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On Simple Solutions of Some Equations of Mathematical Physics

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Abstract. All solutions to the Burgers, Hopf, Helmholtz, Klein–Gordon, sine-Gordon, Schrödinger, and Monge–Ampere equations having analytical complexity one (simple solutions) are described. It turns out that all simple solutions of the Burgers and Hopf equation are represented by elementary functions. An example of a family of solutions of complexity two to the Burgers equation is presented. Simple solutions to the Helmholtz (or Klein–Gordon) equation are expressed in terms of Bessel functions and elementary functions. For the Laplace and wave equations, an explicit description is given for the simple solutions that are expressed in terms of Jacobi elliptic functions. Open problems of the theory of analytic complexity (the analytical spectrum of an equation) are discussed.

Among the questions that arose in connection with the discussion of the 13th Hilbert's problem, much of them remains open [1, 2], despite the existing successes. The notion of analytical complexity [3], developed by the author of the present paper, is related to these questions.

One can act on an analytic function of two variables $z(x,y)$ using functions of one variable. This naturally gives rise to the pseudogroup $\mathcal{G}$ which acts as follows. If $g=(a,b,c)\in\mathcal{G}$, where (a,b,c) are three nonconstant analytic functions of one variable, and $z(x,y)$ is an analytic function of two variables, then

$$(g\circ z)(x,y)=c(z(a(x),b(y))).$$

This pseudogroup plays the fundamental role in problems of measuring the complexity of analytic functions of two variables. From the point of view of the theory of analytical complexity, the functions z and $g\circ z$ are indistinguishable (equivalent).

In terms of this pseudogroup, one can, in particular, formulate a unique property of arithmetic operations. Since all four arithmetic operations are equivalent modulo $\mathcal{G}$, and this property can be formulated in terms of the pseudogroup, it follows that we are talking about a unique property of all functions of this class. By defining this class as the orbit of the function $z=x+y$, we obtain its description in the form $Cl_1=\{z(x,y)=c(a(x)+b(y))\}$. Further, for every analytic function $z(x,y)$, we can consider the pseudosubgroup $Stab(z)=\{g\in\mathcal{G}:(g\circ z)(x,y)=z(x,y)\}$; let $d(z)$ be the dimension of $Stab(z)$, understood as the dimension of the corresponding Lie algebra. The following theorem was proved in [5]. For an arbitrary analytic function $z(x,y)$ depending on both the variables (the partial derivatives are not equal to zero identically), $d(z)$ can take only three values: 0, 1, and 3. Moreover, the dimension of the stabilizer takes the value 3 if and only if the function z belongs to $Cl_1\setminus Cl_0$. This theorem shows that the functions in Cl_1 and only these functions have the maximum internal symmetry. From the point of view of the theory of analytic complexity [3], these are exactly the functions of analytical complexity one. Note that

$$Stab(x+y)=\{a(x)=k\,x+l,\quad b(y)=k\,y+m,\quad c(t)=\frac{t-(l+m)}{k}\}$$

Thus, the analytic functions of the form $z(x,y)=c(a(x)+b(y))$ with nonconstant (a,b,c) and only these functions have the maximum finite-dimensional symmetry. One can also consider this kind of symmetry from the following point of view. The action of the pseudogroup $\mathcal{G}$ can be represented as a composition of actions of two commuting pseudogroups $\mathcal{G}_p$ and $\mathcal{G}_q$. Here $\mathcal{G}_p$ acts on the preimage, on the independent variables (x,y), and $\mathcal{G}_q$ acts on the image, on the dependent variable z,

$$z(x,y)\to p_{ab}\circ z(x,y)=z(a(x),b(y)),\qquad z(x,y)\to q_c\circ z(x,y)=c(z(x,y)).$$

Then $Stab(z)$ (the stabilizer of the function z) is formed by the transformations in $\mathcal{G}=\mathcal{G}_p\cdot\mathcal{G}_q$ that have the property

$$p\circ z=q^{-1}\circ z,$$

i.e., the transformations for which the action of the p-component is inverse to the action of the q-component.

For obvious reasons, for functions of one variable, there is no significant difference between the action on the preimage and on the image. Substantial analogs are possible here only at the level of actions of discrete groups. And they are well known. These are periodic and doubly periodic or automorphic functions.

The notion of analytical complexity, in particular, enables one to look at well known partial differential equations from a new point of view. In view of the noted uniqueness of the class of functions of complexity one, whenever there is interest in any differential equation for functions of two variables, the following question arises.

What is the structure of the solutions of this equation that have complexity one?

Earlier, the answer was obtained for the following equations: Laplace, wave, heat, Liouville, and Korteweg–de Vries. In this paper, we answer this question for the following series of equations: Burgers, Hopf, Helmholtz, Klein–Gordon, sine-Gordon, Schrödinger, and Monge–Ampere. For the Laplace equation and the wave equation, we give also a more explicit and detailed description than that given in [4].

As is known, using the Hopf–Cole transform, one can reduce the Burgers equation to the heat equation. However, despite the fact that all solutions of complexity one to the heat equation are explicitly described (they are expressed through the special function erf, the error function), this does not allow us to obtain directly a description of such solutions for the Burgers equation.

For each of the equations under consideration, the calculation of solutions of complexity one is a consideration of the tree of logical possibilities, which is accompanied by a description of the differential-algebraic calculations performed in the Maple system.

The **Burgers equation** has the form

$$z'_y + z\,z'_x + z''_{xx} = 0 \tag{1}$$

Let $z = c(a(x) + b(y))$, where (a, b, c) are nonconstant analytic functions of one variable. Substituting, we obtain

$$a_1^2\,c_2 + a_1\,c_0\,c_1 + a_2\,c_1 + b_1\,c_1 = 0$$

The subscripts denote the orders of the derivatives. In this notation, the assumption that a, b, and c are nonconstant means that a_1, b_1, and c_1 are not identically zero. We have

$$-\frac{c_2}{c_1} = \frac{a_1c_0 + a_2 + b_1}{{a_1}^2} \tag{2}$$

Since the left-hand side of (2) is a function of $(a(x) + b(y))$, it follows that the right-hand side must belong to the kernel of the operator

$$L = b'(y)\frac{\partial}{\partial x} - a'(x)\frac{\partial}{\partial y} \tag{3}$$

We obtain

$$a_1a_2b_1c_0 + {a_1}^2b_2 - a_1a_3b_1 + 2\,{a_2}^2b_1 + 2\,a_2{b_1}^2 = 0 \tag{4}$$

Case 1. If $a_2 = 0$, then it follows from (4) that $b_2 = 0$, i.e., a and b are linear, i,e,, $a(x)+b(y) = k\,x+l\,y+\gamma$, where k and l are nonzero constants. We seek z in the form $z = c(x + \lambda\,y)$; then it follows from (1) that

$$\lambda\,c'(t) + c(t)\,c'(t) + c''(t) = 0.$$

Solving this equation, we obtain

$$z = 2\,\mu\,\mathrm{th}(\mu\,(x + \lambda\,y + \nu)) - \lambda.$$

Case 2. Let $a_2 \neq 0$. Then it follows from (4) that

$$c_0 = -\frac{{a_1}^2b_2 - a_1a_3b_1 + 2\,{a_2}^2b_1 + 2\,a_2{b_1}^2}{a_1a_2b_1} \tag{5}$$

Applying the operator L to (5), we obtain

$$a_1{}^4a_2b_1b_3 - a_1{}^4a_2b_2{}^2 + a_1{}^3a_3b_1{}^2b_2 + a_1{}^2a_2{}^2b_1{}^2b_2 + \\ a_1{}^2a_2a_4b_1{}^3 - a_1{}^2a_3{}^2b_1{}^3 - 2\,a_1a_2{}^2a_3b_1{}^3 + 2\,a_2{}^4b_1{}^3 + 2\,a_2{}^3b_1{}^4 = 0 \tag{6}$$

This relation enables us to conclude that, if $b_2 = 0$, then a_2 also vanishes, i.e., in our case, not only a_2 but also b_2 do not vanish. Writing out the fact that expression (5) for c_0 agrees with expression (2) for c_2/c_1, we obtain

$$a_1{}^4b_1{}^2b_4a_2 - 4\,a_1{}^4b_1b_2b_3a_2 + 3\,a_1{}^4b_2{}^3a_2 - a_1{}^4b_1{}^2b_2b_3 + a_1{}^4b_1b_2{}^3 + \\ a_1{}^3a_3b_1{}^3b_3 - a_1{}^3a_3b_1{}^2b_2{}^2 + a_1{}^2a_2{}^2b_1{}^3b_3 - a_1{}^2a_2{}^2b_1{}^2b_2{}^2 - a_1{}^2b_1{}^4b_3a_2 - \\ a_1{}^2b_1{}^3b_2{}^2a_2 + 2\,a_1a_2a_3b_1{}^4b_2 - 2\,a_2{}^3b_1{}^4b_2 - 2\,a_2{}^2b_1{}^5b_2 = 0 \tag{7}$$

