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CR-Manifolds of Finite Bloom–Graham Type: the Method of Model Surface

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Abstract. In the paper, the method of model surface is applied to arbitrary CR-manifolds of finite Bloom–Graham type. A family of basic assertions is proved. It is proved also that, for a model surface, the condition that the Bloom–Graham type is constant is a criterion for the holomorphic homogeneity. Distinctions from the case of rigid models treated earlier are clarifies. A series of questions and conjectures is formulated.

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1. INTRODUCTION

At the beginning of CR-geometry, H. Poincaré [1], applying his apparatus of working with power series, studies the properties of the germ of real hypersurface of the two-dimensional complex space that are invariant with respect to holomorphic transformations. It turned out here that the key to understanding the situation is the germ of the 3-dimensional sphere which has several extremal properties. For example, its 8-dimensional local group of holomorphic automorphisms has the maximal dimension if we do not take into account the plane whose group is infinite-dimensional. The equations of the germ of a nondegenerate hypersurface in $\mathbf{C}^2$ (in the coordinates $(z, w = u+i\,v)$) can be written out in the form

$$v = |z|^2 + \text{terms of higher degrees}$$

Here the hypersurface $\{v = |z|^2\}$, which is projectively equivalent to the standard sphere $\{|z|^2 + |w|^2 = 1\}$, is just our model hypersurface.

Later on, this approach was developed and successfully applied to study CR-manifolds of diverse dimensions [2] and codimensions [3]. Here the model surface, which is an analog of the sphere, was somewhat modified. In [4] this approach (the method of model surface) was implemented for an arbitrary real germ in general position (under the assumption of complete nondegeneracy).

Recently the g_+-conjecture was proved in the papers by Sabzevari–Spiro [5] and Gregorovich [6]. Namely, it was proved that the stabilizer of the origin of a completely nondegenerate model surface of highest weight exceeding two contains no nonlinear transformations. This can be expressed in other words as follows. The subalgebra g_+ of fields of positive weight in the Lie algebra of infinitesimal automorphisms is trivial. We note here that, if we get rid of the condition of complete nondegeneracy, then this is not the case. For example, $\{v = |z|^4\}$is a hypersurface in $\mathbf{C}^2$ with the highest weight four and a nontrivial subalgebra g_+. Several interesting examples with g_+ of an arbitrary weight is contained in the paper [6] cited above.

After the proof of the conjecture, an interest to extending the method of model surface to classes of CR-manifolds going beyond the framework of the completely nondegenerate manifolds increased. The generating CR-manifolds of finite type form a natural extension of the class of completely nondegenerate manifolds. In [12], the program of the method of model surface was implemented for the manifolds of an arbitrary finite Bloom–Graham type with some rather burdensome additional condition. The matter concerned the manifolds whose model surface has a condition of rigidness. In the present paper, we get rid of this condition and consider arbitrary manifolds of finite type. Here the Bloom–Graham theorem [8] connecting two views concerning the type of the germ of a CR-manifold, namely, the geometric one (fields and commutators) are analytic (coordinates and equations), is substantial for our presentation. In the present paper, we mainly deal with the analytic definition of the type (see Subsec. 2). For the convenience of the reader, we present here the geometric definition of type.

Let M be a smooth generating CR-submanifold of codimension $K \geqslant 1$ of a complex linear space, let $n \geqslant 1$ be the dimension of the complex tangent, and let $\xi \in M$. Let D_1 be the distribution of complex tangents that is defined on M in a neighborhood of ξ, i.e., $D_1 = T^c_M$. This distribution can be defined by a basis family of $2\,n$ smooth real vector fields. Further, we define an infinite sequence of distributions D_ν defined by induction, $D_{\nu+1} = [D_\nu, D_1] + D_\nu, \quad \nu = 1, 2, \dots$. Let, further, $D_\nu(\xi)$ be the value of D_ν at the point ξ. Thus,

$$T^c\, M_\xi = D_1(\xi) \subset D_2(\xi) \subset \dots \subset D_\nu(\xi) \subset \dots$$

Since this nondecreasing sequence consists of subspaces of TM_ξ, it follows that this sequence is stabilized at some step. If the last subspace coincides with TM_ξ, then we say that M is a manifold of *finite type* at the point ξ (if not, we say that M is of *infinite type* at ξ). Let, further, $d_\nu = \dim_{\mathbf{R}}\, D_\nu(\xi)$, and

$$2\,n = d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{\mu-1} < d_\mu = d_{\mu+1} = \dots = d_\infty$$

Marking all indices $\nu \geqslant 2$ at which a jump of dimension occurs, we obtain a finite strictly increasing sequence $2 \leqslant m_1 < m_2 < \dots < m_l$. Denote by $k_j,\ j = 1, \dots, l$, the value itself of the positive jump $d_{m_j} - d_{m_j - 1}$. This very collection of data,

$$m = ((m_1, k_1), \dots, (m_l, k_l)),$$

plus the indication concerning the finiteness of infiniteness of the type is called the (geometric) type of M at the point ξ. It can readily be seen that, if the type is finite, then the codimension K is the sum of all k_j. For the infinite type, this sum is less than the codimension.

