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Homogeneous Surfaces in $\mathbb{C}^4$ Associated with a 5-Dimensional Completely Nondegenerate Cubic Model Surface of CR-Type $(1, 3)$.

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Abstract. The action on the space $\mathbb{C}^4$ of the 7-dimensional Lie group of infinitesimal holomorphic automorphisms of a completely nondegenerate cubic model surface Q of CR-type $(1, 3)$ is considered. All orbits of the given action are found and their biholomorphic classification is given. One of the orbits coincides with the surface Q (the 5-dimensional orbit), two orbits are 6-dimensional, and the remaining part of the space $\mathbb{C}^4$, the complement to the above orbits, foliates into 7-dimensional real orbits. It is also proved that the algebra of infinitesimal holomorphic automorphisms of all orbits, except for the twelve holomorphically degenerate orbits, coincides with the algebra of the surface Q.

Let M_ξ be the germ of a smooth real generating subvariety of the space $\mathbb{C}^N$. Let n be its CR-dimension and let d be the codimension. In this case, $n + d = N$. We call the pair (n, d) the CR-type.

Let $\operatorname{aut} M_\xi$ be the Lie algebra consisting of germs of real vector fields generating one-parameter groups of holomorphic transformations in a neighborhood of the point ξ that preserve M_ξ; let $\operatorname{aut}_\xi M_\xi$ be the subalgebra of $\operatorname{aut} M_\xi$ consisting of the fields $X \in \operatorname{aut} M_\xi$ such that $X(\xi) = 0$; let $\operatorname{Aut} M_\xi$ be the local group generated by he fields in $\operatorname{aut} M_\xi$; let $\operatorname{Aut}_\xi M_\xi$ consist of transformations $\phi \in \operatorname{Aut} M_\xi$ such that $\phi(\xi) = \xi$. If M_ξ is of finite Bloom–Graham type, then one can associate with this germ a real algebraic surface Q of the same type, which is the tangent model surface [5]. The model surfaces are interesting for many reasons. For example

$$\dim \operatorname{Aut} M_\xi \leqslant \dim \operatorname{Aut} Q.$$

In [3], [4], the action of the group of holomorphic automorphisms of a model surface of CR-type $(1, 2)$ in $\mathbb{C}^3$ was studied and all its orbits were described. In this paper, we consider a similar question related to the space $\mathbb{C}^4$. Namely, for the model surface Q of CR-type $(1, 3)$ we study the action of the 7-dimensional group of its automorphisms $\operatorname{Aut} Q$ on the space $\mathbb{C}^4$ and calculate all its orbits.

Up to biholomorphic equivalence, there is only one completely nondegenerate 5-dimensional model surface Q of type $(1, 3)$. In the coordinates $(z, w_j := u_j + iv_j)$, $j = 1, 2, 3$, of the space $\mathbb{C}^4$, Q is given by the relations

$$\left\{\begin{array}{l} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + z\overline{z}^2, \\ v_3 = -i\,(z^2\overline{z} - z\overline{z}^2). \end{array}\right. \tag{1}$$

The local group of all holomorphic transformations of $\mathbb{C}^4$ that are invertible at the origin and preserve the germ Q at the origin is the 7-dimensional Lie group $\operatorname{Aut} Q$ of triangular-polynomial transformations of the form

$$\begin{aligned} z &\mapsto \lambda\,\gamma\, z + p, \\ w_1 &\mapsto 2\,i\,\lambda\,\gamma\,\overline{p}\,z + \lambda^2 w_1 + i\,|p|^2 + q_1, \\ w_2 &\mapsto 2\,i\,\lambda\,\gamma\big(2\,|p|^2 + \overline{p}^2\big)\,z + 2\,i\,\lambda^2\gamma^2\overline{p}\,z^2 \\ &\qquad + 4\,\lambda^2 \mathrm{Re} p\, w_1 + \lambda^3\big(\mathrm{Re}\gamma\, w_2 - \mathrm{Im}\gamma\, w_3\big) + 2\,i\,\mathrm{Re}(p^2\overline{p}) + q_2, \\ w_3 &\mapsto 2\,\lambda\,\gamma\big(2\,|p|^2 - \overline{p}^2\big)\,z + 2\,\lambda^2\gamma^2\overline{p}\,z^2 \\ &\qquad + 4\,\lambda^2 \mathrm{Im} p\, w_1 + \lambda^3\big(\mathrm{Im}\gamma\, w_2 + \mathrm{Re}\gamma\, w_3\big) + 2\,i\,\mathrm{Im}(p^2\overline{p}) + q_3, \end{aligned} \tag{2}$$

where $\gamma, p \in \mathbb{C}$ and $\lambda, q_1, q_2, q_3 \in \mathbb{R}$ for $|\gamma| = 1$ and $\lambda > 0$. In what follows, we denote this group by G. It can readily be seen that the orbit of the origin of the action of the group G coincides with Q.