Note that these relations depend explicitly neither on the independent variables (x, y) nor on the unknown functions (a, b) themselves nd depend on their derivatives only. Using this fact, we lower the differential orders of the relations, passing to the following variables: $a_1 = A,\ a_2 = P(A),\ b_1 = B,\ b_2 = Q(B)$, i.e., the first derivatives are the new independent variables, while the second derivatives are the unknown functions. Then the derivatives of higher orders can be expressed accordingly. For example, $a_3 = P'(A)\,P(A)$. Here, as above, we write $P' = p_1,\ P'' = p_2, ..., Q' = q_1, ...$ Relations (6) and (7) become

$$A^2B^3p_0{}^2p_2 + A^4Bq_1q_0 + A^3B^2p_1q_0 - 2\,AB^3p_0{}^2p_1 - \\ A^4q_0{}^2 + A^2B^2p_0q_0 + 2\,B^4p_0{}^2 + 2\,B^3p_0{}^3 = 0 \tag{8}$$

$$-A^3B^3p_0{}^3p_1p_2 - A^5Bp_0q_1q_0p_1 + A^4B^2p_0{}^2q_0p_2 - A^4B^2p_0q_0p_1{}^2 + A^2B^4p_0{}^3p_2 + \\ A^2B^3p_0{}^4p_2 + 2\,A^2B^3p_0{}^3p_1{}^2 + A^5Bq_0{}^2p_1 + A^5p_0q_0{}^2p_1 + A^4Bp_0{}^2q_1q_0 + \\ A^3B^3p_0q_0p_1 - 4\,AB^4p_0{}^3p_1 - 4\,AB^3p_0{}^4p_1 - A^4Bp_0q_0{}^2 - A^4p_0{}^2q_0{}^2 + \\ A^2B^3p_0{}^2q_0 - A^2B^2p_0{}^3q_0 + 2\,B^5p_0{}^3 + 4\,B^4p_0{}^4 + 2\,B^3p_0{}^5 = 0 \tag{9}$$

Since $q_0 = b_2 \neq 0$, it follows that, expressing q_1 from (9) and writing out that this expression does not depend on A, we obtain

$$\left(-A^4p_2 + 2\,A^2p_0\right)q_0 - A^3Bp_0{}^2p_3 - 2\,A^3Bp_0p_1p_2 + 4\,A^2Bp_0{}^2p_2 + \\ 4\,A^2Bp_0p_1{}^2 - 4\,AB^2p_0p_1 - 12\,ABp_0{}^2p_1 + 8\,B^2p_0{}^2 + 8\,Bp_0{}^3 = 0 \tag{10}$$

An alternative arises. Either $Q = m\,B^2 + n\,B + l$ or $\left(-A^2p_2 + 2\,p_0\right) = 0$.

Case 2.1. Q depends on B quadratically, i.e., $Q = m\,B^2 + n\,B + l$. Let us substitute the expression for Q into (8) and (9). Distinguishing the term which is free of B in (8), we see that $l = 0$. Further, let us distinguish the term at B^5 in (9) and, using it, express p_1 in terms of p_0. Differentiating, we obtain an expression for p_2 in terms of p_0. Substitute these expressions into the term of (8) at B^4; we obtain $A^2m\left(A^2m + 2\,P\,(A)\right) = 0$. We have the following alternative: either $m = 0$ (*Case 2.1.1*) or $P(A) = -m\,A^2/2$ (*Case 2.1.2*).

Case 2.1.1. Distinguishing the coefficient at B^5 in (9), we see that $a_2 = P(A) = 0$, which contradicts the condition of Case 2.

Case 2.1.2. The substitution into (8) and (9) gives a necessary and sufficient condition $m\,n = 0$. However, $m = 0$ means that $a_2 = P(A) = 0$, and therefore $n = 0$ and $Q(B) = m\,B^2$. Writing out (5) in the variables (P, Q), we obtain

$$c_0 = -\frac{-Ap_1p_0B + A^2q_0 + 2\,p_0B^2 + 2\,p_0{}^2B}{Ap_0B}.$$

Substituting our P and Q, we obtain $c_0 = 0$. A contradiction.

Case 2.2. $\left(-A^2p_2 + 2\,p_0\right) = 0$. Solving this equation, we obtain the general solution in the form $P = m\,A^2 + n/A$. Substituting this solution into (8) and separating the term that is free of A, we obtain $n = 0$, i.e., $P = m\,A^2$. Substituting this into (9), separating the term at A^6, and taking into account that $m \neq 0$, we see that either $Q = -m\,B^2$ or $Q = -2\,m\,B^2$.

Case 2.2.1. $Q = -m\,B^2$. Solving the ordinary differential equations thus obtained, we see that

$$a(x) = \ln (x\alpha + \mu)\,\delta, \quad b(y) = -\ln (\beta\,y + \nu)\,\delta.$$

Substituting this into (5), we obtain

$$z = c(a+b) = \frac{(x\alpha + \mu)\,\beta}{\alpha\;(\beta\,y + \nu)}$$

Case 2.2.2. $Q = -2\,m\,B^2$. Substituting $(P(A) = m\,A^2,\; Q(B) = -2\,m\,B^2)$ into the equation for c_0, we see that $c_0 = 0$. A contradiction.

For the sake of completeness, consider also solutions of (1) depending on only one variable. In our terminology, these are functions of complexity zero. Let $z = z(x)$; then (1) gives an ordinary differential equation of the form $z\,z' + z'' = 0$; solving it, we obtain

$$z(x) = \frac{1}{k}\mathrm{th}\left(\frac{x+\alpha}{2\,k}\right).$$

If the solution of (1) does not depend on x, then it is obviously constant.

Thus, we have proved the following theorem.

Theorem 1: (a)*Every solution of the Burgers equation (1) of the form* $z = c(a(x)+b(y))$, *where* z *depends on both the variables, belongs to one of the following two families*;
(I) $z_1 = 2\,\mu\,\mathrm{th}(\mu\,(x + \lambda\,y + \alpha)) - \lambda$,
(II) $z_2(x,y) = \frac{x+\alpha}{y+\beta}$.
(b) *The solutions of the Burgers equation that depend on only one variable have the form*
(III) $z_3(x) = 2\,\mu\,\mathrm{th}\,(\mu(x+\alpha))$,
(IV) $z_4(y) = \beta$.
Here $\lambda\,\mu \neq 0$.
The first family has 3 *parameters, the second one has* 2 *parameters, the third one has* 2 *parameters, and the fourth one has* 1 *parameter.*
Corollary 2 : All solutions of the Burgers equation of complexity at most one are elementary functions.

Are there solutions to the Burgers equation with a complexity greater than one? The answer is positive.

Example 3: Consider the following family of solutions of equation (1):

$$z = \frac{4\,x + 2\,A}{x^2 + A\,x - 2\,y + B}.$$

It can readily be seen that the complexity of this expression does not exceed two [3]. We claim that the complexity is greater than one. To this end, we calculate the value of the differential operator

$$\delta(z) = (\ln(\frac{z'_x}{z'_y}))''_{xy} = -8\,\frac{A + 2\,x}{(A^2 + 2\,Ax + 2\,x^2 - 2\,B + 4\,y)^2}.$$

A function z has the complexity not exceeding 1 if and only if $\delta(z) \equiv 0$ (see [3]). Our calculation shows that this condition fails to hold, and therefore the complexity of z is greater than 1, and thus is equal to 2. Here the complexity is equal to 2 independently of the values of A and B. In particular, the function

$$z = \frac{4\,x}{x^2 - 2\,y}$$

is also a solution of complexity two.

Hopf equation. If we equip equation (1) with a parameter ν which corresponds to the viscosity of the liquid in the hydrodynamical interpretation, i.e., represent the equation in the form

$$z'_y + z\,z'_x - \nu\,z''_{xx} = 0,$$

then, passing to the limit as ν tends to zero, we obtain the Hopf equation

$$z'_y + z\,z'_x = 0 \tag{11}$$

What can be said about solutions to this equation of complexity one?

Proposition 4: Every solution of the form $z = c(a(x) + b(y))$ of the Hopf equation (11) has the form

$$z_1(x, y) = \frac{x+\alpha}{y+\beta} \quad \text{or } z_0 = \gamma.$$

Proof: After the substitution, we obtain

$$c_1\,(c_0 a_1 + b_1) = 0,$$

i.e., $c_0 = -b_1/a_1$. Since c is a function of $a + b$, we can write $L(c) = 0$, i.e.,

$$b_2 {a_1}^2 + {b_1}^2 a_2 = 0.$$

In other way,

$$\frac{a_2}{a_1^2} = -\frac{b_2}{b_1^2} = m,$$

where m is a nonzero constant. Solving the ordinary differential equations, we obtain

$$a(x) + b(y) = \frac{1}{m}\ln(\frac{x+\alpha}{y+\beta}) + \gamma.$$

We can now be sure that we should seek the solution in the form

$$z = c(\frac{x+\alpha}{y+\beta}).$$

Substituting this expression into (11), we see immediately that a unique nonconstant solution is $c(t) = t$.

To complete the proof, note that, if one of the first derivatives of z is equal to zero, then the second one is also equal to zero. This completes the proof of the proposition.

We see that all solutions of complexity one of the Hopf equation coincide with the third the family of solutions of the Burgers equation z_3. The fact that the whole family z_3 turns out to be also a family of solutions of the Hopf equation follows from the fact that the second derivative of z_3 with respect to x is zero. However, our little calculation proves that there are no other solutions of complexity one for the Hopf equation.

