ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 28, No. 3, 2021, pp. 275–283.

On the Group of Holomorphic Automorphisms of a Model Surface

V. K. Beloshapka[*,1]

[*]M. V. Lomonosov Moscow State University, Moscow, Russia
E-mail: [1]vkb@strogino.ru

Received May 6, 2021; Revised May 10, 2021; Accepted May 10, 2021

Abstract. In this paper, it is proved that the group $\operatorname{Aut} Q$ of all holomorphic automorphisms a holomorphically homogeneous nondegenerate model surface Q is a subgroup of the group of birational isomorphisms of the ambient complex space (the Cremona group) of uniformly bounded degree. The degree is estimated in terms of the dimension of the ambient space (Theorem 4). It is shown that no condition of the theorem can be weakened. In the paper, the question of the connectivity of $\operatorname{Aut} Q$ is also considered (Theorem 7). This paper is directly adjacent to the previous paper of the author [7].

DOI 10.1134/S1061920821030018

1. INTRODUCTION

In the papers [2] and [7], as well as in other papers by the author, CR-manifolds were studied using a uniform approach, namely, the *model surface method*. Here the class of manifolds under consideration was determined using a certain "nondegeneracy" condition. In [2], this was the condition that the germ of the CR-manifold is completely nondegenerate. The complete nondegeneracy condition is a general position condition in the sense that any germ of positive CR-dimension can be made completely nondegenerate by a small smooth deformation. However, the Bloom–Graham type of such a germ cannot be arbitrary. In [7], the same program of the model surface method (a standard set of statements) was implemented for a class of CR-germs of arbitrary *finite* Bloom–Graham type with the condition of *holomorphic nondegeneracy*. Under this pair of conditions, we call the germ *nondegenerate*. The standard set of statements mentioned above includes a statement about the birationality of any automorphism of a model surface. In [7], this statement was omitted. It was announced in the list of open questions (Conjecture 5). In the present paper, this gap is eliminated (Theorem 1), i.e., it is proved that the birationality and the boundedness of degrees remain valid under two conditions: the nondegeneracy and the holomorphic homogeneity. In the previous versions of this assertion, the holomorphic homogeneity was not indicated as an independent condition. This is related to the fact that all completely nondegenerate model surfaces are homogeneous. In a new context of nondegenerate surfaces, this is no longer the case. In [7] a criterion of homogeneity of a model surface was given. Namely, it was shown that the constancy of the Bloom–Graham type, which is obviously a necessary condition for homogeneity, is sufficient for model surfaces. Further, it was shown there that the set of weights of a homogeneous model surface is completely nonarbitrary. Namely, it was shown that this set has the following form: $m_1 = 2$, $m_2 = 3, \dots, m_l = l + 1$.

The first assertion about the birationality of the automorphisms of any model surface was proved by A. Tumanov [4] using W. Kaup's trick [3]. This was done then for the simplest Bloom–Graham type, namely, for $m = (2, k)$ (the model quadric of codimension k). Our construction is essentially a multiple recursive use of Kaup–Tumanov's reasoning. Note also that the possibility to prove the birationality for a model surface of an arbitrary Bloom–Graham type according to this scheme is related to an important structural feature of the model surfaces, namely, to their "triangular" property.

The second main result of this paper (Theorem 2) is a description of the topological structure of the group of holomorphic automorphisms of the model surface. This result is only partly new. In all versions of the model surface method, starting from [1], the following simple consequence of the basic constructions was implied. Since the subgroup $\mathscr{G}_+$ (the nonlinear automorphisms preserving the origin) is parameterized by the kernel of the homological operator, it follows that this subgroup, as well as every linear space, is connected and simply connected (part (a) of the theorem). We present the proof here for two reasons. First, starting from the paper [7], the model surface method works in a much broader context. Second, having an explicitly stated assertion is an opportunity for referencing. The other assertions of the content of the theorem (parts (b) and (c)) are new.

2. BIRATIONALITY

To make the proof of the general statement clear, we preface this proof with an analysis of a particular case. Namely, we consider the first type which goes beyond the quadratic models $m = \{(2,k),(3,K)\}$. Here we do not assume that the surface is completely nondegenerate. For the completely nondegenerate case, the multiplicity k must be equal to n^2, where n is the CR-dimension. In the general case, we have $1 \leqslant k \leqslant n^2$. The nondegenerate model surfaces of this type were considered in detail in [6], and the birationality result was announced there.

Thus, the model surface Q of the Bloom–Graham type $m = \{(2,k),(3,K)\}$ is the surface in the space $\mathbf{C}^n \times \mathbf{C}^k \times \mathbf{C}^K$ with the coordinates $(z,\, w = u + i\,v,\, W = U + i\,V)$ which is given by the equations

$$v = \Phi(z,\,\bar z), \quad V = 2\,\mathrm{Re}\Psi(z,z,\bar z), \tag{1}$$

where Φ and Ψ are vector-valued forms linear in each of their arguments. In this situation, the finiteness of the type is equivalent to the fact that the coordinates of these forms are linearly independent. We introduce the weights of the variables as follows: $[z] = 1, \quad [w] = 2, \quad [W] = 3$. This grading naturally extends to complex and real power series. With the help of an additional agreement

