-4} + 2\overline{K}(\overline{z}, \overline{z}\,)\Delta(g_{\mu-5}) + \big(2\overline{R}(\overline{z}, \overline{z}\,)\zeta + S(z, \overline{z}\,)\overline{\zeta}\,\big)g_{\mu-5} \\ + \big(2\overline{P}(\overline{z}, \overline{z}, z) + \overline{Q}(\overline{z}, \overline{z}, \overline{z}\,)\big)g_{\mu-6}\big], \quad \text{where} \quad w = u + i\langle z, \overline{z}\,\rangle = 0. \end{aligned} \tag{3.9}$$

Lemma 10. *The dimension of the space of solutions of (3.9) does not exceed the dimension of the space of solutions of $\mathcal{L}(f, g, h) = 0$:*

$$\begin{aligned} \mathcal{L}(f, g, h) = 2\operatorname{Re}[ih + i\Delta(h)] + 2\operatorname{Re}[2\langle f, \overline{z}\,\rangle + 2\langle \Delta f, \overline{z}\,\rangle + 4K(f, z)\overline{\zeta}\,] \\ + 2\operatorname{Re}[2\overline{K}(\overline{z}, \overline{z}\,)g + 2\overline{K}(\overline{z}, \overline{z}\,)\Delta(g) + 2\overline{R}(\overline{z}, \overline{z}\,)\zeta g \\ + S(z, \overline{z}\,)\overline{\zeta}\, g], \quad \textit{where} \quad w = u + i\langle z, \overline{z}\,\rangle = 0. \end{aligned} \tag{3.10}$$

Proof. This follows immediately from Theorem 1. □

Note that the recursion described by the operator $\mathcal{L}$ begins with $\mu = 5$. Since we are interested in the dimension of the kernel of L on V_4 in the old weighted grading, Lemma 10 needs a slight correction. Let $\widetilde{V}_5$ be the space of all tuples (f, g, h), where $f = \widetilde{O}(3)$, $g = \widetilde{O}(2)$, $h = \widetilde{O}(5)$ with respect to the new weights. We see that the following lemma holds.

Lemma 11. *If $\phi = (f, g, h) \in V_4 \cap \operatorname{Ker} \mathcal{L}$, then $\phi \in \widetilde{V}_5$.*

Proof. If $\phi \in V_4$, then $\phi = \chi + \psi$, where $\psi \in \widetilde{V}_5$ and $\chi = (0, 0, \gamma\zeta^4)$. Separating the component of weight four in the relation $\mathcal{L}(\chi + \psi) = 0$, we obtain $\mathcal{L}(\chi) = 0$. It follows that $\chi = 0$. □

We proceed to estimate the dimension of the kernel of $\mathcal{L}$. Note that the operator depends on the parameters (k, m) satisfying the constraints listed in Lemma 4 (admissible values) and on the three unconstrained coefficients of the quadratic form $R(z, z) = r_1 z_1^2 + r_2 z_1 z_2 + r_3 z_2^2$. Note also that, independently of the values of the parameters, $\operatorname{Ker} \mathcal{L}$ contains a two-dimensional subspace (of trivial solutions), which is however disjoint from $\widetilde{V}_5$:

$$\begin{aligned} &(f_1 = f_2 = g = 0,\ h = t_1), && t_1 \in \mathbf{R}, \\ &(f_1 = t_2 z_1,\ f_2 = t_2 z_2,\ g = 0,\ h = t_2^2 w), && t_2 \in \mathbf{R}. \end{aligned} \tag{3.11}$$

Lemma 12. *Suppose that* $\phi = (f, g, h) \in \widetilde{V}_5 \cap \operatorname{Ker} \mathcal{L}$.

a) *If* $(k = 1, m = 0)$ (*the pair* 9^*) *in Lemma* 6) *and* $R(z, z) = r_1 z_1^2$, *then*

$$f_1 = i\overline{n}_1 z_1^1, \qquad f_2 = 2i\overline{n}_1 z_1 z_2 - \overline{n}_2 z_1^2 + n_1 w,$$
$$g = \frac{n_2 z_1 - i n_1 z_2 + 2i\overline{n}_1 z_1 \zeta}{1 + 2\overline{r}_1 \zeta}, \qquad h = 2i\overline{n}_1 z_1 w,$$

where n_1 *and* n_2 *are complex numbers and*, *accordingly*, $\dim(\widetilde{V}_5 \cap \operatorname{Ker} \mathcal{L}) = 4$.

b) *Otherwise* $\phi = 0$ *and*, *accordingly*, $\dim(\widetilde{V}_5 \cap \operatorname{Ker} \mathcal{L}) = 0$.