Helmholtz equation. There are two differential equations,

$$z_{xx} - z_{yy} + m^2\,z = 0, \quad z_{xx} + z_{yy} + m^2\,z = 0,$$

which are transformed into each other by the transformation $y \to i\,y$, which does not change the complexity. The first equation is known under diverse names: Klein–Gordon, Klein–Gordon–Fock, Klein–Fock, and Schrödinger–Gordon, and the other is called the homogeneous Helmholtz equation. For us, these are two forms of one and the same equation. After the change

$$(x \to m\,x, \;\; y \to m\,y)$$

we may assume that $m = 1$,

$$z_{xx} + z_{yy} + z = 0 \tag{12}$$

Writing out equation (12) for $z = c(a(x) + b(y))$, we obtain

$$c_2(a_1^2 + b_1^2) + c_1(a_2 + b_2) + c_0 = 0 \tag{13}$$

Applying the operator L to (13) (see (3)), we obtain

$$c_2\, L(a_1^2+b_1^2)+c_1\, L(a_2+b_2)=0 \tag{14}$$

Case 1. If

$$L({a_1}^2+{b_1}^2)=2\, a_1\, b_1\, (a_2-b_2)=0,$$

then, taking into account that a_1 and b_1 do not vanish identically, we conclude that $a_2=b_2$, and hence are constant, i.e.,

$$a(x)=\lambda\, x^2+\mu_1\, x+\mu_0, \quad b(y)=\lambda\, y^2+\nu_1\, y+\nu_0.$$

Case 1.1. If $\lambda=0$, then we can seek z in the form $c(\mu\, x+\nu\, y)$. The equation for $c(t)$ becomes

$$(\mu^2+\nu^2)\, c''(t)+c(t)=0.$$

Hence we obtain

$$z=k\, e^{\mu\, x+\nu\, y}+l\, e^{-(\mu\, x+\nu\, y)}, \quad \text{where } \mu^2+\nu^2=-1.$$

Case 1.2. If $\lambda\neq 0$, then we can seek z in the form $z=c((x-\alpha)^2+(y-\beta)^2)$. The equation for $c(t)$ becomes

$$4\, t\, c''(t)+4\, c'(t)+c(t)=0.$$

If $J_\nu(t)$ and $Y_\nu(t)$ are Bessel functions with the value of the parameter equal to ν, then

$$z=k\, J_0\,(\sqrt{(x-\alpha)^2+(y-\beta)^2)})+l\, Y_0\,(\sqrt{(x-\alpha)^2+(y-\beta)^2)}),$$

Case 2. If

$$L({a_1}^2+{b_1}^2)=2\, a_1\, b_1\, (a_2-b_2)\neq 0,$$

the it follows from (14) that $c_2=\varphi\, c_1$, where

$$\varphi=\frac{a_1 b_3-b_1 a_3}{2\, b_1\, a_1\, (a_2-b_2)} \tag{15}$$

Writing out the equation $L(\varphi)=0$, we obtain

$$\begin{aligned} -{a_1}^3 b_4 b_1 a_2+{a_1}^3 b_3 b_2 a_2-{a_1}^3 b_4 b_1 b_2+{a_1}^3 {b_3}^2 b_1+\\ {a_1}^3 b_3 {b_2}^2+{b_1}^3 a_4 a_1 a_2-{b_1}^3 {a_3}^2 a_1+\\ {b_1}^3 a_4 a_1 b_2-{b_1}^3 a_3 {a_2}^2-{b_1}^3 a_3 a_2 b_2=0 \end{aligned} \tag{16}$$

Substituting the relation $c_2=\varphi\, c_1$ into (13), we obtain $c_1\,\psi+c_0=0$, where

$$\psi=(a_1^2+b_1^2)\,\varphi+(a_2+b_2).$$

The relation $L(\psi)=0$ follows from $L(\varphi)=0$. Writing out the consistency condition for φ and ψ, we obtain

$$\begin{aligned} {a_1}^5 {b_3}^2-4\, {a_1}^4 a_3 b_1 b_3+4\, {a_1}^3 {a_2}^2 {b_1}^2+6\, {a_1}^3 {a_2}^2 b_1 b_3-\\ 2\, {a_1}^3 a_2 a_4 {b_1}^2-8\, {a_1}^3 a_2 {b_1}^2 b_2-4\, {a_1}^3 a_2 b_1 b_2 b_3+3\, {a_1}^3 {a_3}^2 {b_1}^2+\\ 2\, {a_1}^3 a_4 {b_1}^2 b_2+4\, {a_1}^3 {b_1}^2 {b_2}^2+{a_1}^3 {b_1}^2 {b_3}^2-2\, {a_1}^3 b_1 {b_2}^2 b_3-\\ 6\, {a_1}^2 a_2 a_3 {b_1}^2 b_2-4\, {a_1}^2 a_3 {b_1}^3 b_3+6\, {a_1}^2 a_3 {b_1}^2 {b_2}^2-2\, a_1 a_2 a_4 {b_1}^4+\\ 3\, a_1 {a_3}^2 {b_1}^4+2\, a_1 a_4 {b_1}^4 b_2+2\, {a_2}^2 a_3 {b_1}^4-2\, a_2 a_3 {b_1}^4 b_2=0 \end{aligned} \tag{17}$$

As in the study of the Burgers equation, we introduce new variables

$$a_1=A,\ a_2=p(A)=p_0,\ a_3=p'(A)\, p(A)=p_1\, p_0, \ldots$$
$$b_1=B,\ b_2=q(B)=q_0,\ b_3=q'(B)\, q(B)=q_1\, q_0, \ldots$$

Then (16) becomes

$$\begin{aligned} A^3\left(q_2q_0+{q_1}^2\right)q_0Bp_0-A^3q_1{q_0}^2p_0-A^3\left(q_2q_0+{q_1}^2\right){q_0}^2B+ \\ A^3{q_1}^2{q_0}^2B+A^3q_1{q_0}^3+B^3\left(p_2p_0+{p_1}^2\right){p_0}^2A-B^3{p_1}^2{p_0}^2A- \\ B^3\left(p_0p_2+{p_1}^2\right)p_0Aq_0-B^3p_1{p_0}^3+B^3p_1{p_0}^2q_0=0 \end{aligned} \tag{18}$$

and (17) becomes

$$\begin{aligned} A^5{q_1}^2{q_0}^2-4\,A^4p_1p_0Bq_1q_0+4\,A^3{p_0}^2B^2+6\,A^3{p_0}^2Bq_1q_0- \\ 2\,A^3{p_0}^2\left(p_0p_2+{p_1}^2\right)B^2-8\,A^3p_0B^2q_0-4\,A^3p_0B{q_0}^2q_1+ \\ 3\,A^3{p_1}^2{p_0}^2B^2+2\,A^3\left(p_0p_2+{p_1}^2\right)p_0B^2q_0+4\,A^3B^2{q_0}^2+ \\ A^3B^2{q_1}^2{q_0}^2-2\,A^3B{q_0}^3q_1-6\,A^2{p_0}^2p_1B^2q_0-4\,A^2p_1p_0B^3q_1q_0+ \\ 6\,A^2p_1p_0B^2{q_0}^2-2\,A{p_0}^2\left(p_0p_2+{p_1}^2\right)B^4+3\,A{p_1}^2{p_0}^2B^4+ \\ 2\,A\left(p_0p_2+{p_1}^2\right)p_0B^4q_0+2\,{p_0}^3p_1B^4-2\,{p_0}^2p_1B^4q_0=0 \end{aligned} \tag{19}$$

A simple analysis of relations (18) and (19) shows that, if $a_2 = p_0 = 0$, then also $b_2 = q_0 = 0$. In the framework of Case 2, this is impossible. Therefore, assume that $p_0 \neq 0$ and $q_0 \neq 0$. If $q_1 = 0$, i.e., q_0 is constant, then, eliminating p_1 and p_2 we see that $p_0 = q_0$. A similar colclusion can be obtained from the assumption that $p_1 = 0$, i.e., in the framework of our case, we have $p_1 \neq 0$ and $q_1 \neq 0$. The coefficient in (18) at the quantity q_2 is $A^3B{q_0}^2\left(p_0-q_0\right)$ This coefficient cannot vanish in the framework of our case. Let us express $q_2\,q_0^2$ and differentiate with respect to A. We obtain

$$\begin{aligned} &-A^4{q_0}^2{q_1}^2p_1+A^2B^2p_3{p_0}^4+2\,A^2B^2{p_0}^3p_2p_1+2\,A^2B^2p_3q_0{p_0}^3 \\ &\quad+6\,A^2B^2q_0{p_0}^2p_2p_1+A^2B^2p_3{q_0}^2{p_0}^2+4\,A^2B^2{q_0}^2p_0p_2p_1+A^2B^2{q_0}^2{p_1}^3 \\ &\quad-3\,AB^2{p_0}^4p_2-2\,AB^2{p_0}^3{p_1}^2-6\,AB^2{p_0}^3p_2q_0-6\,AB^2{p_0}^2{p_1}^2q_0 \\ &\quad-3\,AB^2{p_0}^2p_2{q_0}^2-4\,AB^2p_0{p_1}^2{q_0}^2+3\,B^2{p_0}^4p_1+6\,B^2{p_0}^3p_1q_0+3\,B^2{p_0}^2p_1{q_0}^2=0. \end{aligned} \tag{20}$$