$$[\frac{\partial}{\partial z}] = -1, \quad [\frac{\partial}{\partial w}] = -2, \quad [\frac{\partial}{\partial W}] = -3,$$

the grading is also extended to vector fields with analytic coefficients. Then the Lie algebra of infinitesimal holomorphic automorphisms in a neighborhood of the origin of $\mathrm{aut}\,Q$ becomes a graded Lie algebra. If Q is nondegenerate, then the Lie algebra becomes a finite-dimensional and finitely graded algebra of the form

$$\mathrm{aut}\,Q = g_{-3} + g_{-2} + g_{-1} + g_0 + g_1 + \cdots + g_\delta.$$

This algebra is formed by the vector fields of the form

$$X = 2\,\mathrm{Re}\left(f(z,w,W)\,\frac{\partial}{\partial z} + g(z,w,W)\,\frac{\partial}{\partial w} + h(z,w,W)\,\frac{\partial}{\partial W} \right),$$

where the coefficients of the fields are holomorphic in a neighborhood of the origin, and the fields satisfy the tangency condition, i.e.,

$$\mathrm{Im}\,g = 2\mathrm{Re}\Phi(f,\,\bar z), \quad \mathrm{Im}\,h = 2\mathrm{Re}(2\,\Psi(f,z,\bar z) + \Psi(z,z,\bar f)), \tag{2}$$

for $w = u + i\,\Phi(z,\,\bar z),\ W = U + 2\,i\,\mathrm{Re}(\Psi(z,z,\bar z)$.

The fact that all g_j for $j > \delta$ are equal to zero means, in particular, that the coefficients of the fields are polynomials whose degrees do not exceed $d = \delta + 3$.

It is clear that the fields of the weights (-2) and (-3) are fields of the form

$$X_{-3} = 2\,\mathrm{Re}(\mu\,\frac{\partial}{\partial W}), \quad X_{-2} = 2\,\mathrm{Re}(\nu\,\frac{\partial}{\partial w}),$$

where μ and ν are arbitrary constant vectors in $\mathbf{R}^K$ and $\mathbf{R}^k$, respectively. A field of weight (-1) is a field with the coefficients zero, one, and two, respectively, of the weights, i.e.,

$$f = const = p \in \mathbf{C}^n, \quad g = a(z,p), \quad h = \alpha(z,z,p) + \beta(w,p).$$

Substituting these coefficients into the tangency condition, we obtain

$$f = p, \quad g = 2\,i\,\Phi(z,\,\bar p), \quad h = 2\,i\,\Psi(z,z,\,\bar p) + \beta(w,p),$$

where $\beta(w,p)$ is a real linear form which is determined by the relation

$$\beta(\Phi(z,\bar z),p) = 4\,\mathrm{Re}\Psi(p,z,\bar z). \tag{3}$$

The uniqueness of the solution to this equation with respect to β is ensured by the linear independence of the coefficients Φ, and the solvability for every p is a condition which is equivalent to the holomorphic homogeneity of Q.

To the subalgebra $g_- = g_{-3} + g_{-2} + g_{-1}$ there corresponds the Lie subgroup $\mathscr{G}_-$ consisting of triangular-quadratic shifts. The holomorphic homogeneity of Q is equivalent to the fact that $\mathscr{G}_-$ acts transitively on Q, which enables us to identify the model surface Q and the Lie group $\mathscr{G}_-$ as CR-varieties. Note also that, for any Φ and Ψ, the component g_0 contains a field of the form

$$X_0 = 2\,\mathrm{Re}\left(z\,\frac{\partial}{\partial z} + 2\,w\,\frac{\partial}{\partial w} + 3\,W\,\frac{\partial}{\partial W}\right),$$

Let

$$\chi = (z \to F(z,w,W),\ w \to G(z,w,W),\ W \to H(z,w,W))$$

be an automorphism of Q. Replacing χ by its composition with an appropriate transformation in $\mathscr{G}_-$, we can assume that χ leaves the origin fixed. Here we use the holomorphic homogeneity of Q. The differential of the automorphism χ takes vector fields in a neighborhood of the origin in $\operatorname{aut} Q$ to vector fields in $\operatorname{aut} Q$. Writing out that vector fields with coordinates (A,B,C) and (R,S,T) are connected by the mapping χ, we obtain

$$\begin{bmatrix} F_z & F_w & F_W \\ G_z & G_w & G_W \\ H_z & H_w & H_W \end{bmatrix}^{-1} \cdot \begin{bmatrix} A(F,G,H) \\ B(F,G,H) \\ C(F,G,H) \end{bmatrix} = \begin{bmatrix} R \\ S \\ T \end{bmatrix} \tag{4}$$