Proof. This is a standard but long calculation performed by means of computer algebra (Maple). The calculation is done separately for the pairs $1^*)$–$6^*)$ and for $7^*)$–$9^*)$. To study the pairs $1^*)$–$3^*)$ and $4^*)$–$6^*)$ in a unified way, we introduce a parameter $\varepsilon = \pm 1$ accounting for the signature of the Levi form. We also put

$$\big(f_1(0,0,0,u), f_2(0,0,0,u)\big) = \big(a_1(u), a_2(u)\big) = a(u),$$
$$g(0,0,0,u) = b(u), \qquad h(0,0,0,u) = c(u),$$
$$\frac{\partial f_1}{\partial z_1}(0,0,0,u) = a_{11}(u), \qquad \frac{\partial f_2}{\partial z_1}(0,0,0,u) = a_{21}(u),$$
$$\frac{\partial f_1}{\partial z_2}(0,0,0,u) = a_{12}(u), \qquad \frac{\partial f_2}{\partial z_2}(0,0,0,u) = a_{22}(u),$$
$$\frac{\partial g}{\partial z_1}(0,0,0,u) = b_1(u), \qquad \frac{\partial g}{\partial z_2}(0,0,0,u) = b_2(u), \qquad \frac{\partial^2 g}{\partial z_1^2}(0,0,0,u) = B(u).$$

The scheme of calculation differs by small details in the first and second case. We describe it in the second case (of the pairs $7^*)$–$9^*)$).

Step 1. Putting $\overline{z} = 0$, $\overline{\zeta} = 0$ in the relation

$$\mathcal{L}(f_1, f_2, g, h) = 0, \tag{3.12}$$

we can express $h(z_1, z_2, \zeta, u)$ in terms of $(a_1(u), a_2(u), b(u), c(u))$. This expression is of the form

$$h(z_1, z_2, \zeta, u) = \overline{c}\,(u) + 2i\langle z, \overline{a}\,(u)\rangle + 2i\overline{b}\,(u) K(z, z).$$

Substituting $(z = 0,\ \zeta = 0)$, we see that $\overline{c}\,(u) = c(u)$.

Step 2. Substitute the resulting value of h into (3.12), calculate $\mathcal{L}'_{\overline{z}_1}$ and $\mathcal{L}'_{\overline{z}_2}$, substitute $\overline{z} = 0$, $\overline{\zeta} = 0$, and use the resulting relations to express f_1 and f_2

in terms of $(a_1(u), a_2(u), b(u), c(u), a_{11}, a_{12}, a_{21}, a_{22})$. These expressions are of the form

$$\begin{aligned} f_1 &= a_1(u) + 2i\overline{b}'(u)mz_1z_2^2 + 2i\overline{b}'(u)z_1^3 - 4\overline{m}\, z_1\zeta\overline{b}\,(u) + c'(u)z_1 + 2i\overline{a}_2'(u)z_1^2 \\ &\quad + 2i\overline{a}_1'(u)z_1z_2 - \overline{b}_2(u)mz_2^2 - 2\zeta\overline{a}_2(u)\overline{m} - \overline{b}_2(u)z_1^2 - \overline{a}_{22}(u)z_1 - \overline{a}_{12}(u)z_2, \\ f_2 &= a_2(u) + 2i\overline{b}'(u)mz_2^3 + 2i\overline{b}'(u)z_1^2z_2 - \overline{b}_1(u)mz_2^2 - 4mz_2\zeta\overline{b}\,(u) + c'(u)z_2 \\ &\quad + 2i\overline{a}_2'(u)z_1z_2 + 2i\overline{a}_1'(u)z_2^2 - \overline{b}_1(u)z_1^2 - \overline{a}_{11}(u)z_2 - \overline{a}_{21}(u)z_1 - 2\zeta\overline{a}_1(u). \end{aligned}$$