In [8], these data are written out in another format, namely,

$$m = (m_1, \dots, m_1, m_2, \dots, m_2, \dots, m_l, \dots, m_l),$$

and here the number of repetitions of m_j is equal to k_j and, if the type is infinite, then the symbol ∞ is posed at the end of the sequence.

Further, for every M_ξ (the germ of a real submanifold of a complex space at a point ξ), we introduce the following objects:

$$\operatorname{aut} M_\xi, \quad \operatorname{aut}_\xi M_\xi, \quad \operatorname{Aut}\, M_\xi, \quad \operatorname{Aut}_\xi\, M_\xi,$$

Here $\operatorname{aut} M_\xi$ stands for the Lie subalgebra of germs of vector fields at ξ tangent to M_ξ and generating local 1-parameter groups of holomorphic transformations of M_ξ. In the coordinates (z, w) these fields have the form

$$X = 2\,\mathrm{Re}\Big(f(z,w)\,\frac{\partial}{\partial\, z} + g(z,w)\,\frac{\partial}{\partial\, w}\Big), \tag{1}$$

where f and g are germs holomorphic at ξ and $\operatorname{aut}_\xi M_\xi$ is the Lie subalgebra of $\operatorname{aut} M_\xi$ consisting of the fields vanishing at the point ξ. Each of these Lie algebras generates a local group of holomorphic transformations of M_ξ, $\operatorname{Aut} M_\xi$, and $\operatorname{Aut}_\xi M_\xi$, respectively (the local automorphisms of the germ and the stabilizer of the point in the local group of automorphisms).

2. ANALYSIS OF THE LOWER COMPONENTS OF THE MAPPING AND THE BLOOM–GRAHAM THEOREM

Let M be a smooth generating real submanifold of a complex space $\mathbf{C}^N$ of a positive CR-dimension n and a positive real codimension K. In other words, M has the CR-type (n, K) at every point, and here $N = n + K$. Let M_ξ be the germ at the point ξ. Let M_ξ be a generating germ of a CR-submanifold of finite type

$$\begin{aligned} m &= (m_1, \dots, m_1, m_2, \dots, m_2, \dots, m_l, \dots, m_l) \\ &= ((m_1, k_1), (m_2, k_2), \dots, (m_l, k_l)), \end{aligned}$$

where m_j and k_j are positive integers, and $2 \leqslant m_1 < m_2 < \dots < m_l$. As was said above, the main result of [8] is the equivalence of the geometric and analytic definitions of the type of manifold at a point. To formulate the analytic definition, we partition the coordinates of the ambient complex space $\mathbf{C}^{n+K}$ into groups,

$$z \in \mathbf{C}^n,\ w_1 \in \mathbf{C}^{k_1}, \dots, w_l \in \mathbf{C}^{k_l}, \quad k_1 + \dots + k_l = K.$$

Here the variables are equipped with weights: $[z] = 1,\ [w_j] = m_j$. The complex conjugate variables $\bar{z}$ and $\bar{w}_j$ obtain the same weights, and correspondingly $u_j = \operatorname{Re} w_j,\ v_j = \operatorname{Im} w_j$. This agreement enables us to extend the grading to power series in these variables. If we set in addition

$$\left[\frac{\partial}{\partial z}\right] = \left[\frac{\partial}{\partial \bar{z}}\right] = -1, \quad \left[\frac{\partial}{\partial w_j}\right] = \left[\frac{\partial}{\partial \bar{w}_j}\right] = -m_j,$$

then we can extend the grading to vector fields. Denote a power series containing no terms of weight μ and lower than that by $o(\mu)$.

Let the equations of the germ M_ξ be represented in the form

$$v_j = \Phi_j(z, \bar{z}, u_1, \dots, u_{j-1}) + o(m_j), \quad j = 1, \dots, l \tag{2}$$

where the real vector-valued form Φ_j (i.e., its coordinates) have the homogeneous weight m_j.