Let $(e_1,\dots,e_n)$ be the standard basis of the space $\mathbf{C}^n$, $(\nu_1,\dots,\nu_k)$ the standard basis of $\mathbf{R}^k$, and $(\mu_1,\dots,\mu_K)$ of $\mathbf{R}^K$. Choosing these values for the parameters defining fields in g_{-1}, g_{-2}, g_{-3}, we obtain fields generating g_-. Let us substitute all these fields into (4) instead of (A,B,C) and write out the result thus obtained in the block-matrix form. We have

$$\begin{bmatrix} F_z & F_w & F_W \\ G_z & G_w & G_W \\ H_z & H_w & H_W \end{bmatrix}^{-1} \cdot \begin{bmatrix} E_n & 0 & 0 \\ 2i\Phi(F,E_n) & E_k & 0 \\ 2i\Psi(F,F,E_n) + \beta(G,E_n) & 0 & E_K \end{bmatrix} = P \tag{5}$$

where (E_n, E_k, E_K) are unit matrices of the corresponding sizes, and P is a matrix of size $N \times N$ composed of the vectors (R,S,T) that are images of vectors in g_-, i.e., its elements are polynomials of degree at most d. By the relation (5), this matrix is nondegenerate and the elements of the inverse matrix $M = P^{-1}$ are rational functions of degree at most $d\,N$. Let

$$M = P^{-1} = \begin{bmatrix} M_1^1 & M_2^1 & M_3^1 \\ M_1^2 & M_2^2 & M_3^2 \\ M_1^3 & M_2^3 & M_3^3 \end{bmatrix}.$$

Represent (5) in the form

$$\begin{bmatrix} F_z & F_w & F_W \\ G_z & G_w & G_W \\ H_z & H_w & H_W \end{bmatrix} = \begin{bmatrix} E_n & 0 & 0 \\ 2i\Phi(F,E_n) & E_k & 0 \\ 2i\Psi(F,F,E_n) + \beta(G,E_n) & 0 & E_K \end{bmatrix} \cdot \begin{bmatrix} M_1^1 & M_2^1 & M_3^1 \\ M_1^2 & M_2^2 & M_3^2 \\ M_1^3 & M_2^3 & M_3^3 \end{bmatrix} \tag{6}$$

From the first block row of this relation, we obtain

$$\operatorname{grad} F = (F_z, F_w, F_W) = (M_1^1, M_2^1, M_3^1),$$

i.e., $\operatorname{grad} F$ is rational, of degree at most $d\,N$.

The degree of a rational function is the maximum of the degrees of the numerator and denominator. The weight (weighted degree) is calculated similarly. For further calculations, note that any arithmetic operations with two rational functions of degrees d_1 and d_2 give a rational function whose degree does not exceed $d_1 + d_2$.

Lemma 1. *Let* $\deg R_1 = d_1$, $\deg R_2 = d_2$; *then* $\deg(R_1 \diamond R_2) \leqslant (d_1 + d_2)$, *where* $\diamond$ *stands for any of the four arithmetic operations. The same is true for the weight of a rational function.*

Substitute the field X_0 into (5) instead of (A,B,C); we obtain

$$\begin{bmatrix} F \\ 2\,G \\ 3\,H \end{bmatrix} = \begin{bmatrix} F_z & F_w & F_W \\ G_z & G_w & G_W \\ H_z & H_w & H_W \end{bmatrix} \begin{bmatrix} R \\ S \\ T \end{bmatrix} \tag{7}$$

The first block coordinate of this relation has the form $F = F_z\, R + F_w\, S + F_W\, T$, whence we see that F is rational, of degree at most $d\,(N+1)$. Returning to (6), from the second block line we obtain

$$G_z = 2i\Phi(F, E_n)M_1^1 + M_1^2,$$
$$G_w = 2i\Phi(F, E_n)M_2^1 + M_2^2,$$
$$G_W = 2i\Phi(F, E_n)M_3^1 + M_3^2.$$

Therefore, $\operatorname{grad} G$ is rational, of degree at most $d\,(3N+1)$. Now the second coordinate (7) implies that G is rational, of degree at most $d\,(3N+2)$. Similarly, from the third block row in (6) we obtain

$$H_z = (2i\Psi(F, F, E_n) + \beta(G, E_n))M_1^1 + M_1^3,$$
$$H_w = (2i\Psi(F, F, E_n) + \beta(G, E_n))M_2^1 + M_2^3,$$
$$H_W = (2i\Psi(F, F, E_n) + \beta(G, E_n))M_3^1 + M_3^3.$$

Whence it follows that $\operatorname{grad} H$ is rational, of degree at most $d\,(7N+6)$. Now it follows from the third coordinate of (7) that H is rational, and the degree does not exceed $7\,d\,(N+1)$.

Thus, every automorphism χ that preserves the origin is a rational mapping of degree at most $7\,d\,(N+1)$. An arbitrary automorphism Q has the form $\eta(\chi)$, where $\eta \in \mathscr{G}_-$ is a quadratic-triangular transformation. Therefore, the final estimate for the degree of an arbitrary automorphism is $14\,d\,(N+1)$.

Lemma 2. *Let $Q \in \mathbf{C}^N$ be a nondegenerate model surface. Then the coefficients of the vector fields forming* $\operatorname{aut} Q$ *are polynomials of degree at most N^3.*

Proof. Let Q be a model surface of CR-dimension n and of codimension κ ($N = n + \kappa$) and $\operatorname{aut} Q = g_{-l} + \cdots + g_0 + \cdots + g_\delta$, i.e., δ is the highest weight. It follows immediately from the results of [5] that the automorphism of Q is uniquely is determined by its $n\,(\kappa+1)$-jet at a point. This implies that the same can be claimed about the coefficients of the fields in $\operatorname{aut} Q$. Since, together with every vector field X, the algebra $\operatorname{aut} Q$ contains each of its graded components, it follows that this algebra cannot contain weight components whose weight exceeds $l\,n\,(\kappa+1)$, i.e., $d \leqslant \delta \leqslant l\,n\,(\kappa+1) \leqslant n\,\kappa\,(\kappa+1) \leqslant N^3$.