Substituting $(z = 0,\ \zeta = 0)$, we obtain

$$a_{22}(u) = c'(u) - \overline{a}_{11}(u), \qquad \operatorname{Re} a_{12}(u) = \operatorname{Re} a_{21}(u) = 0.$$

Step 3. Substitute the resulting values of f_1 and f_2 into (3.12), calculate $\mathcal{L}''_{\overline{z}_1^2}$, substitute $\overline{z} = 0$, $\overline{\zeta} = 0$ and use the resulting relations to express g in terms of $(a_1(u), a_2(u), b(u), c(u), a_{11}, a_{12}, a_{21}, a_{22}, b_1(u), b_2(u), B(u))$:

$$\begin{aligned} g = \frac{1}{2(2\overline{r}_1\zeta + 1)} &\big(2\,b(u) + 4i\overline{b}_1'(u)z_2z_1^2 - \overline{B}(u)mz_2^2 + 12i\zeta\overline{a}_1'(u)z_2 - 2c'(u)\zeta \\ &+ 4a_{22}(u)\zeta + 2b_1(u)z_1 - \overline{B}(u)z_1^2 + 2b_2(u)z_2 - 8\overline{b}_1(u)m\zeta z_2 \\ &- 2i\overline{a}_{12}'(u)z_2z_1 + 2i\overline{a}_{12}'(u)z_1z_2 + 4i\overline{a}_2'(u)\zeta z_1 + 20imz_2^2\zeta\overline{b}'(u) - 2ia_{22}(u)z_2^2 \\ &+ 4i\overline{b}'(u)\zeta z_1^2 + 4\overline{b}''(u)z_2^4m + 4\overline{b}''(u)z_2^2z_1^2 + 4\overline{a}_2'(u)z_2^2z_1 \\ &+ 2i\overline{a}_{11}'(u)z_2^2 + 4i\overline{b}_1'(u)z_2^3m + 2\overline{a}_1''(u)z_2^3\big). \end{aligned}$$

Substituting $(z = 0,\ \zeta = 0)$, we obtain $B = -(1/2)\overline{B}$. It follows that $B = 0$.
Step 4. Substituting the resulting expression for g into (3.12), we obtain a real polynomial in $(z, \overline{z}, \zeta, \overline{\zeta}\,)$ whose coefficients are differential polynomials of the functions (introduced above) of the variable u and their derivatives. Equating all the coefficients to zero and analyzing the resulting system of ordinary differential equations enables us to complete the proof of the lemma. □

§ 4. 2-nondegenerate hypersurfaces of a special form

By Lemma 12, a non-trivial kernel occurs only for a special class of 2-nondegenerate hypersurfaces that can be given by an equation of the following form near any point:

$$v = 2\operatorname{Re}(z_1\overline{z}_2) + 2\operatorname{Re}(z_1^2\overline{\zeta}\,) + 2\operatorname{Re}(r_1z_1^2\overline{\zeta}^{\,2}) + \cdots .$$

The change of variable $\zeta \to \zeta + \overline{r}_1\zeta^2$ brings the equation to the form

$$v = 2\operatorname{Re}(z_1\overline{z}_2) + 2\operatorname{Re}(z_1^2\overline{\zeta}\,) + \text{monomials of new weight at least 7.} \tag{4.1}$$

To study these hypersurfaces, we permute the coordinates and change the weights once again. We now put

$$[z_1] = 2, \qquad [z_2] = [\zeta] = 1, \qquad [w] = [u] = 3.$$

Then the hypersurface (a weighted model surface) is given by

$$Q = \bigl\{v = 2\,\mathrm{Re}(z_1\overline{\zeta} + z_2\overline{\zeta}^{\,2})\bigr\}. \tag{4.2}$$

With our choice of weights, Q is the graph of a quasi-homogeneous real polynomial of weight 3. This enables us to use recursion of depth one and obtain a complete answer. We also note that using weighted model surfaces is a long practice (see [4], [10], [11]), and this technique is completely standard.