The Lie algebra $\operatorname{aut} Q$ corresponding to the Lie group G is generated by the fields

$$
\begin{aligned}
X_1 &:= 2\operatorname{Re}(\partial_{w_1}),\\
X_2 &:= 2\operatorname{Re}(\partial_{w_2}),\\
X_3 &:= 2\operatorname{Re}(\partial_{w_3}),\\
X_4 &:= 2\operatorname{Re}(\partial_z + (2iz)\,\partial_{w_1} + (2iz^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3}),\\
X_5 &:= 2\operatorname{Re}(i\,\partial_z + (2z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2iz^2 - 4w_1)\,\partial_{w_3}),\\
X_6 &:= 2\operatorname{Re}(z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3}),\\
X_7 &:= 2\operatorname{Re}(iz\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}).
\end{aligned}
\tag{3}
$$

In what follows, we denote this Lie algebra by g.

If we introduce the weights of the variables and derivations as follows: $[z] = 1$, $[w_1] = 2$, $[w_2] = [w_3] = 3$, $[\partial_z] = -1$, $[\partial_{w_1}] = -2$, $[\partial_{w_2}] = [\partial_{w_3}] = -3$, then g becomes a graded Lie algebra of the form

$$
\operatorname{aut} Q = \underbrace{g_{-3} + g_{-2} + g_{-1}}_{g_-} + g_0,
\tag{4}
$$

where g_t is formed by the fields of weight t. According to this grading, $[X_2] = [X_3] = -3$, $[X_1] = -2$, $[X_4] = [X_5] = -1$, and $[X_6] = [X_7] = 0$.

Denote by G_- and G_0 the Lie groups corresponding to the Lie algebras g_- and g_0. Here G_0 is the stabilizer of the automorphism group of the surface Q at the origin. This subgroup consists of the transformations

$$
\begin{gathered}
z \mapsto \lambda\gamma\, z, \quad w_1 \mapsto \lambda^2 w_1,\\
w_2 \mapsto \lambda^3\big(\operatorname{Re}\gamma\, w_2 - \operatorname{Im}\gamma\, w_3\big), \quad w_3 \mapsto \lambda^3\big(\operatorname{Im}\gamma\, w_2 + \operatorname{Re}\gamma\, w_3\big),
\end{gathered}
\tag{5}
$$

where $\gamma \in \mathbb{C}$, $|\gamma| = 1$, and $\lambda > 0$.

Let us present a small list of obvious properties of the objects introduced above.

Proposition 1. (a) $(X_1, X_2, X_3, X_4, X_5)$ is a basis of g_-, and (X_6, X_7) is a basis of g_0;
(b) $G = G_- \ltimes G_0$. Topologically, G_- is $\mathbb{R}^5$. The group G_0 is isomorphic to the multiplicative group of the complex numbers $\mathbb{C}^*$. Thus, G is connected (but not simply connected) and is generated by the fields in g.
(c) The group G_- acts transitively on Q, i.e., Q coincides with the orbit of the origin. Moreover, it is possible to equip G_- with the structure of a CT-manifold embedded in $\mathbb{C}^4$ and equivalent to Q.

Write

$$
\begin{aligned}
\Phi(z, \overline{z}, v_1, v_2, v_3) &:= v_1 - z\overline{z},\\
\Psi_1(z, \overline{z}, v_1, v_2, v_3) &:= v_2 + z^2\overline{z} + z\overline{z}^2 - 2\,v_1(z + \overline{z}),\\
\Psi_2(z, \overline{z}, v_1, v_2, v_3) &:= v_3 - i\,(z^2\overline{z} - z\overline{z}^2) + 2i\,v_1(z - \overline{z}),
\end{aligned}
$$

then $Q = \{\Phi = \Psi_1 = \Psi_2 = 0\}$.

Let us state the main result of the present paper.

Theorem 1. *(a) Up to locally biholomorphic equivalence, all orbits of the action of $G = \operatorname{Aut} Q$ on the space $\mathbb{C}^4$ decompose into five types presented in the following table:*

dim	Orbit	Relations	dim aut	Degeneracy
5	Q	$\Phi = \Psi_1 = \Psi_2 = 0$	7	Completely nondegenerate
6	$+M^6$ / $-M^6$	$\Phi > 0,\ \Psi_1 = \Psi_2 = 0$ / $\Phi < 0,\ \Psi_1 = \Psi_2 = 0$	7	Completely nondegenerate
7	M^7	$\Psi_1 = 0\ \Psi_2 > 0$	∞	Holomorphically degenerate
7	$O_{\mathbf{m}}, \mathbf{m} > 0$ / $O_{\mathbf{m}}, \mathbf{m} < 0$	$\mathbf{m}\Phi^3 = \Psi_1^2 + \Psi_2^2$, $\Phi > 0, \Psi_1 > 0, \Psi_2 > 0$ / $\mathbf{m}\Phi^3 = \Psi_1^2 + \Psi_2^2$, $\Phi < 0, \Psi_1 > 0, \Psi_2 > 0$	7 / $7,\ \mathbf{m} \neq -\frac{32}{9}$ $\infty,\ \mathbf{m} \neq -\frac{32}{9}$	Levi nondegenerate / Levi nondegenerate for $\mathbf{m} \neq -\frac{32}{9}$ Holomorphically degenerate for $\mathbf{m} = -\frac{32}{9}$
7	S^7	$\Phi = 0, \Psi_1 > 0, \Psi_2 > 0$	∞	Holomorphically degenerate