In particular, our argument shows that, for a surface Q of type $m = ((2,k),(3,K))$ we have $d \leqslant 3\,n\,(k+K+1)$.

Statement 3. *Let Q be a nondegenerate holomorphically homogeneous model surface of CR-dimension n, of Bloom–Graham type $m = ((2,k),(3,K))$, and of codimension $k+K$; then* $\operatorname{Aut} Q$ *consists of birational transformations of the space $\mathbf{C}^N$, where $N = n+k+K$, whose degree does not exceed*

$$D(n,k,K) \leqslant 42\,n\,(k+K+1)(n+k+K+1) \ \ \textit{or} \ \ D(N) \leqslant \frac{21}{2}\,(N^2-1)(N+3).$$

Let now Q be an arbitrary nondegenerate holomorphically homogeneous model surface. As shown in [7], the set of weights under the condition of homogeneity is an interval of the positive integers $(2, 3, \ldots, l)$. In this case, it is convenient for us to denote the coordinates of the ambient space as follows:

$$(z, w_2, w_3, \ldots, w_l), \quad z \in \mathbf{C}^n,\ w_j = u_j + i\,v_j \in \mathbf{C}^{k_j}.$$

The weights are assigned to the variables as follows: $[z] = [\bar z] = 1$, $[w_j] = [u_j] = j$, $j = 2, \ldots, l$. This convention introduces the grading of power series and vector fields. Now the equations of Q can be represented in the form

$$v_j = \Phi_j(z, \bar z, u_2, \ldots, u_{j-1}), \quad j = 2, \ldots, l \tag{8}$$

where the real vector-valued form Φ_j is homogeneous of weight j and is written in the reduced form (see [7]). Since the algebra $\operatorname{aut} Q = g_- + g_0 + g_+$ is nondegenerate, it is finite-dimensional, finitely graded, and consists of fields with polynomial coefficients of uniformly bounded degree. The condition of holomorphic homogeneity of Q is equivalent to the fact that $\dim g_- = \dim Q$. Let us consider in more detail the structure of the subalgebra $g_- = g_{-l} + g_{-l+1} + \cdots + g_{-2} + g_{-1}$ It consists of fields of the form

$$X = 2\operatorname{Re}\left(f(z, w_2, \ldots, w_l)\,\frac{\partial}{\partial z} + g_2(z, w_2, \ldots, w_l)\,\frac{\partial}{\partial w_2} + \cdots + g_l(z, w_2, \ldots, w_l)\,\frac{\partial}{\partial w_l}\right),$$

satisfying the tangency condition, namely,

$$\operatorname{Im} g_j = d\Phi_j(z,\bar z,u_2,\dots,u_{j-1})(f,\bar f,\operatorname{Re}(g_2),\dots,\operatorname{Re}(g_l)), \quad \text{for} \quad w_j = u_j + i\,\Phi_j(z,\bar z,u_2,\dots,u_{j-1}),\ j=2,\dots,l.$$

Or, simply, in the coordinates, $X = (f(z,w),\ g_2(z,w),\dots,g_l(x,w))$. If $X_j \in g_j$, then we write

$$X_j = (f_j(z,w), g_{2,j}(z,w),\dots,g_{l,j}(z,w)).$$

Here $[f_j] = j+1,\ [g_{\nu,j}] = \nu+j$, and the degrees, respectively, do not exceed the weights. For all $-l \leqslant j \leqslant -2$,

$$f_j(z,w) = g_{j,2}(z,w) = \cdots = g_{j,-j-1}(z,w) = 0,$$
$$g_{j,-j}(z,w) = \beta_j \in \mathbf{R}^{k_j}, \quad g_{j,\nu}(z,w) = g_{j,\nu}(z,w,\beta_j),\ -j+1 \leqslant \nu \leqslant l,$$

where $g_{j,\nu}(z,w,\beta_j)$ is a polynomial in (z,w) of weight $j+\nu$ which is linear in β_j. Correspondingly, for $j=-1$ we have

$$f_{-1} = p \in \mathbf{C}^n, \quad g_{-1,\nu}(z,w) = g_{-1,\nu}(z,w,p), \quad 2 \leqslant \nu \leqslant l,$$

where $g_{-1,\nu}(z,w,p)$ is a polynomial in (z,w), of weight $\nu-1$, which is real linear in p. The fact that the tangency conditions are uniquely solvable with respect to $g_{j,\nu}(z,w,\beta_j)$ for any chosen β_j and with respect to $g_{-1,\nu}(z,w,p)$ for a chosen p is a direct consequence of the holomorphic homogeneity of Q. Note also that, for any Φ_j, the component g_0 contains a field of the form

$$X_0 = 2\operatorname{Re}\left(z\,\frac{\partial}{\partial z} + 2\,w_2\,\frac{\partial}{\partial w_2} + \cdots + l\,w_l\,\frac{\partial}{\partial w_l}\right) \tag{9}$$