The holomorphic homogeneity of Q is guaranteed by the subgroup $\mathcal{Q}$ of automorphisms of Q consisting of the transformations

$$\begin{gathered} z_1 \to a + z_1, \qquad z_2 \to b + 2\overline{a}\,\zeta + z_2, \qquad \zeta \to c + \zeta, \\ w \to d + 2i\bigl(a\overline{b} + a^2\overline{c} + (\overline{b} + 2a\overline{c}\,)z_1 + \overline{a}z_2 + \overline{a}^2\zeta + \overline{c}\,z_1^2\bigr) + w, \end{gathered} \tag{4.3}$$

where (a,b,c,d) is an arbitrary point of Q.

Let Γ_0 be the germ (at the origin) of a hypersurface of the form

$$v = 2\,\mathrm{Re}(z_1\overline{\zeta} + z_2\overline{\zeta}^{\,2}) + O(4), \tag{4.4}$$

where $O(4)$ stands for the terms of weight at least four. We consider a map $\phi = (f,g,h,e)$ of this germ to another germ of this form such that

$$f = z_1 + f_3 + \cdots, \quad g = z_2 + g_2 + \cdots, \quad h = \zeta + h_2 + \cdots, \quad e = w + e_4 + \cdots \tag{4.5}$$

(the subscripts denote the weights of the components). Separating the μth weighted component in the analytic relation expressing the fact that the map send the first hypersurface to the second, we obtain

$$-\,\mathrm{Im}\,e_\mu + 2\,\mathrm{Re}(f_{\mu-1}\overline{\zeta} + g_{\mu-2}\overline{\zeta}^{\,2} + h_{\mu-2}(\overline{z}_1 + 2\overline{z}_2\zeta)) = \cdots,$$

where $w = u + 2i\,\mathrm{Re}(z_1\overline{\zeta} + z_2\overline{\zeta}^{\,2})$ and the dots stand for an expression depending only on components of smaller weight (that is, smaller than $\mu-1$ for f, smaller than $\mu-2$ for g and h, and smaller than μ for e).

Thus we see that the dimension of the family of maps of the form (4.5) is controlled by the dimension of the kernel of the homological operator

$$L(f,g,h,e) = 2\,\mathrm{Re}\bigl(ie + 2f\overline{\zeta} + 2g\overline{\zeta}^{\,2} + 2h(\overline{z}_1 + 2\overline{z}_2\zeta)\bigr) \tag{4.6}$$

with $w = u + 2i\,\mathrm{Re}(z_1\overline{\zeta} + z_2\overline{\zeta}^{\,2})$.

On the other hand, let

$$X = 2\,\mathrm{Re}\biggl(f\,\frac{\partial}{\partial z_1} + g\,\frac{\partial}{\partial z_2} + h\,\frac{\partial}{\partial \zeta} + e\,\frac{\partial}{\partial w}\biggr)$$

be the germ (at the origin) of a vector field such that (f,g,h,e) are holomorphic at the origin. Then the equality $L(f,g,h,e) = 0$ holds if and only if X belongs to the Lie algebra $\mathrm{aut}\,Q$ of infinitesimal automorphisms of Q at the origin.

The weights of the coordinates naturally extend to the differentiations with respect to these coordinates. The differentiation with respect to z_1 is of weight (-2), and those with respect to z_2 and ζ (resp. w) are of weight (-1) (resp. (-3)). This makes $\operatorname{aut} Q$ a graded Lie algebra of the form $g_{-3}+g_{-2}+\cdots$. The Lie subalgebra g_0 contains the grading field

$$X_0 = 2\operatorname{Re}\biggl(2z_1\,\frac{\partial}{\partial z_1} + z_2\,\frac{\partial}{\partial z_2} + \zeta\,\frac{\partial}{\partial \zeta} + 3w\,\frac{\partial}{\partial w}\biggr).$$

All the graded components of an element of the algebra are also elements of the algebra. An argument of Kaup [12] enables us to assert that the Lie algebra $\operatorname{aut} Q$ is then finitely graded (polynomial). But we shall calculate this algebra explicitly instead of using this assertion.