To every type M^7, S^7,, and $O_{\mathbf{m}}$ there correspond four holomorphically equivalent orbits. The orbits of the other types are presented in the table explicitly.
(b) The orbits of different types are pairwise holomorphically nonequivalent. The same holds for the orbits $O_{\mathbf{m}}$ for different $\mathbf{m}$.
(c) None of the orbits, except for Q, is not spherical (i.e., is not equivalent to its tangent model surface).
(d) All orbits are of finite Bloom–Graham type. The Bloom–Graham type of Q is $(2,3,3)$, for $\pm M^6$ it is $(2,2)$, and the type of $M^7, O_{\mathbf{m}}, S^7$ is- (2). All CR-types possible in $\mathbb{C}^4$ are present among the orbits, namely, the type of Q is equal to $(1,3)$, the type of $\pm M^6$ is $(2,2)$, and the type of $M^7, O_{\mathbf{m}}$, and S^7 s $(3,1)$.

The figure below shows conventionally the partition of the space into orbits. We give a picture in $\mathbb{R}^3$ with the coordinates (Ψ_1, Ψ_2, Φ).

The proof of this theorem is given below.

It is clear that the orbits of the action of the group G are at least five-dimensional and at most seven-dimensional.

Let us formulate the following quite obvious auxiliary statement:

Lemma 1. *Let B be a smooth manifold on which a connected Lie group $\mathscr{L}$ acts by diffeomorphisms, and let l be the Lie algebra of $\mathscr{L}$ consisting of vector fields. Let K_j be a connected component of the set of points of B at which the Lie algebra l has the rank j. If K_j is a submanifold of dimension j, then K_j is the orbit of the action of the group $\mathscr{L}$.*

Proof. Since the rank of l at an arbitrary point of K_j is equal to j, it follows that the action of G on K_j is locally transitive. Therefore, the orbit of an arbitrary point is open in K_j. Since K_j is connected, this implies that the orbit coincides with K_j. This completes the proof of the lemma. □

Passing to the real coordinates $z = x + iy$ and $w_j = u_j + iv_j,\ j = 1,2,3$, we write out the generators of the algebra g in the form of the matrix

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -2y & 4u_1 - 4xy & 2x^2 - 2y^2 & 2x & 2x^2 - 2y^2 + 4v_1 & 4xy \\ 0 & 1 & 2x & 2x^2 - 2y^2 & 4xy + 4u_1 & 2y & 4xy & 2y^2 - 2x^2 + 4v_1 \\ x & y & 2u_1 & 3u_2 & 3u_3 & 2v_1 & 3v_2 & 3v_3 \\ -y & x & 0 & -u_3 & u_2 & 0 & -v_3 & v_2 \end{pmatrix}}_{\mathbf{A}} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_{u_1} \\ \partial_{u_2} \\ \partial_{u_3} \\ \partial_{v_1} \\ \partial_{v_2} \\ \partial_{v_3} \end{pmatrix}. \tag{6}$$

Let us describe the partition of the space $\mathbb{C}^4$ into subsets R_j, $j = 5, 6, 7$ of constant rank of the matrix $\mathbf{A}$. We make two preliminary remarks.

Any subsets of the space $\mathbb{R}^{nu}$, with coordinates $x = (x_1, \dots, x_\nu)$. that re given by finite systems of equations and inequalities for polynomials in x, and also their finite unions, intersections and complements, are said to be *semi-algebraic* sets. If a set S is given by the system of relations $F(x) = 0$, $G(x) > 0$, and $H(x) < 0$, then by $-S$ we denote the semi-algebraic set defined by the system $F(x) = 0$, $G(x) < 0$, and $H(x) > 0$. When considering the sets S and $-S$ simultaneously, we use the notation $\pm S$.

As a result of calculations carried out using the computer system MAPLE, as well as the algorithm described in [6], we obtain $R_6 = \{\Phi \neq 0,\ \Psi_1 = \Psi_2 = 0\}$. Obviously, $R_7 = \mathbb{C}^4 \setminus (R_5 \bigcup R_6)$. As noted above, Q is an orbit consisting of the points at which the rank of the matrix $\mathbf{A}$ is equal to five, i.e., $R_5 = Q$. Thus, we obtain the following proposition.

Proposition 2. The space $\mathbb{R}^8 = \mathbb{C}^4$ is decomposed into three subsets R_5, R_6, R_7. These are G-invariant semi-algebraic sets of the form

$$R_5 = \{\Phi = \Psi_1 = \Psi_2 = 0\} = Q,$$
$$R_6 = \{\Phi \neq 0,\ \Psi_1 = \Psi_2 = 0\},$$
$$R_7 = \{\Psi_1 \neq 0 \text{ or } \Psi_2 \neq 0\}.$$

Let us pass to the proof of the theorem.