The tangent model surface of the germ M_ξ is a real algebraic surface Q given by the relations

$$v_j = \Phi_j(z, \bar{z}, u_1, \dots, u_{j-1}), \quad j = 1, \dots, l \tag{3}$$

The condition that the type is finite is a condition of nondegeneracy nature. To obtain the condition that the type is finite in terms of forms Φ, one must make additional triangular-polynomial transformations of coordinates which do not change the form of the equations and the weights of the forms Φ and change these forms themselves (the reduction of Q to the standard form, Theorem 6.2 on p. 230 of [8]).

Every coordinate of every vector-valued form Φ_j is a linear combination of monomials of the form

$$z_1^{\alpha_1} \dots z_n^{\alpha_n}\, \bar{z}_1^{\gamma_1} \dots \bar{z}_n^{\gamma_n}\, u_{11}^{\beta_{11}} \dots u_{1k_1}^{\beta_{1k_1}} \dots u_{(j-1)1}^{\beta_{(j-1)1}} \dots u_{(j-1)k_{j-1}}^{\beta_{(j-1)k_{j-1}}}$$

Let us formulate two conditions on the forms Φ.

(I) The coordinates of the forms do not contain monomials of the form

$$z_1{}^{\alpha_1} \dots, z_n{}^{\alpha_n}\, u_{11}^{\beta_{11}} \dots u_{1k_1}^{\beta_{1k_1}} \dots u_{(j-1)1}^{\beta_{(j-1)1}} \dots, u_{(j-1)k_{j-1}}^{\beta_{(j-1)k_{j-1}}}$$

and their conjugates for any α and β

To formulate the other condition, we need a general indexing of all coordinates of all vector-valued forms Φ. Namely, let $(\phi_1, \dots, \phi_{k_1})$ be the coordinate forms of Φ_1, further, let $(\phi_{k_1+1}, \dots, \phi_{k_1+k_2})$ be the coordinate forms of Φ_2, and so on, up to the coordinates of Φ_l. Thus, a complete ordered family of all coordinate forms is $(\phi_1, \dots, \phi_K)$.

(II) For every $1 \leqslant J \leqslant K,$ the form ϕ_J contains no summands of the form

$$c\, \phi_j\, u_{11}^{\beta_{11}}, \dots, u_{1k_1}^{\beta_{1k_1}}, \dots, u_{(j-1)1}^{\beta_{(j-1)1}}, \dots, u_{(j-1)k_{j-1}}^{\beta_{(j-1)k_{j-1}}}$$

for all $j,$ such that $j < J$ and c is a nonzero constant.

We say that the equations of the germ M_ξ and the surface Q are written out in the *standard Bloom–Graham form* if the coordinate forms satisfy conditions (I) and (II). Here we say that the manifold M at the point ξ and the surface Q at zero have the *finite Bloom–Graham type* m if no coordinate form ϕ_j is identically equal to zero.

Remark 1. Condition (II), in contrast to condition (I), has recurrent nature.

Remark 2. The indexing of the weights $m_1, \dots, m_l$ and, correspondingly. of the vector coordinates $w_1, \dots, w_l$ has geometric interpretation (these are the indices of the elements of the embedded sequence of subspaces of the tangent space at which the dimension of the germ grows, see [8]). For this reason, the indexing of these objects is invariant with respect to locally biholomorphic or CR transformations. At the same time, the choice of indexing of the scalar coordinates of the vector coordinate w_j itself is arbitrary and not holomorphically invariant.

Therefore, it is convenient for us to change the condition of [8] making it holomorphically invariant. Namely, we assume that condition (II) holds for a pair of indices $j < J$ only if they are related to *different* weight groups, i.e., are scalar coordinates of different weighted vector coordinates (have different weights). This is not required for the coordinates of the same weight. We keep the term *standard form* for the Bloom–Graham formulation and use the term *reduced form* for our condition. We denote by condition (II') the weakened condition (II) itself.

Here the condition that the germ of a surface given by equations (2) has a given finite type m in the reduced form differs from the similar condition for the standard form. Let us formulate both the conditions.

A proposition from [8] (the condition of finite type for the standard form)**.**
(a) The germ of a surface written by equations (2) *in the standard form has a given finite type* m *if and only if there are no identically zero forms among the scalar coordinate forms* $(\phi_1, \dots, \phi_K)$.
(b) Every germ of finite type can be written in such a form.

The following proposition is an immediate consequence of this assertion.