We proceed to calculate the Lie algebra $\operatorname{aut} Q$, which coincides with the kernel of the operator (4.6). The procedure is similar to that described in the proof of Lemma 12, but the calculation is simpler.

We introduce the following notation:

$$f(0,0,0,u) = a(u), \qquad g(0,0,0,u) = b(u), \qquad h(0,0,0,u) = c(u),$$
$$e(0,0,0,u) = d(u), \qquad \frac{\partial f}{\partial z_1}(0,0,0,u) = a_1(u), \qquad \frac{\partial f}{\partial \zeta}(0,0,0,u) = a_3(u),$$
$$\frac{\partial g}{\partial z_1}(0,0,0,u) = b_1(u), \qquad \frac{\partial g}{\partial \zeta}(0,0,0,u) = b_3(u),$$
$$\frac{\partial h}{\partial z_1}(0,0,0,u) = c_1(u), \qquad \frac{\partial h}{\partial \zeta}(0,0,0,u) = c_3(u), \qquad \frac{\partial^2 g}{\partial z_1^2}(0,0,0,u) = B(u).$$

Putting $\overline{z}_1 = 0$, $\overline{z}_2 = 0$, $\overline{\zeta} = 0$ in the relation

$$\mathcal{L}(f,g,h,e) = 0, \tag{4.7}$$

we obtain an expression for h. It is a polynomial of degree two in (z_1, z_2, ζ) whose coefficients depend on $(a(u), b(u), c(u), d(u))$. Substituting this value of h into (4.7), we calculate $\mathcal{L}'_{\overline{\zeta}}$ and substitute $\overline{z} = 0$, $\overline{\zeta} = 0$. The resulting relation yields an expression for f. It is a polynomial of degree three whose coefficients depend on $(a, a', b', c', d, a_3, b_3, c_3)$. We calculate $\mathcal{L}'_{\overline{z}_1}$ and substitute $\overline{z} = 0$, $\overline{\zeta} = 0$. The resulting relation yields an expression for h. It is a polynomial of degree two whose coefficients depend on $(a, a', b', c, c', a_1, b_1, c_1)$. Substituting these values of f and h into (4.7), we calculate $\mathcal{L}''_{\overline{\zeta}^2}$ and put $\overline{z} = 0$, $\overline{\zeta} = 0$. The resulting relation yields an expression for g. It is a polynomial of degree four whose coefficients depend on $(a', a'', b, b', b'', c', c'', d', a_1', b_1, b_3, b_1', c_1', c_3', B)$.

Further analysis of (4.7) yields that

$$\operatorname{Im} d = \operatorname{Re} c_1 = \operatorname{Re} B = 0, \qquad b_3 = ic', \qquad c_3 = d' - \overline{a}_1,$$
$$a' = b' = c'' = d'' = a_1' = a_3 = b_1' = c_1' = B' = 0.$$

We calculate the number of free real parameters:

$$a-2,\quad b-2,\quad c-4,\quad d-2,\quad a_1-2,\quad a_3-0,\quad b_1-2,\quad b_3-0,\quad c_1-1,\quad c_3-0,\quad B-1.$$

Thus we see that the dimension of $\operatorname{aut} Q$ does not exceed 16.

On the other hand, it is easy to find some lower-weight components of $\operatorname{aut} Q$. These components are

$$
\begin{aligned}
g_{-3} &= \{(0,\, 0,\, 0,\, d)\},\\
g_{-2} &= \{(a,\, 0,\, 0,\, 2i\overline{a}\,\zeta)\},\\
g_{-1} &= \{(-2\overline{c}\,z_2 + ie\zeta,\, b,\, c,\, 2i\overline{c}\,z_1 + 2i\overline{b}\,\zeta^2)\},\\
g_0 &= \{(\alpha_1 z_1 - \overline{\alpha}_2\zeta^2,\, (2\alpha_2 - \alpha_3)z_2 + \alpha_2\zeta,\, (\alpha_3 - \overline{\alpha}_1)\zeta,\, \alpha_3 w)\},\\
g_1 &= \{(2i\overline{\beta}_1 z_1\zeta + \beta_1 w,\, 2i\overline{\beta}_1 z_2\zeta - i\beta_1 z_1 + i\beta_2\zeta^2,\, i\overline{\beta}_1\zeta^2,\, 2i\overline{\beta}_1\zeta w)\},\\
&\qquad a, b, c, \alpha_1, \alpha_2, \beta_1 \in \mathbf{C}, \qquad d, e, \alpha_3, \beta_2 \in \mathbf{R}.
\end{aligned}
\tag{4.8}
$$