This relation has the form

$$(q'(B)\,q(B))^2 = B^2\,(n(A)\,q(B)^2 + m(A)\,q(B) + l(A)).$$

After differentiating with respect to A, we obtain

$$n'(A)\,q(B)^2 + m'(A)\,q(B) + l'(A) = 0.$$

If q_0 is a constant, then $q_1 = q_2 = 0$; in this case, eliminating p_1 and p_2, one can show that p_0 is constant and is equal to q_0. A similar conclusion holds if $p_0 is constant$. This is impossible in the framework of Case 2, and therefore we assume that both p_0 and q_0 are nonconstant. Hence $(n(A), m(A), l(A))$ are constant. Using (20) to obtain expressions for the coefficients in terms of $p(A)$ and equating these expressions to the values of the constants (n, m, l), we obtain

$$\begin{aligned} &A^2B^2{p_0}^4p_3+2\,A^2B^2{p_0}^3p_2p_1-3\,AB^2{p_0}^4p_2, \\ &\qquad-2\,AB^2{p_0}^3{p_1}^2+3\,B^2{p_0}^4p_1-lA^4p_1=0, \\ &\qquad-2\,A^2B^2{p_0}^3p_3-6\,A^2B^2{p_0}^2p_1p_2+6\,AB^2{p_0}^3p_2, \\ &\qquad+6\,AB^2{p_0}^2{p_1}^2-mA^4p_1-6\,B^2{p_0}^3p_1=0, \\ &A^2B^2{p_0}^2p_3+4\,A^2B^2p_0p_1p_2+A^2B^2{p_1}^3, \\ &\qquad-nA^4p_1-3\,AB^2{p_0}^2p_2-4\,AB^2p_0{p_1}^2+3\,B^2{p_0}^2p_1=0. \end{aligned}$$

Eliminating p_3 and p_2 from these relations, we see that $p(A)$ satisfies the same relation as $q(B)$, with the only replacement of B by A, i.e., we have

$$\begin{aligned} (p'(A)\,p(A))^2 &= A^2\,(n\,p(A)^2 + m\,p(A) + l), \\ (q'(B)\,q(B))^2 &= B^2\,(n\,q(B)^2 + m\,q(B) + l) \end{aligned} \tag{21}$$

Substituting (21) into (18), we see that the necessary conditions(21) turn out to be sufficient for the validity of (18).

Case 2.1. Let $n \neq 0$. Solving the first equation of (21) as an equation with separated variables, after the integration we obtain

$$A + const = \frac{\sqrt{n\,(p\,(A))^2 + m\,p\,(A) + l}}{n} -$$
$$(1/2)\,\frac{m}{n^{3/2}}\ln\left(\frac{m/2 + p\,(A)\,n}{\sqrt{n}} + \sqrt{n\,(p\,(A))^2 + m\,p\,(A) + l}\right)$$

The function $q(B)$ satisfies a similar equation. This implies that $p(A)$ and $q(B)$ are transcendent functions. Using (21), one can eliminate all derivatives from relation (19) and obtain an algebraic polynomial connecting the variables $(p(A), A, q(B), B)$. It follows from the transcendence of p and q that all coefficients of this polynomial are equal to zero. Solving the system of equations thus obtained with respect to (n, m, l), we see that it has solutions for $n = 0$ only. A contradiction.

Case 2.2.1. Suppose now that $n = 0$, $m \neq 0$. Relations (21) become

$$(p'(A)\,p(A))^2 = A^2\,(m\,p(A) + l),$$
$$(q'(B)\,q(B))^2 = B^2\,(m\,q(B) + l) \tag{22}$$

The solutions of these differential equations with separated variables are algebraic functions satisfying the following relations:

$$ep = -9\,A^4 m^4 - 18\,A^2 \alpha\,m^4 + 16\,m^3 p^3 - 9\,\alpha^2 m^4 - 48\,l m^2 p^2 + 64\,l^3$$
$$eq = -9\,B^4 m^4 - 18\,B^2 \beta\,m^4 + 16\,m^3 q^3 - 9\,\beta^2 m^4 - 48\,l m^2 q^2 + 64\,l^3$$

Using the relations $ep = eq = 0$, we can express the derivatives (p_2, q_2, p_1, q_1) using (p_0, q_0) and transform (19) into a polynomial relation connecting $(p_0, q_0, A, B, \alpha, \beta, m, l)$. Calculating the resultant, we exclude the function q_0 from this relation and from the relation $eq = 0$. The resulting relation is a polynomial in B of degree 23 of the form

$$\sum r_j(p_0, A, \alpha,\, \beta,\, m,\, l)\,B^j.$$

Let usequate its coefficients to zero. Eliminate p_0 from the relations $r_2 = 0$ and $ep = 0$; we obtain a relation which is a polynomial in A of degree 12. This polynomial can turn out to be identically zero only for $l = \beta = 0$. It follows from the symmetry $(p, A, \alpha) \leftrightarrow (q, B, \beta)$ that $\alpha = 0$. However, a verification of the solutions thus obtained shows that the validity of (18) and (19) is possible for $m = 0$ only. A contradiction.

Case 2.2.2. Let now $n = 0$, $m = 0$, $l = \lambda^2 \neq 0$. We have

$$p'(A)\,p(A) = \lambda\,A, \quad q'(B)\,q(B) = \pm\lambda\,B.$$

Whence, integrating, we obtain

$$ep = -p_0^2 + \lambda\,A^2 + \alpha^2 = 0, \quad eq = -q_0^2 \pm \lambda\,B^2 + \beta^2 = 0.$$

Case 2.2.2.1. Let the sign at λ be the minus. In this case, expressing the derivatives and substituting them into (19), we obtain

$${q_0}^2{p_0}^3\left(2\,A^2\lambda^2 + 2\,B^2\lambda^2 - 2\,\lambda\,{p_0}^2 + 2\,\lambda\,{q_0}^2 + {p_0}^2 - 2\,p_0 q_0 + {q_0}^2\right) = 0.$$

This implies that $l = \lambda^2 = 0$. A contradiction.

Case 2.2.2.2. Let the sign at λ^2 be the plus. In this case, expressing the derivatives and substituting them into (19), we obtain

$${q_0}^2{p_0}^3\,(p_0 - q_0)^2\,(\lambda + 1) = 0,$$

i.e., $\lambda = -1$. Solving equations (22) in this case, we obtain

$$a(x) = \alpha\,\cos\,(x + \gamma) + \mu, \quad b(y) = \beta\,\cos\,(y + \delta) + \nu.$$

Substituting these a and b into (15), we obtain $c''(t) = 0$, i.e., c is linear.

Finally, we have obtained the following description of the solutions of complexity one.

Theorem 5: *Every solution of the form $z = c(a(x) + b(y))$ of the Helmholtz equation (12), where z depends on both the variables, belongs to one of the following three families*:

$$(I) \quad z_1(x, y) = k\, e^{m\,x+n\,y} + l\, e^{-m\,x-n\,y}, \quad \text{where } m^2 + n^2 = 1, \quad m\,n \neq 0,$$
$$(II) \quad z_2(x, y) = k\ J_0\,(\sqrt{(x-\alpha)^2 + (y-\beta)^2)}) + l\ Y_0\,(\sqrt{(x-\alpha)^2 + (y-\beta)^2)}),$$
$$\text{where } (J_\nu, Y_\nu) \text{ are the Bessel functions},$$
$$(III) \quad z_3(x, y) = k\, cos(x+\alpha) + l\, cos(y+\beta), \quad k\,l \neq 0.$$

The solutions of the Helmholtz equation depending on only one variable are the functions of family (I) with $m\,n = 0$. Moreover, they coincide with family (III) for $k\,l = 0$.

Our consideration can readily be extended to equations of the form

$$z_{xx} + z_{yy} + \varphi(z) = 0 \tag{23}$$

If $\varphi(z) = -sin(z)$, then this equation is called the **sine-Gordon equation**. The part of the consideration related to the functions $a(x)$ and $b(y)$ remains the same. Here two cases are possible: $z_1 = c(\lambda\,x + \mu\,y)$ and $z_2 = c((x-\alpha)^2 + (y-\beta)^2))$. In the first case, the equation for c has the form

$$(\lambda^2 + \mu^2)\, c''(t) = \varphi(c(t)). \tag{24}$$

In the other case, the equation becomes

$$4\,t\,c''(t) + 4\,c'(t) = \varphi(c(t)). \tag{25}$$

These two families exist for an arbitrary choice of φ. The third family is possible for linear functions $c(t)$ only, and therefore these solutions re possible for $\varphi(z) = z + m$ only.

Corollary 6 : *Every solution of the form $z = c(a(x) + b(y))$ of equation (23), where z depends on both the variable. belongs to one o two families*:

$$(I) \quad z_1(x, y) = c(m\,x + n\,y), \quad \text{where } c(t) \text{satisfies equation } (24),$$
$$(II) \quad z_2(x, y) = c((x-\alpha)^2 + (y-\beta)^2)), \quad \text{where } c(t) \text{satisfies equation } (25),$$

If we set $\varphi(z) = z^2$, then, for the first family of functions, $c(t)$ can be expressed using the Weierstrass $\wp$-function. If $\varphi(z) = e^z$, then $c(t)$ is am elementary function (logarithms and tangents) both for the first and the second family).

Schrödinger equation. Consider the equation

$$z_y = z_{xx} + z. \tag{26}$$

Writing out equation (26) for $z = c(a(x) + b(y))$, we obtain

$$c_2 {a_1}^2 + c_1 a_2 - c_1 b_1 + c_0 = 0 \tag{27}$$

Applying the operator L to (27), we obtain

$$2\,b_1 c_2 a_1 a_2 + a_1 c_1 b_2 + b_1 c_1 a_3 = 0 \tag{28}$$

Case 1. If $a_2 = 0$, i.e., a is linear, then it follows from (28) that b is also linear. We seek c in the form $z = c(x + 2\,\mu\,y)$ and obtain $2\,\mu\,c'(t) = c''(t) + c(t)$. The characteristic values are

$$\nu_{1,2}(\mu) = \mu \pm \sqrt{\mu^2 - 1}.$$

Case 1.1. If $\mu = \pm 1$, then $c(t) = k\, e^{\pm t} + l\, t\, e^{\pm t}$. Correspondingly,

$$z = k\, e^{\pm x + 2\, y} + l\, (x \pm 2\, y)\, e^{\pm x + 2\, y}$$

(the signs agree).