Proposition 3 (the condition of finite type for the reduced form)**.**
(a) The germ of a surface written by equations (2) *in the reduced form has a given finite type* m *if and only if the coordinate forms* $(\phi_1, \dots, \phi_K)$ *are linearly independent.*
(b) Every germ of finite type can be written in such a form.

Proof. When passing from the reduced form to the standard form, the transformed coordinate weighted forms ϕ_j are linear combinations of the original forms. Therefore, the appearance of the identical zero is possible only if the original forms have a linear dependence.

Remark 4. Here it is clear that, if among the forms there are no identically zero ones, then a linear dependence is possible only within the forms of the same weight group.

The parameters $l \geqslant 1,\ m_l \geqslant 2$ have two interpretations. In the analytic representation, l is the number of distinct weights, and m_l is the maximal weight in the representation of equations (2) in the standard or reduced form. In the geometric representation, l is the number of jumps of the dimension in the sequence of subspaces beginning at the complex tangent at ξ and ending by the full tangent space, and m_l is the depth of the bracket construction needed to obtain the full tangent. We call the number $\mu = m_l$ the *highest weight*. In the case of an infinite type, we set $\mu = \infty$.

Let there be a locally invertible holomorphic mapping

$$(z \to f(z,w),\ w_j \to g_j(z,w)),\ j = 1, \dots, l$$

of a germ M_0 at the origin of a finite type m given by equations in the standard form

$$v_j = \Phi_j(z, \bar{z}, u_1, \dots, u_{j-1}) + F_j(z, \bar{z}, u), \quad j = 1, \dots, l \tag{4}$$

and another germ $\tilde{M}_0$ of the same kind,

$$v_j = \tilde{\Phi}_j(z, \bar{z}, u_1, \dots, u_{j-1}) + \tilde{F}_j(z, \bar{z}, u), \quad j = 1, \dots, l \tag{5}$$

where F_j and $\tilde{F}_j$ are $o(m_j)$. Let Q and $\tilde{Q}$ be their model surfaces given by the equations

$$\begin{aligned} v_j &= \Phi_j(z, \bar{z}, u_1, \ldots, u_{j-1}), \quad j = 1, \ldots, l, \\ v_j &= \tilde{\Phi}_j(z, \bar{z}, u_1, \ldots, u_{j-1}), \quad j = 1, \ldots, l, \end{aligned} \tag{6}$$

respectively.

Below we shall use the expansions

$$f = \sum_1^\infty f_\mu, \; g_j = \sum_1^\infty g_{j\mu}, \; F_j = \sum_{m_j+1}^\infty F_{j\mu}, \; \tilde{F}_j = \sum_{m_j+1}^\infty \tilde{F}_{j\mu},$$

where $f_\mu, \; g_{j\mu}, \; F_\mu, \; \tilde{F}_\mu$ are the components of the weight μ.

Writing out the condition that the image of M_0 is contained in $\tilde{M}_0$, we obtain the identity

$$\begin{aligned} &\operatorname{Im} g_j = \tilde{\Phi}_j(f, \bar{f}, \operatorname{Re} g_1, \ldots, \operatorname{Re} g_{j-1}) + \tilde{F}_j(f, \bar{f}, \operatorname{Re} g), \quad j = 1, \ldots, l, \\ &\quad \text{for } w = u + i(\Phi + F) \end{aligned} \tag{7}$$

Consider the lower components of (7).

Let us begin with the group of variables w_1. In the weights from 1 to $(m_1 - 1)$ we obtain $\operatorname{Im} g_{1\nu} = 0$, where $1 \leqslant \nu \leqslant m_1 - 1$. Since every homogeneous form of weight $\nu < m_1$ is a holomorphic form of the variable z, we conclude that $g_{11} = g_{12} = \cdots = g_{1(m_1-1)} = 0$.

In the weight m_1 we have $g_{1m_1} = a(z) + \rho_1 w_1$, $f_1 = C z$, where ρ_1 and C are linear and $a(z)$ is a holomorphic homogeneous form of degree m_1. We see that

$$\operatorname{Im} (a(z) + \rho_1 (u_1 + i \Phi_1)) = \tilde{\Phi}_1(C z, \overline{C z})$$

Separating the component holomorphic with respect to z in this relation and taking into account that Φ_1 and $\tilde{\Phi}_1$ contain no holomorphic summands, we obtain $a(z) = 0$. We obtain $\operatorname{Im} \rho_1 u_1 = 0$ from the linear component with respect to u_1. After this, we have

$$\rho_1 \Phi_1(z, \bar{z}) = \tilde{\Phi}_1(C z, \overline{C z})$$

Note that this relation is equivalent to the condition that the mapping $(z \to C z, w_1 \to \rho_1 w_1)$ takes the "truncated" surface $v_1 = \Phi_1(z, \bar{z})$ of the space $\mathbf{C}^{n+k_1}$ to another "truncated" surface $v_1 = \tilde{\Phi}_1(z, \bar{z})$.