We see that the dimension of the sum of these five components is equal to 16. This the algebra is calculated. In what follows it is convenient to represent g_{-1} as a direct sum $g'_{-1} + g''_{-1}$, where

$$
g'_{-1} = \{(-2\overline{c}\,z_2,\, b,\, c,\, 2i\overline{c}\,z_1 + 2i\overline{b}\,\zeta^2)\}, \qquad g''_{-1} = \{(ie\zeta,\, 0,\, 0,\, 0)\}.
$$

We now state the result obtained.

Theorem 2. a) *The algebra* $\operatorname{aut} Q$ *is the sum* $g_{-3} + g_{-2} + g_{-1} + g_0 + g_1$ *of five graded components. The components are described in* (4.8), $\dim \operatorname{aut} Q = 16$.

b) *The stabilizer* $\operatorname{aut}_0 Q$ *of the origin in* $\operatorname{aut} Q$ (*that is, the set of all vector fields belonging to the algebra and vanishing at the origin*) *is* $g''_{-1} + g_0 + g_1$. *Its dimension is equal to* 9.

c) *The subalgebra* $g_{-3} + g_{-2} + g'_{-1}$ *is the Lie algebra of the subgroup* $\mathcal{Q}$ (*the group of 'shifts'*). *There is a natural one-to-one correspondence between* Q *and* $\mathcal{Q}$. *This enables us to endow* $\mathcal{Q}$ *with the structure of a* CR-*manifold* (*a hypersurface in* $\mathbf{C}^4$).

d) *If* Γ_0 *is the germ of a hypersurface of the form* (4.4), *then we have the following bounds for the whole algebra and for the stabilizer of the origin*:

$$
\dim \operatorname{aut} \Gamma_0 \leqslant 16, \qquad \dim \operatorname{aut}_0 \Gamma_0 \leqslant 9.
$$

For completeness, we present the automorphisms generated by these fields.

The algebra $g_{-3} + g_{-2} + g'_{-1}$ corresponds to the 'shift' group $\mathcal{Q}$. It is parametrized by the quadruples (a, b, c, d) and, accordingly, $\dim gs = 7$. The subgroup $\mathcal{Q}$, which guarantees the holomorphic homogeneity of Q, consists of transformations of the form

$$
\begin{aligned}
&z_1 \to A + z_1, \qquad z_2 \to B + 2\overline{A}\,\zeta + z_2, \qquad \zeta \to C + \zeta,\\
&w \to D + 2i(A\overline{B} + A^2\overline{C} + (\overline{B} + 2A\overline{C}\,)z_1 + \overline{A}\,z_2 + \overline{A}^{\,2}\zeta + \overline{C}\,z_1^2) + w,
\end{aligned}
\tag{4.9}
$$

where (A, B, C, D) is an arbitrary point of Q.

We have $\dim g''_{-1} = 1$. The field $(i\zeta, 0, 0, 0)$ generates a subgroup of the form

$$
z_1 \to z_1 + it\zeta, \qquad z_2 \to z_2, \qquad \zeta \to \zeta, \qquad w \to w.
$$

The algebra g_0 is parametrized by the triples $(\alpha_1, \alpha_2, \alpha_3)$ and, accordingly, $\dim g_0 = 5$. To compute the group G_0 corresponding to g_0, we put $\gamma = \alpha_1 + \overline{\alpha}_1 - \alpha_3$.