Proof of the theorem. As was said above, the set R_5 is the orbit coinciding with Q.

The set R_6 consists of two connected components $\pm M^6$, where

$$M^6 = \{\ \Psi_1 = 0, \quad \Psi_2 = 0, \quad \Phi > 0\}\ .$$

which are real semi-algebraic generators of a CR-manifold of CR-type $(2, 2)$. According to Lemma 2, each of the generators is an orbit of the actions of the group G. Note that the rank six of the algebra g at the points of R_6 is wnsured by the six vector fields $(X_1, X_2, X_3, X_4, X_5, X_6)$.

The open set R_7 is given by the condition that at least one of the functions Ψ_1 or Ψ_2 is nonzero. Consider first a subset of R_7 of the form $\{\Psi_1 = 0,\ \Psi_2 \neq 0\} \bigcup \{\Psi_1 \neq 0,\ \Psi_2 = 0\}$. This is the union of four connected 7-dimensional hypersurfaces $\pm M^7$ and $\pm N^7$ given by the systems

$$M^7 = \{\ \Psi_1 = 0,\ \Psi_2 > 0\ \}, \quad N^7 = \{\ \Psi_1 > 0,\ \Psi_2 = 0\ \}.$$

It follows from Lemma 2 that these hypersurfacea are orbits of the action of G. The holomorphic transformation

$$(z, w_1, w_2, w_3) \mapsto (-z, w_1, -w_2, -w_3), \tag{7}$$

implements the following biholomorphic equivalence:

$$M^7 \sim -M^7, \qquad N^7 \sim -N^7.$$

Considering the holomorphic transformation

$$(z, w_1, w_2, w_3) \mapsto (-iz, w_1, w_3, -w_2), \tag{8}$$

we see also that M^7 and N^7 are equivalent. Thus, all four orbits indicated above are holomorphically equivalent. For every holomorphic function $f(z, w_1, w_2, w_3)$, the holomorphic vector field $f\partial_{w_3}$ is tangent to M^7. Therefore, M^7 and thus all four orbits treated above are *holomorphically degenerate*, whence we see that their algebras of infinitesimal automorphisms are *infinite-dimensional* (see [1]).

The remaining part has the form

$$\mathscr{P} = \{\ \Psi_1 \neq 0,\ \Psi_2 \neq 0\ \}. \tag{9}$$

Let $(a, b_1, b_2, b_3) = \mathbf{p}_0 \in \mathscr{P}$. We have

$$\begin{cases} \mathrm{Im} b_2 \neq -(a^2\overline{a} + a\overline{a}^2) + 2\,\mathrm{Im} b_1 (a + \overline{a}), \\ \mathrm{Im} b_3 \neq i\,(a^2\overline{a} - a\overline{a}^2) - 2\,i\,\mathrm{Im} b_1 (a - \overline{a}). \end{cases} \tag{10}$$

The orbit $\mathscr{O}_{\mathbf{p}_0}$ of a point $\mathbf{p}_0$ is the set of points $\mathbf{p} = (z, w_1, w_2, w_3) \in \mathbb{C}^4$ of the form

$$\begin{aligned} z &= \lambda\,\gamma\,a + p, \\ w_1 &= 2\,i\,\lambda\,\gamma\,\overline{p}\,a + \lambda^2 b_1 + i\,|p|^2 + q_1, \\ w_2 &= 2\,i\,\lambda\,\gamma\big(2\,|p|^2 + \overline{p}^2\big)\,a + 2\,i\,\lambda^2\gamma^2\overline{p}\,a^2 \\ &\qquad + 4\,\lambda^2 \mathrm{Re} p\, b_1 + \lambda^3\big(\mathrm{Re}\gamma\, b_2 - \mathrm{Im}\gamma\, b_3\big) + 2\,i\,\mathrm{Re}(p^2\overline{p}) + q_2, \\ w_3 &= 2\,\lambda\,\gamma\big(2\,|p|^2 - \overline{p}^2\big)\,a + 2\,\lambda^2\gamma^2\overline{p}\,a^2 \\ &\qquad + 4\,\lambda^2 \mathrm{Im} p\, b_1 + \lambda^3\big(\mathrm{Im}\gamma\, b_2 + \mathrm{Re}\gamma\, b_3\big) + 2\,i\,\mathrm{Im}(p^2\overline{p}) + q_3, \end{aligned} \tag{11}$$

for some $\gamma, p \in \mathbb{C}$ and $\lambda, q_1, q_2, q_3 \in \mathbb{R}$ for $|\gamma| = 1$ and $\lambda > 0$. Since the group contains all real shifts with respect to the variables w_1, w_2, w_3, it follows that we can eliminate the parameters q_1, q_2, q_3, passing to the imaginary parts of the second, third, and fourth equations of (11). It follows from the first expression in (11) that $p = z - \lambda\gamma a$. Substituting this expression for p into the imaginary part of the second expression (11), we obtain

$$v_1 - z\overline{z} = \lambda^2\,(\mathrm{Im} b_1 - a\overline{a}). \tag{12}$$

This means that $v_1 - z\overline{z}$ is a relative invariant of the action of the group.