Case 1.2. If $\mu \neq \pm 1$, then $c(t) = k\, e^{\nu_1(\mu)\, t} + l\, e^{\nu_2(\mu)\, t}$. Correspondingly,

$$z = k\, e^{\nu_1(\mu)\,(x + 2\,\mu y)} + l\;\, e^{\nu_2(\mu)\,(x + 2\,\mu\, y)}.$$

Case 2. Let now $a_2 \neq 0, b_2 \neq 0$. Then $c_2 = \varphi c_1$, where

$$\varphi = -\frac{a_1 b_2 + a_3 b_1}{2\, a_1 a_2 b_1}.$$

Writing our the relation $L(\varphi) = 0$, we obtain

$$a_1{}^3 b_3 a_2 b_1 - a_1{}^3 b_2{}^2 a_2 + a_3 b_2 a_1{}^2 b_1{}^2 - b_1{}^3 a_4 a_1 a_2 + b_1{}^3 a_3{}^2 a_1 + a_3 b_1{}^3 a_2{}^2 = 0 \tag{29}$$

It follows from (27) that $c_1\, \psi + c_0 = 0$, where $\psi = a_1^2\, \varphi + a_2 - b_1$. The condition $L(\psi) = 0$ coincides with the condition $L(\varphi) = 0$. The consistency condition for φ and ψ becomes

$$\begin{aligned} &\psi'_x + a_1\, \varphi\, \psi\, a_1 \\ &\quad = a_1{}^3 b_2{}^2 + 4\, a_1{}^2 a_3 b_1 b_2 + 4\, a_1 a_2{}^2 b_1{}^2 - 6\, a_1 a_2{}^2 b_1 b_2 - 2\, a_1 a_2 a_4 b_1{}^2 \\ &\quad + 2\, a_1 a_2 b_1{}^2 b_2 + 3\, a_1 a_3{}^2 b_1{}^2 + 2\, a_2 a_3 b_1{}^3 = 0 \end{aligned} \tag{30}$$

Translating the relations thus obtained from the ab- to the pq-notation and taking into account that $a_2 = p_0 \neq 0$, we obtain

$$e_1 = -AB^3 p_0{}^2 p_2 + A^3 q_1 q_0 B + p_1 q_0 A^2 B^2 + p_1 p_0{}^2 B^3 - A^3 q_0{}^2 = 0 \tag{31}$$

$$\begin{aligned} e_3 = q_0{}^2 + \frac{\left(4\, A^2 B p_0 p_1 + 2\, AB^2 p_0 - 6\, AB p_0{}^2\right) q_0}{A^3} + \\ \frac{-2\, AB^2 p_0{}^3 p_2 + A p_1{}^2 p_0{}^2 B^2 + 2\, p_0{}^2 p_1 B^3 + 4\, A p_0{}^2 B^2}{A^3} = 0 \end{aligned} \tag{32}$$

Relation (32) has the form $q_0^2 + m(p_0, p_1, p_2, A, B)\, q_0 + l(p_0, p_1, p_2, A, B) = 0$. Differentiating it with respect to A, we obtain $q_0\, m'_A + l'_A = 0$. There are two possibilities:

Case 2.1. $m'_A = l'_A = 0$. Either relation is linear with respect to B, and therefore we obtain four relations

$$\begin{aligned} A^2 p_0{}^2 p_3 + 2\, A^2 p_0 p_2 p_1 - A^2 p_1{}^3 - 2\, A p_0{}^2 p_2 + A p_0 p_1{}^2 - 4\, A^2 p_1 + 4\, A p_0 &= 0, \\ -A p_0 p_2 - 2\, A p_1{}^2 + 3\, p_0 p_1 &= 0, \\ 2\, A^2 p_0 p_2 + 2\, A^2 p_1{}^2 - 8\, A p_0 p_1 + 6\, p_0{}^2 &= 0, \\ A p_1 - 2\, p_0 &= 0. \end{aligned}$$

We obtain immediately from the last equation that $p(A) = \lambda\, A^2$. Substituting this into each of the remaining equations, we obtain $\lambda = 0$, which does not correspond Case 2.

Case 2.2. $(l'_A/m'_A)'_A = 0$. Separating the coefficients at powers of B in this expression, we obtain

$$
\begin{aligned}
r_0 = {} & 2\,A^5{p_0}^4p_2p_4 - 2\,A^5{p_0}^4{p_3}^2 + 2\,A^5{p_0}^3{p_1}^2p_4 + 4\,A^5{p_0}^3{p_2}^3 \\
& +12\,A^5{p_0}^2{p_1}^3p_3 - 6\,A^5{p_0}^2{p_1}^2{p_2}^2 + 6\,A^5p_0{p_1}^4p_2 - 2\,A^5{p_1}^6 \\
& -8\,A^4{p_0}^4p_1p_4 + 6\,A^4{p_0}^4p_2p_3 - 40\,A^4{p_0}^3{p_1}^2p_3 - 12\,A^4{p_0}^2{p_1}^3p_2 \\
& +6\,A^4p_0{p_1}^5 + 8\,A^5{p_0}^2p_1p_3 - 8\,A^5{p_0}^2{p_2}^2 + 8\,A^5p_0{p_1}^2p_2 \\
& -8\,A^5{p_1}^4 + 6\,A^3{p_0}^5p_4 + 34\,A^3{p_0}^4p_1p_3 + 4\,A^3{p_0}^4{p_2}^2 \\
& +10\,A^3{p_0}^3{p_1}^2p_2 - 6\,A^3{p_0}^2{p_1}^4 - 8\,A^4{p_0}^3p_3 + 24\,A^4p_0{p_1}^3 \\
& -6\,A^2{p_0}^5p_3 - 4\,A^2{p_0}^4p_1p_2 + 2\,A^2{p_0}^3{p_1}^3 - 8\,A^3{p_0}^3p_2 \\
& -24\,A^3{p_0}^2{p_1}^2 + 8\,A^2{p_0}^3p_1 = 0, \\
r_1 = {} & A^4{p_0}^3p_1p_4 - A^4{p_0}^3p_2p_3 + 7\,A^4{p_0}^2{p_1}^2p_3 - 6\,A^4{p_0}^2p_1{p_2}^2 \\
& -2\,A^4p_0{p_1}^3p_2 - 5\,A^4{p_1}^5 - 2\,A^3{p_0}^4p_4 - 8\,A^3{p_0}^3p_1p_3 \\
& +33\,A^3{p_0}^2{p_1}^2p_2 + 22\,A^3p_0{p_1}^4 - 4\,A^4{p_1}^3 - 4\,A^2{p_0}^4p_3 \\
& -48\,A^2{p_0}^3p_1p_2 - 47\,A^2{p_0}^2{p_1}^3 + 4\,A^3{p_0}^2p_2 + 8\,A^3p_0{p_1}^2 \\
& +18\,Ap_0^4p_2 + 48\,A{p_0}^3{p_1}^2 - 4\,A^2{p_0}^2p_1 - 18\,{p_0}^4p_1 = 0, \\
r_2 = {} & -A^3{p_0}^2p_1p_3 + A^3{p_0}^2{p_2}^2 - 4\,A^3p_0{p_1}^2p_2 - 2\,A^3{p_1}^4 \\
& +2\,A^2{p_0}^3p_3 + 11\,A^2{p_0}^2p_1p_2 + 8\,A^2p_0{p_1}^3 - 6\,A{p_0}^3p_2 \\
& -12\,A{p_0}^2{p_1}^2 + 6\,{p_0}^3p_1 = 0.
\end{aligned}
$$

Eliminating p_4 from r_0 and r_1, we obtain the relation $R(p_3, p_2, p_1, p_0, A) = 0$. Eliminating p_3 from R and r_2, we obtain a differential polynomial which factorizes into a product of powers of three factors:

$$
\begin{aligned}
R_1 = {} & Ap_1 - 2\,p_0, \\
R_2 = {} & 2\,A^3p_0{p_1}^2p_2 + 2\,A^3{p_1}^4 - 8\,A^2{p_0}^2p_1p_2 - 7\,A^2p_0{p_1}^3 \\
& +5\,A{p_0}^3p_2 + 10\,A{p_0}^2{p_1}^2 - 5\,{p_0}^3p_1, \\
R_3 = {} & 3\,A^3{p_0}^2{p_2}^2 + 4\,A^3p_0{p_1}^2p_2 + A^3{p_1}^4 - 9\,A^2{p_0}^2p_1p_2 \\
& -11\,A^2p_0{p_1}^3 - 4\,A^3{p_1}^2 + 4\,A{p_0}^3p_2 + 22\,A{p_0}^2{p_1}^2 \\
& +12\,A^2p_0p_1 - 12\,{p_0}^3p_1 - 8\,A{p_0}^2.
\end{aligned}
$$

Case 2.2.1. Let $R_1 = 0$, i.e., $p(A) = \lambda\,A^2$. Then, substituting this relation nto (31) and (32), we see that $\lambda = 0$. A contradiction.