Let us pass to the coordinate w_2. The components $g_{2\nu}$, where $\nu < m_2$, are expressions of the form $\sum \psi_{\alpha\beta}(z, w_1)$, where $\psi_{\alpha\beta}(z, w_1)$ is a holomorphic multilinear form of degree α with respect to z and β with respect to w_1, and $\alpha + m_1 \beta = \nu$. The components of identity (7) of the weights $\nu < m_2$ give

$$\operatorname{Im} (\sum \psi_{\alpha\beta}(z, u_1 + i \Phi_1)) = 0,$$

This implies that $g_{2\nu} = 0$ for $\nu < m_2$. This fact could be proved immediately from the identity thus obtained. However, we use another way. Indeed, $g_{2\nu}$ is a holomorphic function on the "truncated" generating manifold $Q(1) = \{(z, w_1) : v_1 = \Phi_1(z, \bar{z})\}$ whose imaginary part is equal to zero. Therefore, $g_{2\nu}$ is constant, and it vanishes since its weight exceeds zero.

In the weight m_2 we have $g_{2m_2} = \sum \psi_{\alpha\beta}(z, w_1) + \rho_2 w_2$, where $\alpha + m_1 \beta = m_2$, and

$$\operatorname{Im} \Big(\sum \psi_{\alpha\beta}(z, u_1 + i \Phi_1) = \tilde{\Phi}_2(C z, \overline{C z}, \rho_1 u_1) - \rho_2(u_2 + i \Phi_2(z, \bar{z}, u_1)) \Big)$$

By condition (I), the right-hand side contains no summands holomorphic with respect to z. Setting $\bar{z} = 0$, we see that, if $\alpha \neq 0$, then $\psi_{\alpha\beta}(z, u_1) = 0$, and $\psi_{0\beta}(u_1)$ is a real form of weight m_2, which we denote by $\theta_2(w_1)$, i.e., $g_{2m_2} = \rho_2 w_2 + \theta_2(w_1)$. The relation acquires the form

$$\operatorname{Im} \theta_2(u_1 + i \Phi_1) = \tilde{\Phi}_2(C z, \overline{C z}, \rho_1 u_1) - \rho_2(u_2 + i \Phi_2(z, \bar{z}, u_1)))$$

Note that this relation is equivalent to the condition that the mapping

$$(z \to C\,z, \quad w_1 \to \rho_1\,w_1, \quad w_2 \to \rho_2\,w_2 + \theta_2(w_1))$$

takes the "truncated" surface $\{v_1 = \Phi_1(z,\bar z),\, v_2 = \Phi_2(z,\bar z,u_1)\}$ of the space $\mathbf{C}^{n+k_1+k_2}$ to another "truncated" surface $\{v_1 = \tilde\Phi_1(z,\bar z),\, v_2 = \tilde\Phi_2(z,\bar z,u_1)\}$.

We can proceed in this way up to the last weighted group corresponding to w_l. Let us state the result thus obtained.

Theorem 5. *Let $(z \to f(z,w),\ w_j \to g_j(z,w))$, $j = 1,\dots,l$, be an invertible holomorphic mapping of the germ (4) written in the reduced form onto another germ (5) of the same kind. Then*

(a) This mapping has the form

$$(z \to C\,z + o(1),\ w_j \to \rho_j\,w_j + \theta_j(w_1,\dots,w_{j-1}) + o(m_j),\ j = 1,\dots,l),$$

where $C \in GL(n,\mathbf{C})$, $\rho_j \in GL(k_j,\mathbf{R})$ and where $\theta_j(w_1,\dots,w_{j-1})$ is a homogeneous real form of weight m_j, and, for all $j = 1,\dots,l$,

$$\begin{aligned}&\operatorname{Im}\theta_j(u_1 + i\,\Phi_1(z,\bar z),\dots,u_{j-1} + i\,\Phi_{j-1}(z,\bar z,u_1,\dots,u_{j-2})))\\ &\quad = \tilde\Phi_j(C\,z,\overline{C\,z},\rho_1\,u_1,\dots,\rho_{j-1}\,u_{j-1}) - \rho_j\,\Phi_j(z,\bar z,\,u_1,\dots,u_{j-1})\end{aligned} \tag{8}$$

(b) Here the triangular weighted homogeneous mapping

$$(z \to C\,z,\ w_j \to \rho_j\,w_j + \theta_j(w_1,\dots,w_{j-1}),\ j = 1,\dots,l) \tag{9}$$

takes the model surface Q to the model surface $\tilde Q$.