If $\gamma \neq 0$, we obtain

$$z_1 \to \biggl(z_1 - \overline{\alpha}_2\biggl(\frac{e^{\gamma t}-1}{\gamma}\biggr)\zeta^2\biggr)e^{\alpha_1 t},$$
$$z_2 \to \biggl(z_2 + \alpha_2\biggl(\frac{e^{1-\overline{\gamma}\, t}}{\overline{\gamma}}\biggr)\zeta\biggr)e^{(2\alpha_1-\alpha_3)t},$$
$$\zeta \to \zeta e^{(\alpha_3-\overline{\alpha}_1)t}, \qquad w \to w e^{\alpha_3 t}.$$

The degenerate directions $\gamma = 0$ are obtain by passing to a limit.

The subalgebra g_+ consists of a single component g_1. It is parametrized by the pairs (β_1, β_2) and, accordingly, $\dim g_1 = 3$. The field $(0, i\zeta^2, 0, 0)$ in g_1 ($\beta_1 = 0$, $\beta_2 = 1$) generates the transformation

$$z_1 \to z_1, \qquad z_2 \to z_2 + it\zeta^2, \qquad \zeta \to \zeta, \qquad w \to w. \tag{4.10}$$

The transformations in g_1 with $\beta_2 = 0$ are of the form

$$\begin{gathered} z_1 \to \frac{z_1}{(1-i\overline{\beta}_1\zeta t)^2}, \qquad z_2 \to \frac{z_2 - i\beta_1 z_1 t}{(1-i\overline{\beta}_1\zeta t)^2}, \\ \zeta \to \frac{\zeta}{1-i\overline{\beta}_1\zeta t}, \qquad w \to \frac{w}{(1-i\overline{\beta}_1\zeta t)^2}. \end{gathered} \tag{4.11}$$

The transformations (4.10) and (4.11) generate the group G_+ corresponding to g_+.

The hypersurface Q is remarkable in many ways. It is a common starting point of two (otherwise disjoint) sequences of hypersurfaces in $\mathbf{C}^N$ with $N \geqslant 4$. The first sequence was given by Labovskii [13] as an example of holomorphically homogeneous l-nondegenerate hypersurfaces with arbitrary l. If this hypersurface lies in $\mathbf{C}^N$, then it is uniformly $(N-2)$-nondegenerate.

On the other hand, a recent paper [8] by Zelenko and Sykes contains a description of a series of holomorphically homogeneous 2-nondegenerate hypersurfaces in $\mathbf{C}^N$ with holomorphic automorphism algebra of dimension $(N-1)^2 + 7$ and a proof of optimality of these hypersurfaces in the class of all holomorphically homogeneous ones. Thus there are no holomorphically homogeneous hypersurfaces with automorphism algebra of larger dimension. Note that the technique in their paper is very different from ours. This is differential geometry in the style of Cartan and Tanaka.

Statement 2. a) *Let* Γ *be a real hypersurface which is* 2*-nondegenerate everywhere outside a proper analytic subset. Then, for every point* $\xi \in \Gamma$, *the dimension of the automorphism algebra* $\operatorname{aut}\Gamma_\xi$ *of the germ of* Γ *at* ξ *does not exceed* 17.

b) *But if* Γ *belongs to the special class* (4.1) (*or the larger class* (4.4)) *at a generic point*, *then* $\dim \operatorname{aut}\Gamma_\xi \leqslant 16$.

Proof. The dimension of $\operatorname{aut}\Gamma$ does not exceed the dimension of the hypersurface (which is equal to 7) plus the dimension of the stabilizer of a point. To estimate the stabilizer at any point, it suffices to perform the estimate at a 2-nondegeneracy point. By Theorem 1 and all subsequent lemmas, the dimension of the stabilizer is estimated in terms of the dimension of the lower-order jet (Lemmas 3 and 6) and the dimension (which is equal to zero) of the kernel of $\mathcal{L}$ on $\widetilde{V}_5$. The dimension of the group of parameters (C, ρ, β) does not exceed 3. The parameters (a, α) give another 8. In total, $7+3+8 = 18$. However, to obtain 18, the orbit of the origin must be 7-dimensional. This means holomorphic homogeneity. But then the dimension is less than or equal to 16 by [8]. Therefore we can assume that the dimension of the orbit is smaller than 7. This yields the bound $6 + 3 + 8 = 17$.