Case 2.2.2. Let $R_2 = 0$. Then, eliminating p_3 from $(R_2)'_A = 0$ and $r_2 = 0$, we obtain

$$
\begin{aligned}
R_4 = {} & 3\,A^4p_0{p_1}^2{p_2}^2 - 3\,A^4{p_1}^4p_2 - 14\,A^3{p_0}^2p_1{p_2}^2 + 27\,A^3p_0{p_1}^3p_2 \\
& -A^3{p_1}^5 + 13\,A^2{p_0}^3{p_2}^2 - 71\,A^2{p_0}^2{p_1}^2p_2 + 7\,A^2p_0{p_1}^4 \\
& +68\,A{p_0}^3p_1p_2 - 11\,A{p_0}^2{p_1}^3 - 20\,{p_0}^4p_2 + 5\,{p_0}^3{p_1}^2 = 0.
\end{aligned}
$$

Eliminating p_2 from $R_2 = 0$ and $R_4 = 0$, we obtain a polynomial which factorizes into three factors,

$$
\begin{aligned}
R_5 &= A\,p_1, \\
R_6 &= Ap_1 - p_0, \\
R_7 &= 2\,A^2{p_1}^2 - 6\,Ap_0p_1 + 5\,{p_0}^2.
\end{aligned}
$$

Case 2.2.2.1. Let $R_5 = A\,p_1 = 0$, i.e., $p(A)$ is constant. Then, substituting this into (31) and (32), we see that $q_0 = 0$. A contradiction.

Case 2.2.2.2. Let $R_6 = Ap_1 - p_0 = 0$; then $p(A) = \lambda\,A$. Substituting this relation into (31) and (32), we see that $q(B) = \mu\,B$ and, further, $\lambda\,\mu = 0$. A contradiction.

Case 2.2.2.3. Let $R_7 = 0$. Eliminating p_2 from $(R_7)'_A = 0$ and $R_2 = 0$, we obtain $R_8(p_1, p_0, A)$. Eliminating p_1 from $R_7 = 0$ and $R_8 = 0$, we obtain $80\,A^8 {p_0}^8 = 0$. A contradiction.

Case 2.2.3. Let $R_3 = 0$; then, eliminating p_3 from $(R_3)'_A = 0$ and $r_2 = 0$, we obtain $R_9(p_2, p_1, p_0, A) = 0$. Eliminating p_2 from $R_9 = 0$ and $R_3 = 0$, we obtain a polynomial for which only two of its irreducible factors can vanish in our case:

$$R_{10} = p_0 p_1 + A,$$
$$R_{11} = 60\,A^6 {p_1}^7 + 828\,A^5 p_0 {p_1}^6 + 784\,A^6 {p_1}^5 - 7341\,A^4 {p_0}^2 {p_1}^5$$
$$-5744\,A^5 p_0 {p_1}^4 + 21554\,A^3 {p_0}^3 {p_1}^4 + 16896\,A^4 {p_0}^2 {p_1}^3 - 30350\,A^2 {p_0}^4 {p_1}^3$$
$$-24816\,A^3 {p_0}^3 {p_1}^2 + 20968\,A {p_0}^5 {p_1}^2 - 48\,A^4 {p_0}^2 p_1 + 18088\,A^2 {p_0}^4 p_1$$
$$-5712\,{p_0}^6 p_1 + 96\,A^3 {p_0}^3 - 5168\,A {p_0}^5.$$

Case 2.2.3.1. Let $R_{10} = 0$; then $p(A) = \sqrt{\lambda^2 - A^2}$. Eliminating p_2 from $(R_{10})'_A = 0$ and $R_3 = 0$, we obtain $R_{12}(p_1, p_0, A) = 0$. Eliminating p_1 from $R_{11} = 0$ and $R_{12} = 0$, we obtain the relation $R_{13}(p_0, A) = 0$ which, after substitution the value $p_0 = \sqrt{\lambda^2 - A^2}$, becomes $18630\,A^{31} - 270081\,\lambda^2\,A^{29} \ldots = 0$. A contradiction.

Case 2.2.3.2. Let $R_{11} = 0$. Eliminating p_2 from $(R_{11})'_A = 0$ and $R_3 = 0$, we obtain $R_{14}(p_1, p_0, A) = 0$. Eliminating p_1 from $R_{11} = 0$ and $R_{14} = 0$, we obtain the relation $R_{15}(p_0, A) = 0$ which, after removing the nonzero factor $3735465674926184202\,2400\,A^{91}\,p_0^{32}$, is a homogeneous polynomial with respect to the variables p_0 and A of degree 78. This polynomial factorizes into factors of the form $(p_0 - \lambda_j\,A)$. Thus, its vanishing is possible only for $p(A) = \lambda\,A$, and this possibility was excluded above.

Thus, we have proved the following theorem.

Theorem 7: *Every solution of the form $z = c(a(x) + b(y))$ of the Schrödinger equation (26), where z depends on both the variables, belongs to one of the following two families of elementary functions:*

$$(I) \quad z_1(x,y) = k\,e^{\nu_1(\mu)(x+2\,\mu\,y)} + l\;e^{\nu_2(\mu)(x+2\,\mu\,y)},\ \nu_{1,2}(\mu) = \mu \pm \sqrt{\mu^2 - 1},\ \mu \neq \pm 1,$$
$$(II) \quad z_1(x,y) = k\,e^{\pm\,x+2\,y} + l\,(x \pm 2\,y)\,e^{\pm\,x+2\,y},\ \text{the signs agree,}$$

and at least one of the parameters k and l is nonzero.

The solutions depending on x only have the form $z(x) = k\,e^{ix} + l\,e^{-ix}$; the solutions depending on y only have the form $z(y) = k\,e^y$.

The Homogeneous Monge–Ampere Equation is

$$z''_{xx}\,z''_{yy} - (z_{xy})^2 = 0, \tag{33}$$

Substituting $z = c(a(x) + b(y))$, we obtain

$${a_1}^2 b_2 c_2 + a_2 {b_1}^2 c_2 + a_2 b_2 c_1 = 0 \tag{34}$$

where $c_1 \neq 0$, $a_1 \neq 0$, $b_1 \neq 0$.

Case 1. Let $c_2 = 0$, i.e., $c(t)$ is linear; then $a_2\,b_2 = 0$, i.e., one of these functions is also linear and the other is arbitrary, i.e., $z_1 = k\,x + b(y),\ z_2 = a(x) + l\,y$.

Case 2. Let $c_2 \neq 0$. If here $a_2 = 0$, then $b_2 = 0$, and conversely. Thus, both the functions are linear and $z = c(k\,x + l\,y)$, where c is arbitrary.

Case 3. Let $c_2 \neq 0$, $a_2 \neq 0, b_2 \neq 0$; then

$$\frac{c_2}{c_1} = -\frac{a_2 b_2}{{a_1}^2 b_2 + a_2 {b_1}^2}.$$

Applying the operator L, we obtain

$$-a_1 a_3 {b_2}^2 + {a_2}^2 b_1 b_3 = 0 \text{ or } \frac{b_3\, b_1}{b_2^2} = \frac{a_3\, a_1}{a_2^2} = \lambda = const.$$

Case 3.1. Solving, for $\lambda \neq 0;\ 1;\ 2$, the differential equation

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}t^3} f(t)\right) \frac{\mathrm{d}}{\mathrm{d}t} f(t) - \lambda \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)\right)^2 = 0,$$

we obtain

$$a(x) + b(y) = \mu\, (x+\alpha)^{\frac{\lambda-2}{\lambda-1}} + \nu\, (y+\beta)^{\frac{\lambda-2}{\lambda-1}}$$

(we include the additive constant into $c(t)$). The equation for $c(t)$ becomes

$$\frac{c''(t)}{c'(t)} = \frac{1}{(\lambda-2)\, t}.$$

Solving it, we obtain

$$c(t) = \gamma + \delta\, t^{\frac{\lambda-1}{\lambda-2}}.$$

Finally,

$$z = \gamma + \left(\nu\, (y+\beta)^{\frac{\lambda-2}{\lambda-1}} + \mu\, (x+\alpha)^{\frac{\lambda-2}{\lambda-1}}\right)^{\frac{\lambda-1}{\lambda-2}}.$$

Case 3.2. If $\lambda = 0$, then $a_3 = b_3 = 0$, i.e.,

$$a(x) + b(y) = k\,(x-\alpha)^2 + l\,(y-\beta)^2.$$

Here the equation for $c(t)$ becomes

$$\frac{c''(t)}{c'(t)} + \frac{1}{2\,t} = 0,$$

and its solution has the form $c(t) = \gamma + \delta\,\sqrt{t}$. Finally, we obtain

$$z = \gamma + \delta\,\sqrt{k\,(x-\alpha)^2 + l\,(y-\beta)^2}.$$

Case 3.3. If $\lambda = 1$, then, solving the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) \frac{\mathrm{d}^3}{\mathrm{d}t^3} f(t) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)\right)^2,$$

we obtain

$$f(t) = \frac{\mathrm{e}^{c_2\, c_1} \mathrm{e}^{t\, c_1}}{c_1} + c_3,$$

and therefore

$$a(x) + b(y) = \frac{\mathrm{e}^{c_2 c_1} \mathrm{e}^{x c_1}}{c_1} + c_3 + \frac{\mathrm{e}^{c_5 c_4} \mathrm{e}^{y c_4}}{c_4} + c_6.$$

We can remove the additive constants, and the equation for $c(t)$ becomes $\frac{c''(t)}{c'(t)} + \frac{1}{t} = 0$. Its solution is $c(t) = \mu\, \ln(t) + \gamma$, and we obtain

$$z = \delta\, \ln(\mu\, e^{\alpha x} + \nu\, e^{\beta y}).$$

Case 3.4. If $\lambda = 2$, then, solving the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right) \frac{\mathrm{d}^3}{\mathrm{d}t^3} f(t) = 2\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)\right)^2,$$

we obtain

$$a(x) + b(y) = \mu \ln(\alpha + x) + \nu \ln(\beta + y).$$

Here $c(t) = C_1 + C_2\, e^{\frac{t}{\mu+\nu}}$, and therefore

$$z = \delta\ (\alpha + x)^{\mu} (\beta + y)^{1-\mu} + \gamma$$

Theorem 8: *Every solution of the form $z = c(a(x) + b(y))$ of the Monge–Ampere equation (34), where z depends on both the variables, belongs to one of the following families of functions*:

$$(I)\quad z_{11} = k\,x + b(y),\quad z_{12} = a(x) + l\,y,$$
$$(II)\quad z_2 = c(k\,x + l\,y),$$
$$(III)\quad z_3 = \left(\mu\,(x-\alpha)^{\lambda} + \nu\,(y-\beta)^{\lambda}\right)^{\frac{1}{\lambda}} + \gamma,\quad \lambda \neq 0,\ 1$$
$$(IV)\quad z_4 = \delta\,\ln(\mu\, e^{kx} + \nu\, e^{ly}),$$
$$(V)\quad z_5 = \delta\ (x-\alpha)^{\mu} (y-\beta)^{\nu} + \gamma,\quad \mu + \nu = 1.$$

where $k\,l\,\mu\,\nu\,\delta \neq 0$, $(a(x),\ b(y),\ c(t))$ are arbitrary nonconstant functions.