(c) For every $j = 1,\dots,l-1$, the truncated mapping

$$(z \to C\,z,\ w_\nu \to \rho_\nu\,w_\nu + \theta_\nu(w_1,\dots,w_{\nu-1}),\ \nu = 1,\dots,j)$$

takes the truncated model surface $Q(j) = \{v_\nu = \Phi_\nu,\ \nu = 1,\dots,j\}$ of the space $\mathbf{C}^{n+k_1+\dots+k_j}$ to another truncated model surface $\tilde Q(j) = \{v_\nu = \tilde\Phi_\nu,\ \nu = 1,\dots,j\}$.

(d) The action of holomorphic mappings on the germs of manifolds of a given type generates an action on the space of weighted homogeneous forms $\Phi = (\Phi_1,\dots,\Phi_l)$ of the group of weighted homogeneous triangular polynomial transformations of the form

$$\begin{aligned}\Phi_j(z,\bar z,\,u_1,\dots,u_{j-1}) \to \rho_j\,\Phi_j(&C^{-1}\,z,\overline{C^{-1}\,z},\rho_1^{-1}\,u_1,\ \rho_2^{-1}\,(u_2 - \operatorname{Re}\theta_2(u_1 + i\,\Phi_1)),\dots,\\ &\rho_{j-1}^{-1}\,(u_{j-1} - \operatorname{Re}\theta_{j-1}((u_1 + i\,\Phi_1),\dots,(u_{j-2} + i\,\Phi_{j-2}))))\end{aligned} \tag{10}$$

(e) The stabilizer of zero in the group of automorphisms of the model surface $\operatorname{Aut}_0 Q_0$ contains the 1-parameter subgroup

$$(z \to t\,z,\ w_j \to t^{m_j}\ w_j), \quad t \in \mathbf{R}^* \tag{11}$$

To this subgroup, there corresponds a vector field of weight zero,

$$X_0 = 2\operatorname{Re}\Big(z\,\frac{\partial}{\partial z} + \sum m_j\,w_j\,\frac{\partial}{\partial w_j}\Big), \tag{12}$$

(f) If the model surface Q has finite type at zero, then an element of the stabilizer of this action, i.e., a mapping (10), is determined uniquely by its z-coordinate, i.e., the parameter $C \in GL(n,\mathbf{C})$.

Proof. In part (a), it remains to verify the invertibility of the linear mappings. This follows from the fact that, as shown above, the differential of our holomorphic mapping at zero has a block-triangular form, and the invertibility of the differential requires the invertibility of every diagonal block, i.e., of the mappings $(C, \rho_1, \dots, \rho_l)$. Part (b) was proved above, part (c) follows immediately from (b), part (d) follows from (c). Part (e) is obvious. Let us prove part (f), assuming here that $\tilde{\Phi} = \Phi$. For the definition of (ρ_j, θ_j) we have

$$\mathrm{Im}\,(\rho_j\, w_j + \theta_j(w_1, \dots, w_{j-1})) = \Phi_j(C\,z, \overline{C\,z}, \rho_1\, u_1, \dots, \rho_{j-1}\, u_{j-1}).$$

where $w_j = u_j + i\,\Phi_j(z, \bar{z}, u_1, \dots, u_{j-1})$ for $j = 1, \dots, l$. For a chosen C, these relations enable us to uniquely determine (ρ_j, θ_j) in succession for j from 1 to l. Indeed, let (ρ_ν, θ_ν) be already uniquely defined for $\nu = 1, \dots, j-1$, and there be two pairs (ρ_j, θ_j) and $(\rho_j + \delta\rho_j, \theta_j + \delta\theta_j)$; then

$$\mathrm{Im}\,(\delta\rho_j\, w_j + \delta\theta_j(w_1, \dots, w_{j-1})) = 0, \text{ where} \qquad w_\nu = u_\nu + i\,\Phi_\nu(z, \bar{z}, u_1, \dots, u_\nu), \quad \nu = 1, \dots, j$$

Since the truncated surface

$$Q_j = \{v_\nu = \Phi_\nu(z, \bar{z}, u_1, \dots, u_\nu),\ \nu = 1, \dots, j\}$$

has finite type, it follows that $(\delta\rho_j, \delta\theta_j) = \mathit{const}$, and this constant vanishes since the weight is positive. This completes the proof of the theorem.