Consider the cases $\Phi(\mathbf{p}_0) \neq 0$ and $\Phi(\mathbf{p}_0) = 0$.

Case 1. $\Phi(\mathbf{p}_0) \neq 0$. In this case, it follows from (12) that, for every point $\mathbf{p} = (z, w_1, w_2, w_3) \in \mathscr{O}_{\mathbf{p}_0}$, the sign of the expression $v_1 - z\overline{z}$ coincides with the sign of the expression $\Phi(\mathbf{p}_0)$. Hence, if $\Phi(\mathbf{p}_0) > 0$, then $\mathbf{p}$ can be translated only to points (z, w_1, w_2, w_3) for which $v_1 - z\overline{z} > 0$. A similar assertion holds also in the case of $\Phi(\mathbf{p}_0) < 0$. In this connection, we distinguish two subcases: $\Phi(\mathbf{p}_0) > 0$ and $\Phi(\mathbf{p}_0) < 0$. In turn, the desired orbits are decomposed into two types: the orbits lying in the connected component $v_1 - z\overline{z} > 0$ (the orbits "outside the ball") and lying in the connected component $v_1 - z\overline{z} < 0$ (the orbits "inside the ball"). In each of these cases, we can assign an arbitrary value to b_1. With regard to the assumption $\lambda > 0$, we see from (11) that

$$\lambda = \sqrt{\frac{v_1 - z\overline{z}}{\mathrm{Im}\, b_1 - a\overline{a}}} = \sqrt{\frac{\Phi(\mathbf{p})}{\Phi(\mathbf{p}_0)}}.$$

Let us now use the endomorphisms for p and λ; after simplifications, from the imaginary parts of the third and fourth equations (11) we obtain

$$\left[\begin{array}{l} v_2 + z^2\overline{z} + z\overline{z}^2 - 2\,v_1(z + \overline{z}) = \frac{1}{2}\lambda^3(\Omega\gamma + \overline{\Omega}\,\overline{\gamma}) \\ v_3 - i\,(z^2\overline{z} - z\overline{z}^2) + 2i\,v_1(z - \overline{z}) = -\frac{i}{2}\,\lambda^3(\Omega\gamma - \overline{\Omega}\,\overline{\gamma}), \end{array}\right. \tag{13}$$

where

$$\Omega = \mathrm{Im} b_2 + i\,\mathrm{Im} b_3 + 2\,a^2\overline{a} - 4\,a\,\mathrm{Im} b_1 = \Psi_1(\mathbf{p_0}) + i\,\Psi_2(\mathbf{p_0}).$$

It follows from (10) that $\Omega \neq 0$. Multiplying the second equation by i and adding it to the first one, we see that

$$v_2 + i\,v_3 + 2\,z^2\overline{z} - 4\,z v_1 = \lambda^3\gamma\,\Psi_1(\mathbf{p_0}) + i\,\Psi_2(\mathbf{p_0}).$$

Substituting $\lambda = \sqrt{\frac{v_1 - z\overline{z}}{\Phi(\mathbf{p}_0)}}$, we express the parameter γ:

$$\gamma = \frac{(\Psi_1(\mathbf{p}) + i\,\Psi_2(\mathbf{p}))}{\Psi_1(\mathbf{p_0}) + i\,\Psi_2(\mathbf{p_0})} \cdot \left(\frac{\Phi(\mathbf{p})}{\Phi(\mathbf{p}_0)}\right)^{-\frac{3}{2}}.$$

The condition $\gamma\overline{\gamma} = 1$ imposes restrictions on the modification of the imaginary parts of the variables b_2 and b_3. Namely, the mapping $(\mathrm{Im} b_2, \mathrm{Im} b_3) \mapsto (v_2, v_3)$ is possible only if the parameter γ satisfies the condition

$$(\Psi_1(\mathbf{p})^2 + \Psi_2(\mathbf{p})^2)\big(\Phi(\mathbf{p})\big)^{-3} = (\Psi_1(\mathbf{p}_0)^2 + \Psi_2(\mathbf{p}_0)^2)\Phi(\mathbf{p}_0)^{-3}.$$

It follows from the condition $\lambda \neq 0$ that the expression $v_1 - z\overline{z}$ is nonzero on the orbit of the point $\mathbf{p}_0$. Therefore, on the 8-dimensional subset

$$\mathscr{P}' = \left\{ \ \Phi \neq 0, \ \Psi_1 \neq 0, \ \Psi_2 \neq 0 \ \right\} \tag{14}$$

of the set $\mathscr{P}$, the function

$$F(\mathbf{p}) = (\Psi_1(\mathbf{p})^2 + \Psi_2(\mathbf{p})^2)\big(\Phi(\mathbf{p})\big)^{-3}$$

is invariant under the action of the group G. In particular, for every point $\mathbf{p}_0 \in \mathscr{P}'$, the 7-dil semi-algebraic submanifold given by the level lines of F,