Passing to the proof of birationality, we note that we follow the same Kaup–Tumanov scheme which was shown above. Let

$$\chi = (z \to F(z,w_2,\dots,w_l), \quad w_j \to G^j(z,w_2,\dots,w_l)), \quad j=2,\dots,l$$

be an automorphism of Q. Replacing χ by its composition with an appropriate transformation in $\mathscr{G}_-$, we can assume that χ leaves the origin fixed. The differential of the automorphism χ takes any vector fields in a neighborhood of the origin in $\operatorname{aut} Q$ to vector fields in $\operatorname{aut} Q$. Writing out that the vector fields with the coordinates $(A,B_2,\dots,B_l)$ and $(R,S_2,\dots,S_l)$ are related by the mapping χ, we obtain

$$\begin{bmatrix} F_z & F_{w_2} & \dots & F_{w_l} \\ G^2_z & G^2_{w_2} & \dots & G^2_{w_l} \\ \dots & \dots & \dots & \dots \\ G^l_z & G^l_{w_2} & \dots & G^l_{w_l} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A(F,G^2,\dots,G^l) \\ B_2(F,G^2,\dots,G^l) \\ \\ \dots \\ \\ B_l(F,G^2,\dots,G^l) \end{bmatrix} = \begin{bmatrix} R \\ S_2 \\ \dots \\ S_l \end{bmatrix} \tag{10}$$

Here, as above, we use the block-matrix arithmetic, i.e., we write a square matrix of size $N \times N$ as a block $l \times l$ matrix.

Let $e = (e_1,\dots,e_n)$ be the standard basis of the space $\mathbf{C}^n$ and let $\nu^j = (\nu^j_1,\dots,\nu^j_{k_j})$ be the basis of the space $\mathbf{R}^{k_j}$. By choosing the elements of e as values for parameters defining fields in g_{-1} and the elements ν^j for fields in g_{-j}, we obtain fields generating the entire subalgebra g_-. Arranging all such fields in the form of columns of a block square matrix T, we obtain

$$\begin{bmatrix} E_n & 0 & 0 & \dots & 0 \\ g_{-1,2}(F,E_n) & E_{k_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ g_{-1,l-1}(F,G^2,\dots,E_n) & g_{-2,l-1}(F,G^2,\dots,E_{k_2}) & \dots & E_{k_{l-1}} & 0 \\ g_{-1,l}(F,G^2,\dots,G^{l-1},E_n) & g_{-2,l}(F,G^2,\dots,E_{k_2}) & g_{-3,l}(F,G^2,\dots,E_{k_3}) & \dots & E_{k_l} \end{bmatrix}$$

We denote the block element of the matrix M standing at the intersection of the i-th block row and the j-th block column by M^i_j, i.e.,

$$M = P^{-1} = \begin{bmatrix} M^1_1 & M^1_2 & \dots & M^1_l \\ M^2_1 & M^2_2 & \dots & M^2_l \\ \dots & \dots & \dots & \dots \\ M^l_1 & M^l_2 & \dots & M^l_l \end{bmatrix}.$$

Now we write the relations obtained from (10) in the form of a single matrix equality

$$J = T \cdot P^{-1} = T \cdot M, \tag{11}$$

where J is the Jacobian matrix of the mapping χ and P is the matrix composed of the fields of the form $(R, S_2, \dots, S_l)$ that are images of the basic fields in g_{-1}. If the number d is an upper bound for the degrees of fields in $\operatorname{aut} Q$, then the elements of the matrix $M = P^{-1}$ are rational functions whose degrees do not exceed $N\,d$.

Writing out the first block line of the relation (11), we obtain

$$\operatorname{grad} F = (F_z, F_{w_2}, \dots, F_{w_l}) = (M_1^1, M_2^1, \dots, M_l^1), \tag{12}$$

i.e., $\operatorname{grad} F$ is rational, of degree at most $d\,N$. Further, substitute the field $X_0 \in g_0$ into (10) instead of $(A, B_2, \dots, B_l)$ (see (9)); we obtain

$$\begin{bmatrix} F \\ 2\,G^2 \\ \dots \\ l\,G^l \end{bmatrix} = \begin{bmatrix} F_z & F_{w_2} & \dots & F_{w_l} \\ G_z^2 & G_{w_2}^2 & \dots & G_{w_l}^2 \\ \dots & \dots & \dots & \dots \\ G_z^l & G_{w_2}^l & \dots & G_{w_l}^l \end{bmatrix} \begin{bmatrix} R \\ S_2 \\ \dots \\ S_l \end{bmatrix} \tag{13}$$

Writing out the first block coordinate (13), from (12) we conclude that $F = F_z\,R + F_{w_2}\,S_2 + \dots + F_{w_l}\,S_l$. Since $l \leqslant K \leqslant N$, we conclude that $\deg F \leqslant 2\,d\,N^2$.

Now, writing out the second block line (11), we obtain

$$\begin{aligned} G_z^2 &= g_{-1,2}(F, E_n) M_1^1 + M_1^2, \\ G_{w_2}^2 &= g_{-1,2}(F, E_n) M_2^1 + M_2^2, \\ &\dots \\ G_{w_l}^2 &= g_{-1,2}(F, E_n) M_l^1 + M_l^2. \end{aligned}$$

Therefore, $\operatorname{grad} G^2$ is rational and of degree at most $d\,(2\,N^2 + 2\,N)$. Then it follows from the second coordinate (13) that G^2 is rational and of degree at most $d\,(2\,N^2 + 2\,N + 1)\,N$.