For the case in which the mappings (9) take the model surface Q onto itself, we denote the corresponding subgroup of the group $\mathrm{Aut}_0\, Q_0$ consisting of these mappings by G_0.

Remark 6. (a) As was shown in [7], for the case in which the model surface is rigid (Φ does not depend on u), the nonlinear summands θ_j are absent.

(b) For the occurrence of a nonzero nonlinear summand θ_j, the presence of some integer condition is needed, a "resonance," namely, the existence of a representation of the form

$$m_j = \mu_1\, m_1 + \dots + \mu_{j-1}\, m_{j-1}$$

with nonnegative integer coefficients μ_ν. In the absence of resonances, the subgroup G_0 consists of linear transformations only.

(c) In the general case, transformations in G_0 are said to be *quasilinear.*

3. MODEL SURFACE Q

As noted above, our grading can be extended to germs of vector fields. As a result, the Lie algebra of all real fields analytic in a neighborhood of zero decomposes into the direct sum of its homogeneous components of weights from $(-m_l)$ and higher. Correspondingly, every Lie subalgebra of the Lie algebra inherits this grading. In particular, $\mathrm{aut}\ Q = \sum_{-m_l}^{\infty} g_\nu$. The presence of the grading subgroup (11) in the group of automorphisms of every model surface Q has an obvious but important consequence.

Proposition 7. *(a) Let* $X = \sum_{-\mu}^{\infty} X_\nu \in \mathrm{aut}\ Q$ *be a vector field; then* $X_\nu \in \mathrm{aut}\ Q$ *for all* ν.

(b) The algebra aut Q *is finite-dimensional if and only if it is finitely graded, i.e.,* $\mathrm{aut}\ Q = \sum_{-\mu}^{\delta} g_\nu$. *Here the algebra consists of fields with polynomial coefficients.*

Here by $\delta = \delta(Q)$ (*highest positive weight*) we have denoted the maximal ν such that $g_\nu \neq 0$.

We assume that the equations of Q are written in the reduced form and are linearly independent. This means that the model surface Q is of finite type at the origin, and therefore is minimal. In this case, the criterion for the finite-dimensionality of the Lie algebra $\mathrm{aut}\, Q_0$ is the condition of holomorphic nondegeneracy [10]. In the general case, the condition of holomorphic nondegeneracy does not have a constructive nature. This is a condition for the nonexistence of some vector field. However, since Q is the graph of a real polynomial mapping, it follows that the verification of this condition is reduced to verifying the maximality of the rank of some matrix. In a specific situation, these conditions can be written out explicitly (see [12, 7]).

Definition 8. If a model surface Q has finite type at the origin and is holomorphically nondegenerate, then we shall say that the surface is *nondegenerate* at the origin. Here we say that a CR-manifold M is *nondegenerate at a point* $\xi \in M$ if the model surface Q of the germ M_ξ at the point ξ is nondegenerate at the origin.

For the germ of the manifold M_ξ, the nondegeneracy condition at the point is the sufficient condition of the finite-dimensionality of $\operatorname{aut} M_\xi$ but is certainly not necessary.

Proposition 9. *For a model surface Q, the nondegeneracy condition is a criterion of the finite-dimensionality (finiteness of the grading) of* $\operatorname{aut} Q_0$.

Proof. It can readily be seen that a violation of either of the two requirements of Definition 8 leads to infinite-dimensionality of $\operatorname{aut} Q$.

The surface Q is the graph of a real polynomial equation of the form (3), and therefore Q, at every point, is generating a real algebraic submanifold of $\mathbf{C}^N$ of codimension K without singularities. The dimension of the complex tangent, i.e., the CR-dimension, is the same everywhere and is equal to $n = N - K$. The family of K gradients of the defining equations has the maximum complex rank K at all points of the space. However, the type can depend on a point of Q. In this connection, we introduce the following subsets of Q.

Q^m is the family of points $\xi \in Q$ such that Q at ξ is a manifold of finite type whose Bloom–Graham type is equal to

$$\begin{aligned} m &= (m_1, \dots, m_1, m_2, \dots, m_2, \dots, m_l, \dots, m_l) \\ &= ((m_1, k_1), (m_2, k_2), \dots, (m_l, k_l)); \end{aligned}$$

note that here the codimension is $K = \sum k_j$.