$$F(\mathbf{p}) = |\Psi_1(\mathbf{p_0}) + i\,\Psi_2(\mathbf{p_0})|^2\Phi(\mathbf{p}_0)^{-3},$$

together with three defining inequalities (14) is also invariant. By Lemma 2, every connected component of this semi-algebraic manifold is the orbit of the action of the group G on the points of $\mathscr{P}' \subset \mathbb{C}^4$. This implies that, in accordance with the signs of the three real numbers $(\Phi(\mathbf{p}_0), \Psi_1(\mathbf{p}_0), \Psi_2(\mathbf{p}_0))$, the 7-dimensional orbit $\mathscr{O}_{\mathbf{p}_0}$ belongs to one of the following eight types: $O^7_{\pm,\pm,\pm}$, where the signs mean the choice of the corresponding inequality.

Using the holomorphic transformation (7), we obtain a family of equivalencies

$$O^7_{+++} \equiv O^7_{+--}, \qquad O^7_{-++} \equiv O^7_{---}, \qquad O^7_{++-} \equiv O^7_{+-+}, \qquad O^7_{-+-} \equiv O^7_{--+}.$$

Further, using the holomorphic transformation (8), we obtain the equivalencies

$$O^7_{+++} \equiv O^7_{++-}, \qquad O^7_{-++} \equiv O^7_{-+-}.$$

Therefore, we can consider only two types of 7-dimensional orbits: O^7_{+++} and O^7_{-++}, which differ in the sign of the expression $\Phi(\mathbf{p}_0)$ (in our indexing. this is the sign in the first position). For every point $\mathbf{p}_0 \in \mathscr{P}'$, we define the quantity

$$\mathbf{m} = (\Psi_1(\mathbf{p}_0)^2 + \Psi_2(\mathbf{p}_0)^2)(\Phi(\mathbf{p}_0))^{-3}.$$

Here $\mathbf{m}$ is an invariant of the action of the group G on the submanifold $\mathscr{P}'$. Therefore, we can parameterize the orbits found above using this parameter which can take all real values except for zero. A representative of the orbits at which $\mathbf{m} > 0$ is given by O^7_{+++}, and that of the orbits with $\mathbf{m} < 0$ is given by the orbit O^7_{-++}. Denote by $O_{\mathbf{m}}$ the orbit with the corresponding value of $\mathbf{m}$.

Since the orbits are holomorphically homogeneous, it follows that the Levi nondegeneracy at a point is equivalent to the Levi nondegeneracy everywhere. Evaluating the Levi matrix of the hypersurfaces $O_{\mathbf{m}}$ and evaluating its determinant at an arbitrary point, we see that all orbits $O_{\mathbf{m}}$, except for the case of $\mathbf{m} = -\frac{32}{9}$, are Levi nondegenerate. In the exceptional case, it can be proved that $O_{-\frac{32}{9}}$ is also holomorphically degenerate. Therefore, this orbit is not holomorphically equivalent to any other orbit, and its Lie algebra of infinitesimal holomorphic automorphisms is infinite-dimensional.

The space $\mathbb{R}^8$ is partitioned by singular orbits into 12 parts. Let $D(j,\pm,\pm)$, $j = 1,2,3$, be the domains each of which is defined by three inequalities. The second and third argument of D encodes the sign of Ψ_1 and Ψ_2, respectively, and one of the following inequalities holds in dependence on j:

$$\text{if } j = 1, \quad \text{then } (\Psi_1(\mathbf{p})^2 + \Psi_2(\mathbf{p})^2)(\Phi(\mathbf{p}))^{-3} < -32/9,$$
$$\text{if } j = 2, \quad \text{then } -32/9 < (\Psi_1(\mathbf{p})^2 + \Psi_2(\mathbf{p})^2)(\Phi(\mathbf{p}))^{-3} < 0,$$
$$\text{if } j = 3, \quad \text{then } 0 < (\Psi_1(\mathbf{p})^2 + \Psi_2(\mathbf{p})^2)(\Phi(\mathbf{p}))^{-3}.$$

Each of these domains is fibered into orbits that are Levi nondegenerate hypersurfaces of a fixed signature. Here there are both strictly pseudoconvex and not sigh-definite surfaces.

We claim now that the orbits for different $\mathbf{m}$ are pairwise holomorphically nonequivalent.

In what follows, we need a consideration which was used earlier (see [4, Lemma 3.1]) and whose proof goes back to W. Kaup (see, e.g., [9]). We formulate this consideration in the form of two lemmas.

Lemma 4. For every nonzero value of the parameter $\mathbf{m} \neq -\frac{32}{9}$, the Lie algebra $\operatorname{aut} O_{\mathbf{m}}$ is finite-dimensional and polynomial.