The bound for the degree of G^2 is given here for the convenience of the reader. It is included in the total recursive reasoning presented below.

Thus, now we are ready to describe the general $(j+1)$-th stage of this process, namely, a bound for the degree of G^{j+1}. Here we shall give an upper bound for the degree depending only on $N \geqslant 2$, and this bound is not pretending to be accurate. As we have seen, this stage consists of two steps: a bound based on relation (11) for the degree of the gradient G^{j+1} and a bound based on relation (13) for the degree of the component G^{j+1}.

Thus, let d_1 be a bound for the degree of F obtained by us, i.e., $d_1 = d\,\sigma_1 = 2\,d\,N^2$, and let a value $d_j = d\,\sigma_j$, an estimate of the degree of G^j, be obtained at the j-th stage. Consider the the $(j+1)$-th block row of relation (11). The left-hand side of such a relation contains $\operatorname{grad} G^{j+1}$ and the right-hand side of (11) has an expression in the form of a sum in which the number of summands does not exceed $l \leqslant K \leqslant N$. Here every summand has the form of a product of $g_{\nu,j+1}(F, G^2, \dots, E_{k-\nu})$ and of some block element of the matrix M. Taking into account that the degrees of $g_{\nu,j+1}$ are less than l and the degrees of the elements of M are at most $d\,N$, we can claim that the degree $\operatorname{grad} G^{j+1}$ does not exceed $d\,(l\,\sigma_j + N)\,l$. Further, writing out $(j+1)$-th block coordinate (13), we see that G^{j+1} is a sum of length l, where the degrees of the summands do not exceed $d\,(l\,\sigma_j + N)\,l + d$, whence it follows that $d\,\sigma_{j+1} \leqslant d\,l\,((l\,\sigma_j + N)\,l + 1)$. Thus,

$$\sigma_{j+1} \leqslant l\,((l\,\sigma_j + N)\,l + 1) \leqslant N^3\,(\sigma_j + 2) \leqslant 2\,N^3\,\sigma_j.$$

Taking into account that $\sigma_1 = 2\,N^2 \leqslant 2\,N^3$ and that the length of the sequence $(\sigma_1, \sigma_2, \dots)$ does not exceed N, we see that all degrees do not exceed $d\,(2\,N^3)^N$. According to Lemma 2, we have $d \leqslant N^3$. Therefore, the quantity $N^3\,(2\,N^3)^N$ gives a bound for the degree of the automorphism χ. Now we should recall that an arbitrary automorphism of Q can be obtained from χ by a composition with a triangular polynomial transformation in $\mathscr{G}_-$ whose degree does not exceed $l - 1 \leqslant N$. As a result, we obtain the following theorem.

Theorem 4. *The group* $\operatorname{Aut} Q$ *of holomorphic automorphisms of an arbitrary nondegenerate holomorphically homogeneous model surface* $Q \subset \mathbf{C}^N$ *consists of birational transformations of* $\mathbf{C}^N$*, whose maximal degree does not exceed*

$$D(N) \leqslant N^4\,2^N\,(N^N)^3 \leqslant \exp\,(3\,N\,(\ln(N) + 1)).$$

Theorem 4 has two conditions: the nondegeneracy and the holomorphic homogeneity. Here, in turn, the nondegeneracy condition splits into two conditions: the finiteness of the type and the holomorphic nondegeneracy. If Q is a model surface whose Bloom–Graham type is infinite, then it can be given in the form (8) where, among the coordinates of some weighted homogeneous form Φ_j, there are identical zeros. Then, subjecting the corresponding coordinate of the group w to an arbitrary real-analytic transformation (the other coordinates are kept), we obtain an automorphism Q. That is, in this case, the birationality assertion fails to hold. This assertion also becomes invalid if the second nondegeneracy condition is violated, namely, the holomorphic nondegeneracy. Indeed, in this case, the algebra $\operatorname{aut} Q$ contains nonpolynomial fields, which contradicts the birationality of $\operatorname{Aut} Q$.

We claim that the holomorphic homogeneity is also a necessary condition. A corresponding example was considered in [8].

Example 5. Consider a model hypersurface of the space $\mathbf{C}^2$ of the form $Q = \{v = |z|^4\}$. This hypersurface is not holomorphically homogeneous. Its Bloom-Graham type varies from point to point. Let $\xi = (a, b) \in Q$ and let $m(\xi)$ be the type at the point ξ; then

$$m((a,b)) = (4) \text{ if } \; a = 0$$
$$m((a,b)) = (2) \text{ if } \; a \neq 0$$

Here $\operatorname{aut} Q$ contains the field

$$X = \operatorname{Re}\left(w\, z\, \frac{\partial}{\partial z} + 2\, w^2\, \frac{\partial}{\partial w}\right).$$

To this field, there corresponds the 1-parameter subgroup of $\operatorname{Aut} Q$ formed by the transformations of the form

$$z \to \frac{z}{\sqrt{1 - 2\, t\, w}}, \quad w \to \frac{w}{1 - 2\, t\, w},$$

which are not rational.