These families have the following dimensions: (I) ∞, (II) ∞, (III) 6, (IV) 5, (V) 5. Note also that any function of one variable is a solution to the homogeneous Monge–Ampere equation (34).

The Laplace equation

$$z''_{xx} + z''_{yy} = 0, \tag{35}$$

and also the wave equation $z''_{xx} - z''_{yy} = 0$ related to the Laplace equation by the substitution $y \to i\,y$ were previously considered in [4]. However, the description of the solutions of complexity one given there was insufficiently explicit, and therefore we return to these equations here. From these two equations we choose the Laplace equation for definiteness, and construct a procedure for calculating in another way)not as in [4]). Here we pose the question differently. For what f and g, the general solution to the Laplace equation $z = f(x + i\,y) + g(x - i\,y)$ has the complexity one?

Case 0. If z depends on only one variable, then this means hat $z = k\,x + l$ or $z = k\,y + l$.

Case 1. Note further that, if one of the functions is constant, then, for every choice of the other function, z has the complexity not exceeding one ("holomorphic" and "antiholomorphic" functions).

Case 2. Let

$$f'(x + i\,y) \neq 0,\ g'(x - i\,y) \neq 0,\ f'(x + i\,y) + g'(x - i\,y) \neq 0,\ f'(x + i\,y) - g'(x - i\,y) \neq 0.$$

Let us write out the equation of the first class for $z = f + g$; we obtain

$$(\ln(\frac{z'_x}{z'_y}))''_{xy} = (\ln(\frac{f'(x + i\,y) + g'(x - i\,y)}{i\,f'(x + i\,y) - i\,g'(x - i\,y)}))''_{xy} = 0.$$

This is a fraction whose numerator gives

$${f_1}^3 g_3 + {f_1}^2 f_3 g_1 - 2\,f_1 {f_2}^2 g_1 - f_1 {g_1}^2 g_3 + 2\,f_1 g_1 {g_2}^2 - f_3 {g_1}^3 = 0, \tag{36}$$

and the denominator $(f_1 + g_1)^2\,(f_1 - g_1)^2$ is nonzero. Assume now that $f'(x + i\,y) = A$ is the independent variable and $(f''(x + i\,y))^2 = p(A)$ is the function. Similarly, $g'(x - i\,y) = B$ and $(g''(x - i\,y))^2 = q(B)$. Then (36) becomes

$$A^3 \frac{\mathrm{d}}{\mathrm{d}B} q\,(B) + A^2 B \frac{\mathrm{d}}{\mathrm{d}A} p\,(A) - 4\,Ap\,(A)\,B - AB^2 \frac{\mathrm{d}}{\mathrm{d}B} q\,(B)$$
$$+4\,ABq\,(B) - B^3 \frac{\mathrm{d}}{\mathrm{d}A} p\,(A) = 0. \tag{37}$$

Choosing the values $(B \neq 0,\ q(B),\ q'(B))$ in (37), we obtain an ordinary linear differential equation for $p(A)$. Its general solution has the form

$$\left(A^2 - B^2\right)^2 C + (1/2)\,\frac{A^2 q_1 - B^2 q_1 + 2\,B q_0}{B},$$

where C is an arbitrary constant. Thus, we can write that $p(A) = N\,A^4 + M\,A^2 + L$. Proceeding in the similar way, we obtain $q(B) = n\,B^4 + m\,B^2 + l$. Substituting these expressions into (37), we see that $N = n,\ M = m,\ L = l$ and, for every (n, m, l), this is sufficient for the validity of (37).

Case 2.1. If $n = m = l = 0$, then $f'' = g'' = 0$; here $z = \alpha\,x + \beta\,y + \gamma$.

Case 2.2. If $n = m = 0,\ l = \lambda^2 \neq 0$, then $f'' = g'' = \lambda$; here $f(t) = (\lambda/2)\,t^2 + c_1\,t + c_0$ ($g(t) = (\lambda/2)\,t^2 + c_3\,t + c_4$, respectively). Here $z = f(x + i\,y) + g(x - i\,y) = \lambda((x+\alpha)^2 - (y+\beta)^2) + \gamma$.

Case 2.3.1. If $n = 0,\ m = \mu^2 \neq 0, l = 0$, then, solving the equation $f''(t) = \mu\,(f'(t))$, we obtain $f(t) = c_1\,e^{\mu\,t} + C_1$. We btain similarly $g(t) = c_2\,e^{\mu\,t} + C_2$. Finally,

$$z = f(x + i\,y) + g(x - i\,y) = c_1\,e^{\mu\,(x+i\,y)} + c_2\,e^{\mu\,(x-i\,y)} + c_3$$

Case 2.3.2. If $n = 0,\ m = \mu^2 \neq 0, l = \nu^2 \neq 0$, then, solving the solution $f''(t) = \sqrt{\mu^2\,(f'(t))^2 + \nu^2}$, we obtain

$$f(t) = (1/2)\,\frac{\nu^2}{\mu^2 c_1 \mathrm{e}^{\mu\,t}} + (1/2)\,\frac{c_1 \mathrm{e}^{\mu\,t}}{\mu^2} + c_2.$$

Solving the equation for g, we obtain, respectively,

$$\begin{aligned} z &= f(x + i\,y) + g(x - i\,y) \\ &= (1/2)\,\frac{\nu^2}{\mu^2 \alpha\,\mathrm{e}^{\mu\,(x+iy)}} + (1/2)\,\frac{\alpha\,\mathrm{e}^{\mu\,(x+iy)}}{\mu^2} + (1/2)\,\frac{\nu^2}{\mu^2 \beta\,\mathrm{e}^{\mu\,(x-iy)}} + (1/2)\,\frac{\beta\,\mathrm{e}^{\mu\,(x-iy)}}{\mu^2} + \gamma. \end{aligned}$$

Case 2.4. Let now $n = \delta^2 \neq 0$; then $n\,A^4 + m\,A^2 + l = \delta^2\,(A^2 - \mu^2)(A^2 - \nu^2)$, i.e., the equations for $f(t)$ and $g(t)$ satisfy one and the same equation of the form

$$f''(t) = \delta\,\sqrt{(f'(t)^2 - \mu^2)\,(f'(t)^2 - \nu^2)}. \tag{38}$$

Case 2.4.1. Let $\mu = \nu = 0$; then, solving the equation $f''(t) = \delta\,(f'(t))^2$, we obtain

$$z = -\frac{1}{\delta}\ln((x + i\,y) + c_1) - \frac{1}{\delta}\ln((x - i\,y) + c_2) + c_3 = \lambda\,(\ln((x+\alpha)^2 + (y+\beta)^2)) + \gamma.$$

Case 2.4.2. Let $\mu \neq 0,\ \nu = 0$; then, solving equations (38), we obtain

$$z = \frac{2}{\delta\,i}\left(\arctan\left((1/2)\,\frac{\mathrm{e}^{i(i\delta\,y - \delta\,x - c_2)\mu}}{\mu}\right) + \arctan\left((1/2)\,\frac{\mathrm{e}^{-i(\delta\,x + c_1 + i\delta\,y)\mu}}{\mu}\right)\right) + c_3.$$

Case 2.4.3. Let $\mu^2 = \nu^2 \neq 0$. Solving equation (38), we obtain

$$f(t) = -\frac{\ln\left(-1 + \mathrm{e}^{2\,t\delta\,\mu + 2\,\delta\,\mu\,c_1}\right)}{\delta} + (1/2)\,\frac{\ln\left(\mathrm{e}^{2\,t\delta\,\mu + 2\,\delta\,\mu\,c_1}\right)}{\delta} + c_2.$$

Then

$$\begin{aligned} z &= f(x + i\,y) + g(x - i\,y) \\ &= \frac{1}{\delta}\,\ln\left(\frac{\mathrm{e}^{\delta\,\mu\,(\alpha+\beta+2\,x)}}{\left(-1 + \mathrm{e}^{2\,\delta\,\mu\,(iy+x+\alpha)}\right)\left(-1 + \mathrm{e}^{-2\,\delta\,\mu\,(iy-\beta-x)}\right)}\right) + \gamma. \end{aligned}$$