Proof. The finite-dimensionality of the Lie algebras $\operatorname{aut} O_{\mathbf{m}}$ for $\mathbf{m} \neq -\frac{32}{9}$ follows from the Levi nondegeneracy of the orbits $O_{\mathbf{m}}$. To prove that these Lie algebras are polynomial we note that they contain the field

$$D = z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3},$$

satisfying the equations

$$[D, Y_k] = k\,Y_k$$

for every weighted-homogeneous vector field Y_k of weight k. Suppose that Y belongs to the complexification of the Lie algebra $\operatorname{aut} O_{\mathbf{m}}$ and has a convergent expansion $Y := Y_{-3} + Y_{-2} + Y_{-1} + \cdots$ in the weight components. We have

$$[D, Y] = \sum_{k=-3}^{\infty} kY_k.$$

Let P be the minimal polynomial of the adjoint operator ad_D of the above complexified algebra. Then

$$0 = P(\operatorname{ad}_D)\,(Y) = \sum_{k=-3}^{\infty} P(k)Y_k.$$

However, P has only finitely many roots. Therefore, for a sufficiently large k_o, we have $Y_k \equiv 0$ for all $k > k_o$. Hence, Y is polynomial.

Lemma 2. *For nonzero real values* $\mathbf{m}, \mathbf{m}' \neq -\frac{32}{9}$, *every biholomorphic mapping* $F\ O_{\mathbf{m}} \to O_{\mathbf{m}'}$ *s a birational mapping of the ambient space* $\mathbb{C}^4$.

Proof. The proof of this lemma is based on two facts: (a) the Lie algebras of the infinitesimal automorphisms of both the surfaces are finite-dimensional and polynomial; (b) the complexifications of these algebras contain vector fields that ensure the homogeneity of the corresponding surface; in our case, these are the generators of the algebra g, i.e., the fields $X_1, \ldots, X_7$. The corresponding consideration was presented in the paper [14].

Proposition 3. (a) Two orbits $O_{\mathbf{m}}$ and $O_{\mathbf{m}'}$ are equivalent if and only if $\mathbf{m} = \mathbf{m}'$.
(b) If $\mathbf{m}, \mathbf{m}' \neq -\frac{32}{9}$, then $\operatorname{aut} O_{\mathbf{m}} = g$.
(c) The orbits M^6 and $-M^6$ are not equivalent.
(d) The Lie algebras of the infinitesimal automorphisms of the orbits M^6 and $-M^6$ coincide with g.

Proof. Since $O_{-\frac{32}{9}}$ a unique holomorphically degenerate orbit of the type under consideration, it follows that this orbit is not equivalent to any other orbit $O_{\mathbf{m}}$. Let $F : \mathbb{C}^4 \to \mathbb{C}^4$ be a birational transformation taking the orbit $O_{\mathbf{m}}$ to the orbit $O_{\mathbf{m}'}$ for $\mathbf{m}, \mathbf{m}' \neq -\frac{32}{9}$. Consider the equations

$$\Xi := |v_2 + i\,v_3 + 2\,z^2\overline{z} - 4\,zv_1|^2 - \mathbf{m}\big(v_1 - z\overline{z}\big)^3 = 0$$
$$\Xi' := |v_2 + i\,v_3 + 2\,z^2\overline{z} - 4\,zv_1|^2 - \mathbf{m}'\big(v_1 - z\overline{z}\big)^3 = 0,$$

defining the orbits $O_{\mathbf{m}}$ and $O_{\mathbf{m}'}$ and denote by $\mathscr{H}_{\mathbf{m}}$ and $\mathscr{H}_{\mathbf{m}'}$ the algebraic hypersurfaces given by the equations $\Xi = 0$ and $\Xi' = 0$, respectively. It can readily be seen that $Q \subset \mathscr{H}_{\mathbf{m}}, \mathscr{H}_{\mathbf{m}'}$. Since F takes $O_{\mathbf{m}}$ to $O_{\mathbf{m}'}$, it follows that that F also takes $\mathscr{H}_{\mathbf{m}}$ to $\mathscr{H}_{\mathbf{m}'}$. Writing $z := x + iy$, we consider the gradients

$$\nabla\mathscr{H}_{\mathbf{m}} := (\Xi_x, \Xi_y, \Xi_{v_1}, \Xi_{v_2}, \Xi_{v_3}) \qquad \text{and} \qquad \nabla\mathscr{H}_{\mathbf{m}'} := (\Xi'_x, \Xi'_y, \Xi'_{v_1}, \Xi'_{v_2}, \Xi'_{v_3}).$$

Calculations show that the singular sets of the hypersurfaces $\mathscr{H}_{\mathbf{m}}$ and $\mathscr{H}_{\mathbf{m}'}$, i.e., the sets at which the values of the vectors $\nabla\mathscr{H}_{\mathbf{m}}$ and $\nabla\mathscr{H}_{\mathbf{m}'}$ vanish, coincide with the surface Q. It follows from the birationality of F that the generating manifold Q cannot be completely contained in the singular set of the mapping F, because the singular set of F is a proper analytic subset of the ambient space. Since $F\ \mathscr{H}_{\mathbf{m}} \to \mathscr{H}_{\mathbf{m}'}$ must take the singular set of the manifold $\mathscr{H}_{\mathbf{m}}$ to the singular set of the manifold $\mathscr{H}_{\mathbf{m}'}$, it follows that F is a holomorphic automorphism of the surface Q in a neighborhood of a nonsingular point of the birational mapping F. Thus, $F \in G = \operatorname{Aut} Q$. However, the orbit $O_{\mathbf{m}}$ is invariant with respect to the the action of the group G, and therefore F t this orbit into itself, i.e., $O_{\mathbf{m}} = O_{\mathbf{m}'}$.