In case b) we can use Theorem 2, d). □

Theorem 3. *Suppose that* Γ *is a holomorphically non-degenerate real analytic hypersurface in* $\mathbf{C}^4$ *and* Γ_ξ *is the germ of* Γ *at a point* $\xi \in \Gamma$. *Let* $\operatorname{aut}\Gamma_\xi$ *be the Lie algebra of infinitesimal holomorphic automorphisms of this germ. Then the following assertions hold.*

1) $\dim \operatorname{aut}\Gamma_\xi \leqslant 24$.

2) *If* Γ *is* 2*-nondegenerate everywhere outside a proper analytic subset*, *then* $\dim \operatorname{aut}\Gamma_\xi \leqslant 17$.

3) *If* Γ *is* 3*-nondegenerate everywhere outside a proper analytic subset*, *then* $\dim \operatorname{aut}\Gamma_\xi \leqslant 20$.

Theorem 4. *Let* Γ_ξ *be the germ of an arbitrary real-analytic hypersurface in* $\mathbf{C}^4$ *such that* $\dim \operatorname{aut}\Gamma_\xi = 24$. *Then* Γ_ξ *is equivalent to one of the two standard non-degenerate hyperquadrics* (*hence it is Levi non-degenerate and spherical*).

Proof. If Γ_ξ is not Levi non-degenerate at a generic point, then it follows from Theorem 3 that the dimension does not exceed 20. Thus, Γ_ξ is Levi non-degenerate at a generic point. If it is not spherical there, then the dimension does not exceed 13 by the results in [14]. Hence it is spherical. In this case, a result of Kruglikov [15] yields that if Γ_ξ is not equivalent to a hyperquadric (of any signature), then the dimension is not greater than 17. □

The similar estimates for $\mathbf{C}^2$ and $\mathbf{C}^3$ are 8 and 15. They are also attained only at hyperquadrics. As in $\mathbf{C}^4$, the main difficulty occurs for hypersurfaces which are spherical at a generic point. The result for $\mathbf{C}^2$ (resp. $\mathbf{C}^3$) is due to Kossovskiy and Shafikov [16] (resp. Isaev and Kruglikov [17]).

These results together with the well-known criterion for finite dimension yield the following list of possibilities. Put $d = \dim \operatorname{aut}\Gamma_\xi$. We have

1) $d = \infty$ if and only if Γ is holomorphically degenerate;

2) $d = 24$ if and only if Γ is equivalent to one of the two non-degenerate standard hyperquadrics;

3) $d \leqslant 17$ if Γ_ξ is non-spherical but Γ is spherical at a generic point;

4) $d \leqslant 13$ if Γ is simultaneously 1-nondegenerate (Levi non-degenerate) and non-spherical at a generic point;

5) $d \leqslant 17$ if Γ is 2-nondegenerate at a generic point;

6) $d \leqslant 16$ if Γ is 2-nondegenerate at a generic point and homogeneous (in a neighbourhood of ξ);

7) $d \leqslant 20$ if Γ is 3-nondegenerate at a generic point.

The estimates in parts 1) and 2) of this list are precise. This also holds for part 3). Indeed, [15] contains an example of a hypersurface of this type realizing the bound 17. This also holds for part 4) since [14] contains an example of such a hypersurface with automorphism algebra of dimension 13. Part 6) is also exact and confirmed by an example. Hence the exact bound in part 5) is either 16 or 17. Part 7) is the most indefinite one. We can only say that the maximum is not smaller than 8 and not greater than 20. Hence the following question is appropriate.

Question 1. What are the exact values of the maxima in parts 5) and 7)?

Question 2. Does the same alternative occur for hypersurfaces of dimension 5 and higher? We mean that it is either infinity or not greater than the value for a hyperquadric (that is, $(N+1)^2-1$ in $\mathbf{C}^N$).

This rather old question [4] has a more general version (see Conjectures (5.a) and (5.b) in [18]). Is the maximum dimension of local automorphisms attained at non-degenerate model surfaces?

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