Case 2.4.4. Let now $\mu \neq 0,\ \nu \neq 0,\ \mu^2 \neq \nu^2$; then we can write

$$n\,A^4 + m\,A^2 + l = \delta^2\,\mu^2\,\nu^2\,(1 - \frac{A^2}{\mu^2})\,(1 - k^2\,\frac{A^2}{\mu^2}),$$

where $k^2 = \mu^2/\nu^2 \neq 1$. If $F(t) = f'(t)$, then F satisfies the equation

$$F'(t) = \delta\,\mu\,\nu\,\sqrt{(1 - \frac{F(t)^2}{\mu^2})\,(1 - k^2\,\frac{F(t)^2}{\mu^2})}.$$

Let $\Phi(t) = F(t/(\delta\nu))/\mu$; then Φ satisfies the equation

$$\Phi'(t) = \sqrt{(1 - \Phi(t)^2)\,(1 - k^2\,\Phi(t)^2)}.$$

This implies that $\Phi(t) = \mathrm{sn}(t+c_1, k)$, where sn is the Jacobi elliptic sine. Thus, $f'(t) = F(t) = \mu\,\mathrm{sn}(\delta\,\nu\,t+c_1)$. Since the antiderivative of $\mathrm{sn}\,(t,k)$ has the form

$$\int \mathrm{sn}(t,k)\,dt = \frac{1}{k}\,\ln(\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k)) + c_2,$$

where dn and cn are also well-known Jacobi elliptic functions, we obtain

$$f(t) = \frac{\mu\,\ln\left(\mathrm{dn}\left(\delta\,\nu\,t + c_1, k\right) - k\,\mathrm{cn}\left(\delta\,\nu\,t + c_1, k\right)\right)}{\delta\,\nu\,k} + c_2$$

As a result,

$$\begin{aligned} z &= f(x + i\,y) + g(x - i\,y) \\ &= \frac{\mu\,\ln\left(\mathrm{dn}\left(\delta\,\nu\,\,(x+iy) + c_1, k\right) - k\mathrm{cn}\left(\delta\,\nu\,\,(x+iy) + c_1, k\right)\right)}{\delta\,\nu\,k} \\ &\quad + \frac{\mu\,\ln\left(\mathrm{dn}\left(\delta\,\nu\,\,(x-iy) + c_2, k\right) - k\mathrm{cn}\left(\delta\,\nu\,\,(x-iy) + c_2, k\right)\right)}{\delta\,\nu\,k} + c_3. \end{aligned}$$

Thus, the description of harmonic functions of complexity one looks as follows.

Theorem 9: Every solution of the form $z = c(a(x) + b(y))$ of the Laplace equation (35) belongs to one of the following families:

(I) $z_{11} = k\,x + l$, or $z_{12} = k\,y + l, \quad k \neq 0$,

(II) $z_{21} = g(x + i\,y)$ or $z_{22} = f(x - i\,y)$, $f'\,g' \neq 0$,

(III) $z_3 = \alpha\,x + \beta\,y + \gamma, \quad \alpha\,\beta \neq 0, \quad \beta \neq \pm i\,\alpha$,

(IV) $z_4 = \lambda((x+\alpha)^2 - (y+\beta)^2) + \gamma, \quad \lambda \neq 0$,

(V) $z_5 = \alpha\,e^{\mu\,(x+i\,y)} + \beta\,e^{\mu\,(x-i\,y)} + \gamma$, α or $\beta \neq 0, \quad \mu \neq 0$,

(VI) $z_6 = \lambda\,\ln((x+\alpha)^2 + (y+\beta)^2) + \gamma, \quad \lambda \neq 0$,

(VII) $z_7 = \frac{\nu^2}{2\,\mu^2\alpha\,\mathrm{e}^{\mu\,(x+iy)}} + \frac{\alpha\,\mathrm{e}^{\mu\,(x+iy)}}{2\,\mu^2}$
$+\frac{\nu^2}{2\,\mu^2\beta\,\mathrm{e}^{\mu\,(x-iy)}} + \frac{\beta\,\mathrm{e}^{\mu\,(x-iy)}}{2\,\mu^2} + \gamma, \quad \alpha\,\beta\,\mu\,\nu \neq 0$,

(VIII) $z_8 = \frac{2}{\delta\,i}\left(\mathrm{arctg}\left(\frac{\mathrm{e}^{i(i\delta\,y - \delta\,x - \beta)\mu}}{2\,\mu}\right) + \mathrm{arctg}\left(\frac{\mathrm{e}^{-i(\delta\,x + i\delta\,y + \alpha)\mu}}{2\,\mu}\right)\right) + \gamma, \quad \delta\,\mu \neq 0$,

(IX) $z_9 = \frac{1}{\delta}\,\ln\left(\frac{\mathrm{e}^{\delta\,\mu\,(\alpha+\beta+2\,x)}}{\left(-1+\mathrm{e}^{2\,\delta\,\mu\,(iy+x+\alpha)}\right)\left(-1+\mathrm{e}^{-2\,\delta\,\mu\,(iy-\beta-x)}\right)}\right) + \gamma, \quad \delta\,\mu \neq 0$,

(X) $z_{10} = \frac{1}{\delta}\,\ln\left(\nu\,\mathrm{dn}\left(\delta\,\nu\,\,(x+iy) + \alpha, (\mu/\nu)\right) - \mu\,\mathrm{cn}\left(\delta\,\nu\,\,(x+iy) + \alpha, (\mu/\nu)\right)\right) + \frac{1}{\delta}\,\ln\left(\nu\,\mathrm{dn}\left(\delta\,\nu\,\,(x-iy) + \beta, (\mu/\nu)\right) - \mu\,\mathrm{cn}\left(\delta\,\nu\right.\right.$
$\gamma, \quad \delta\,\nu \neq 0$.

Here the dimensions of these ten families are as follows: (I) 2, (II) ∞, (III) 3, (IV) 4, (V) 4, (VI) 4, (VII) 5, (VIII) 5, (IX) 5, and (X) 6.

Recall that the transformation $(y \to i\,y)$ takes the functions of the ten families described above to solutions of the wave equation whose complexity does nor exceed one.

To every differential relation for a function of two variables we can assign a sequence $\{d_n\}$ (the *analytic spectrum*), where d_n stands for the maximum of the dimensions of the families of solutions of this equation consisting of solutions of the analytic complexity n. Here n takes the values $0, 1, \ldots, \infty$, as well as d_n. Recall that the Cauchy–Kovalevskaya theorem enables us to claim that

$$\sum_{n=0}^{\infty} d_n = \infty.$$

If we sum up the data we have concerning some equations, then we obtain:

$$\begin{aligned}
&\text{the Burgers equation:} \quad d_0 = 2,\ d_1 = 4, d_2 \geqslant 2;\\
&\text{the Hopf equation:} \quad d_0 = 1,\ d_1 = 2;\\
&\text{the heat equation:} \quad d_0 = 2,\ d_1 = 4;\\
&\text{the Schrödinger equation:} \quad d_0 = 2,\ d_1 = 3;\\
&\text{the Helmholtz, Klein–Gordon, sine-Gordon equations, \ldots:} \quad d_0 = 2, d_1 = 4;\\
&\text{the Liouville equation:} \quad d_0 = 0,\ d_1 = 6, d_2 = \infty, d_3 = ... = d_\infty = 0;\\
&\text{the Laplace and wave equations:} \quad d_0 = 2, d_1 = \infty, d_2 = \infty, d_3 = .. = d_\infty = 0;\\
&\text{the Monge–Ampere equation:} \quad d_0 = \infty, d_1 = \infty.
\end{aligned}$$

It is not difficult to show that d_∞ can take only two values: either zero or infinity. Recently, with the help of very witty arguments, M. Stepanova [7] managed to prove that $d_\infty = \infty$ for the heat equation. In the same paper, specific examples of solutions to the heat equation of infinite complexity were constructed.

If, for an equation, we take the equation of the first class,

$$\delta_1(z) = z'_x z'_y (z'''_{xxy} z'_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0,$$

then we shall see that its analytic spectrum has the form $(d_0 = \infty, d_1 = \infty, d_2 = d_3 =, ... = d_\infty = 0)$. If, for an equation, we take an equation of some complexity class and Cl_n, where $n > 1$, then the picture is similar, i.e., all $d_0 = d_1 = \cdots = d_n = \infty$ and $d_{n+1} = \cdots = 0$. However, it should be kept in mind that that this is a system of high-order differential polynomials rather than a single equation. For example, for $n = 2$, the first differential conditions arise in the 11-jet, and the number of equations is not less than 3. Moreover, for a system of equations, i.e., for the case in which there are more than one relation, the sum of the dimensions, i.e., the dimension of the complete solution space, is finite as a rule.

In a more complete form than that of an analytic spectrum, one can represent the information on the interaction of equations and the hierarchy of complexity classes as a kind of a topological space, the (*stratified solution space*). This space, stratified with respect to classes, parameterizes the families of solutions of a given complexity . This space naturally arises when using the topology of the corresponding jets (the number of the jet is the differential order of the family).

Finally, a few questions.

Questions 10: (a) *Is it possible that the sequence $\{d_n\}$ constructed for an irreducible differential polynomial contains a nonzero finite number after infinity?*
(b) *Is it possible that, in this sequence $\{d_n\}$, there are two infinities separated by finite values?*
(c) *Consider the class of irreducible differential polynomials of the differential order p for which $d_1 < \infty$. Is this value bounded on the class? If the answer is 'yes,' then what are differential polynomials on which the maximum is attained?*
(d) *Is the stratified solution space simply connected?*

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