Repeating the above argument in the case of $O_{\mathbf{m}'} = O_{\mathbf{m}}$, we see that every holomorphic automorphism $F : \mathbb{C}^4 \to \mathbb{C}^4$ taking $O_{\mathbf{m}}$ into itself belongs to the automorphism group G. This implies that $\operatorname{Aut} O_{\mathbf{m}} = G$.

Applying the same argument to the mapping of M^6 to $-M^6$, we see that this mapping is contained in G. It can readily be seen that this is impossible. The contradiction thus obtained shows that M^6 and $-M^6$ are not equivalent.

Applying the same argument to the mapping of M^6 into itself, we see that $\operatorname{Aut} M^6 = G$. This completes the proof of the proposition.

Case 2. $\Phi(\mathbf{p}_0) = 0$.

Removing the points $\mathscr{P}'$ given by condition (14) from the submanifold $\mathscr{P}$ given by equation (9), we see that it remains to consider the orbits corresponding to the points of 7-dimensional semi-algebraic manifold given by the conditions

$$\mathscr{Q} : \begin{cases} v_1 = z\overline{z}, \\ v_2 \neq z^2\overline{z} + z\overline{z}^2, \\ v_3 \neq -i\,(z^2\overline{z} - z\overline{z}^2). \end{cases} \tag{15}$$

This manifold is invariant under the action of the group G, since the manifold is a complement to an invariant subspace of the space $\mathbb{C}^4$. The rank of the matrix $\mathbf{A}$ at every point of this submanifold is 7, whence, by Lemma 2, each of its connected components is an orbit of the action of our group. This manifold consists of four connected components $\pm S^7, \pm T^7$, where

$$S^7 : \begin{cases} v_1 = z\overline{z}, \\ v_2 > z^2\overline{z} + z\overline{z}^2, \\ v_3 > -i\,(z^2\overline{z} - z\overline{z}^2), \end{cases} \qquad T^7 : \begin{cases} v_1 = z\overline{z}, \\ v_2 > z^2\overline{z} + z\overline{z}^2, \\ v_3 < -i\,(z^2\overline{z} - z\overline{z}^2). \end{cases}$$

As above, the holomorphic transformation (7) ensures the equivalencies

$$S^7 \equiv -S^7, \qquad T^7 \equiv -T^7.$$

Further, using the holomorphic transformation (8), we see that $S^7 \equiv -T^7$. Therefore, all four given 7-dimensional orbits are holomorphically equivalent.

For every holomorphic function $f(z, w_1, w_2, w_3)$, the holomorphic vector fields $f\partial_{w_2}$ and $f\partial_{w_3}$ are tangent to the orbits $\pm S^7$ and $\pm T^7$. Therefore, ll these orbits are holomorphically degenerate, and their agsof infinitesimal holomorphic automorphisms are infinite-dimensional.

This completes the proof of the theorem. □

Remark 7. By the holomorphic transformation

$$(z, w_1, w_2, w_3) \mapsto (z, w_1, w_2 - 2\,zw_1, w_3 + 2i\,zw_1),$$

the defining equations of M^6 are transformed to the form

$$\mathbb{M}^6 : \begin{cases} v_2 = -i\,(w_1\overline{z} - \overline{w}_1 z) - (z^2\overline{z} + z\overline{z}^2), \\ v_3 = -(w_1\overline{z} + \overline{w}_1 z) + i\,(z^2\overline{z} - z\overline{z}^2), \\ v_1 > z\overline{z}. \end{cases}$$

This manifold is a completely nondegenerate CR-manifold of the type $(2,2)$ with the quadratic model surface (see [2]),

$$Q(2,2) = \begin{cases} v_2 = -i\,(w_1\overline{z} - \overline{w}_1 z), \\ v_3 = -(w_1\overline{z} + \overline{w}_1 z). \end{cases}$$

In [7, 8], this surface (which is called by the authors *elliptic*) was studied in detail. In particular, it turned out that the Lie algebra $\operatorname{aut} Q(2,2)$ of the infinitesimal CR-automorphisms of this surface is 16-dimensional and has the 10-dimensional stabilizer (see [7, Theorem 1]). Since the stabilizer of the surface M^6 is one-dimensional, we can claim that M^6 is nonspherical.

Comparing the results obtained here with the papers [3] and [4], we see that the results are quite similar. The existing differences are completely explainable by the growth of the dimension of space. In [5], the automorphisms of a model surface of an arbitrary finite Bloom–Graham-type were considered. The problem of describing the orbits of the action of the automorphism group of a surface of this kind is of interest in this more general context.

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