The model surfaces obviously fall into the class of real algebraic manifolds. Therefore, we can claim that the automorphisms of any nondegenerate model surface, regardless of its holomorphic homogeneity, are *algebraic* (see [9], Theorem 13.1.4).

The automorphism groups of nondegenerate homogeneous model surfaces show examples of subgroups of the group of birational automorphisms of a complex affine space with the condition of the uniform boundedness of degrees. Such groups are of interest regardless of the CR geometry. In the paper [10], a beautiful construction enabling one to construct such groups is given (see Theorem 3). Unfortunately, this construction does not apply in our situation. In this paper, the following condition is imposed on the group of automorphisms. In the space $\mathbf{C}^N$ on which the group acts by birational transformations, there is a chosen domain that is free of the singularities of the mappings. Although the local group $\operatorname{Aut} Q$ consisting of transformations generated by fields in $\operatorname{aut} Q$ consists of holomorphic mappings in a neighborhood of the origin, we still have no guaranteed size of a neighborhood of the origin on which all transformations are holomorphic. On the other hand, the entire Lie group of the automorphisms of Q may also contain automorphisms having singularities at zero. Note here that, by the generation properties of Q, the model surface cannot entirely be contained in the singular set of any birational transformation. Therefore, the birationality assertion (Theorem 4) applies to all automorphisms of the model surface.

In this connection, two questions arise.

First. Describe the class of homogeneous nondegenerate model surfaces such that their automorphism groups satisfy the condition of the paper [10]. As is well known, this condition is satisfied by all quadratic models (their Bloom–Graham type is equal to (2)) with the condition of positive definiteness. Seemingly, in the general case, we are talking about some analog of positive definiteness.

Second. There is a request for some more general construction that could include the automorphism groups of all homogeneous nondegenerate model surfaces.

The question of estimating the degree can be approached in a more differentiated way. For example, we can consider the quantity $D(m)$, which is the maximum of the degrees of automorphisms over all model surfaces of a chosen finite Bloom–Graham type m. Theorem 4 immediately implies that this quantity is finite. Further, we can, at least in the simplest situations, pose the question concerning the exact values of $D(m)$. For example, if we speak about model hyperquadrics ($m = (2)$), then $D((2)) = 1$. If there are quadratic model surfaces of higher codimension $k > 1$ (quadrics), i.e., $m = (2, \dots, 2) = (2, k)$, then $D((2,k)) \geqslant k$. We do not know examples of quadrics with automorphisms whose degree is greater than the codimension.

3. ON THE TOPOLOGICAL STRUCTURE OF THE GROUP $rmAut\,Q_0$

We note right away that, in contrast to the study of the birationality question in the previous section, the holomorphic homogeneity of Q is not assumed in this section.

Let M_ξ be the germ of a nondegenerate *real algebraic* surface; then, as shown in [9], the group $\operatorname{Aut} M_\xi$ consists of algebraic mappings (Theorem 13.1.5) holomorphic in a neighborhood of ξ and has the structure of a Lie group (Theorem 12.7.18). In particular, this result is applicable to any nondegenerate model surface Q.

Thus, let Q be a nondegenerate model surface. Let us consider the following objects:

g_- is the subalgebra of $\operatorname{aut} Q$ consisting of fields of negative weight, and G_- is the connected Lie group corresponding to g_-;

g_0 is the subalgebra of $\operatorname{aut} Q$ consisting of fields of weight zero, and G_0 is the connected Lie group corresponding to g_0;

g_+ is the subalgebra of $\operatorname{aut} Q$, consisting of fields of positive weight, and G_+ is the connected Lie group corresponding to g_+;

St is the stabilizer of the origin in $\operatorname{Aut} Q_0$.

$\mathscr{G}_-$ is the subgroup of triangular polynomial automorphisms of Q described in [7].

$\mathscr{G}_0$ is the subgroup of the automorphisms of Q (it is described in [7]) such that the action on the coordinate z has the form $(z \to C\,z)$, where C is a nondegenerate linear transformation.

$\mathscr{G}_+$ is the subgroup of automorphisms of Q of the form $(z \to z + o(1),\ w_j \to w_j + o(m_j)),\ j = 1, \ldots, l$.

Statement 6. (a) *Every automorphism of Q can be represented in the form $\tau \circ \sigma$, where $\sigma \in St$, $\tau \in \mathscr{G}_-$. Moreover, σ and τ are defined uniquely, and there is a semidirect decomposition $\operatorname{Aut} Q = \mathscr{G}_- \ltimes St$.*

(b) *Any automorphism of Q in St can be represented in the form $L \circ N$, where $L \in \mathscr{G}_0$ and $N \in \mathscr{G}_+$. Moreover, L and N are defined uniquely, and there is a semidirect decomposition $St = \mathscr{G}_0 \ltimes \mathscr{G}_+$.*

(c) *The group $\mathscr{G}_0$ of quasilinear transformations $(z \to C\,z,\ w_j \to \rho_j(w),\ j = 1, \ldots, l)$ has a faithful representation of the form $(z \to C\,z,\ w_j \to \rho_j(w)) \to (z \to C\,z)$ in $GL(n, \mathbf{C})$ and is isomorphic to a real linear algebraic group.*

Proof. The decompositions in (a) and (b) were discussed in [7]. The assertions about the semi-direct product are verified directly. The faithfulness of the representation in (c) is Theorem 5, part (f) in [7]. This completes the proof of the statement.

Theorem 7. (a) *$\mathscr{G}_+ = G_+$; in particular, the Lie group $\mathscr{G}_+$ is connected and simply connected;*

(b) *The Lie group $\mathscr{G}_0$ has finitely many connected components;*

(c) *$\mathscr{G}_- = G_-$; in particular, the Lie group $\mathscr{G}_-$ is connected. If Q is holomorphically homogeneous, then $\mathscr{G}_-$ is simply connected.*

Proof. The recurrent process of calculating the components of $\mathscr{G}_+$ described in [7] (Poincaré's construction) enables us to recover uniquely an element of $\mathscr{G}_+$ by the parameters contained in the kernel of the homological operator. The set of these parameters is a linear space. All parameters belonging to the kernel of the homological operator are realized by elements of G_+. This completes the proof of part (a).

By part (c) of Statement 6, the group $\mathscr{G}_0$ is isomorphic to a real linear algebraic group. Item (b) is proved.

The group $\mathscr{G}_-$ consists of transformations S_ξ that are uniquely determined by the choice of the point $\xi = (a, b_1, \ldots, b_l)$ of the same Bloom–Graham type as that of the origin. The process of constructing S_ξ is described in [7] (see the proof of Theorem 17). In essence, this construction is a process of reducing a surface to the standard form at the point ξ. In the coordinates associated with a point ξ, every coordinate form has the form of a sum of weight components. The form of the leading component coincides with that at the origin, and the lower ones depend on the point ξ. The fact that a point ξ is a point of the same type as the origin means that, in the process of reduction, all components of the lower weights are reduced to zero. Here it is clear from the construction that the component of every weight is reduced separately. Therefore, the transformation $a \to t\,a,\ b_j \to t^{m_j}\,b_j,\ t > 0$, takes ξ to a point of the same Bloom–Graham type. Letting t tend to zero, we see that the set of points of the same type as the origin is connected. This implies the contractibility and, in particular, the connectivity of $\mathscr{G}_-$. If Q is holomorphically homogeneous, then $\mathscr{G}_-$ is equivalent to Q both topologically and as a CR-manifold. In turn, Q is the graph of a polynomial mapping over a linear space. This completes the proof of the theorem.

Corollary 8. *The group $\operatorname{Aut} Q_0$ is contractible on $\mathscr{G}_0$, and therefore the number of connected components and all their homotopy groups coincide.*

Example 9. (a) If $Q = \{v = |z|^2\}$ is the projective sphere in $\mathbf{C}^2$, then $\mathscr{G}_0 = \mathbf{C}^*$ and, correspondingly, $\operatorname{Aut} Q$ is not simply connected.

(b) If $Q = \{\operatorname{Im} w = |z_1|^2 - |z_2|^2\}$ is a hyperquadric in $\mathbf{C}^3$ with indefinite Levy form, then $\mathscr{G}_0$ and, correspondingly, $\operatorname{Aut} Q_0$ have two connected components.

REFERENCES

[1] V. K. Beloshapka, "On holomorphic transformations of a quadric", *Math. USSR-Sb.*, **72**:1 (1991), 189–205.

[2] V. K. Beloshapka, "Universal Models For Real Submanifolds", *Math. Notes*, **75**:4 (2004), 475–488.

[3] W. Kaup, "Einige Bemerkungen uber polynomiale Vektorfelder, Jordanalgebren und die Automorphismen von Siegelschen Gebieten", *Math. Ann.*, **204** (1973), 131–144.

[4] A. E. Tumanov, "Finite-Dimensionality of the Group of CR Automorphisms of a Standard CR Manifold and Proper Holomorphic Mappings of Siegel Domains", *Izv. Akad. Nauk SSSR Ser. Mat.*, **52**:3 (1988), 651–659.

[5] D. Zaitsev, "Germs of Local Automorphisms of Real-Analytic CR-Structures and Analytic Dependence of k-Jets", *Math. Res. Lett.*, **4** (1997), 1–20.

[6] V. K. Beloshapka, "Cubic Model CR-Manifolds without the Assumption of Complete Nondegeneracy", *Russ. J. Math. Phys.*, **25**:2 (2018), 148–157.

[7] V. K. Beloshapka, "CR-Manifolds of Finite Bloom-Graham Type: the Method of Model Surface", *Russ. J. Math. Phys.*, **27**:2 (2020), 155–174.

[8] V. K. Beloshapka, "Autumorphisms of Degenerate Hypersufaces in $\mathbf{C}^2$ and a Dimention Conjecture", *Russ. J. Math. Phys.*, **4**:3 (1997), 393–396.

[9] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "CR Automorphisms of Real Analytic CR in Complex Space", *Comm. Anal. Geom.*, **6** (1998), 291–315.

[10] A. Huckleberry and D. Zaitsev, *Actions of Groups of Birationally Extendible Automorphisms, Geometric Complex Analysis*, Edited by Junjiro Noguchi et al. World Scientific, Singapore, 1995.