Q^μ is the family of points $\xi \in Q$ such that Q at ξ has the highest weight equal to μ, where $2 \leqslant \mu < \infty$.

All the characteristics under consideration are holomorphically invariant, and therefore they are constant on every orbit of the action of the group of holomorphic automorphisms of Q. As shown in [4], all completely nondegenerate model surfaces are holomorphically homogeneous, i.e., the group of holomorphic automorphisms acts transitively on them. Therefore, in the case of complete nondegeneracy we have $Q = Q^m$, where m is the type of Q at the origin. If the complete nondegeneracy is absent, then the property of holomorphic homogeneity can be lost, and the picture becomes more diverse.

To state the following result, we need a new definition. A subset of the space $\mathbf{R}^N$ with the coordinate x is said to be *semi-algebraic* if it is given by a condition of the form

$$p_\alpha(x) = 0,\ \alpha \in A, \quad q_\beta(x) \neq 0,\ \beta \in B,$$

where A and B are finite families of indices and p_α and q_β are real polynomials. This term is customarily used for a wider class of sets (more-and-less relationships are used). We use this term due to the lack of a more suitable one.

Theorem 10. *Let Q be a nondegenerate model surface.*

(a) If $\mu(0)$ is the highest weight of Q at the origin, then $\mu(\xi) \leqslant \mu(0)$ for all ξ, i.e., $\mu(0)$ is the maximal value of the highest degree and, correspondingly,

$$Q = \bigcup_{\nu=2}^{\mu(0)} Q^\nu.$$

In particular, Q has no points of infinite type.

(b) Q^ν and Q^m are real semi-algebraic subsets of Q.

(c) Both $\mu(\xi)$ and $m(\xi)$ take on Q a finite family of values only.

(d) Let $\mu_{\min}$ be the minimal value of the highest weight over all points of Q and let $Q^{\mu_{\min}}$ be the set of points with this highest weight. Then $Q^{\mu_{\min}}$ is an open subset of Q.

Proof. The Bloom–Graham type determination procedure for Q consists of the weighted re-decomposition of the right-hand sides of the equations at a new point and the reduction of the relations thus obtained to the standard form (or to the reduced form). Both the procedures do not increase the weights of the right-hand sides of the equations. From the viewpoint of geometric definition of type, the fact that the weight of every variable does not increase can be explained as follows. When re-decomposing the right-hand sides of the equations at a new point ξ, the highest weight component (with respect to the old weight) at the new point remains unchanged. Let all truncated model surfaces $Q(\nu)$ for $\nu < j$ be of finite type. Using a basis of vector fields generating the distribution of complex tangents D^1, we see that the finiteness condition of the type of $Q(j)$ at the point ξ remains the same, i.e., the weight of a variable can drop but cannot increase. This proves (a).

By (a), to verify that a point belongs to a chosen finite type, it is sufficient to study only finitely many conditions of linear dependence and independence of fields with polynomial coefficients. Further, the set of points of a chosen highest weight is a finite union of sets of definite types, i.e., a finite union of semi-algebraic sets. This proves (b).

The inequality $\mu(\xi) \leqslant \mu(0)$ implies the finiteness of the values $\mu(\xi)$ and, for a bounded highest weight, there are only finitely many values of the types $\mu(\xi)$. This proves (c).

The highest weight is the number of nested operations of taking the brackets for base fields in the complex tangent that are sufficient to obtain the full tangent. This condition can be written out as a condition of completeness of the rank for brackets of a given depth. If this condition holds at a point, then it holds in a neighborhood. Due to minimality, the depth cannot decrease. This proves (d). The theorem is proved.

We note also that the minimal degree of the equations of Q, i.e., $m_1(\xi)$, is obviously equal to 2 on an open dense set.

With every point ξ of the model surface Q are connected characteristics, namely, the type at the point $m(\xi)$ and the highest weight at the point $\mu(\xi)$. The type that we originally associated with Q and which occurs in the equations of Q is the type of Q at the origin $m(0)$.

We can readily show that the type of points of a model surface need not be constant. Let a hypersurface in $\mathbf{C}^2$ be given by the equation $v = \mathrm{Re}(z^2\,\bar z)$. This is a model surface, and its type is $m(0) = (3)$. If we consider the same hypersurface at a point $\xi = (a, b)$, then it can be written in the form $v = \mathrm{Re}(a)\,|z|^2 + \mathrm{Re}(z^2\,\bar z)$. Therefore, its type is $m(\xi) = (2)$ if $\mathrm{Re}(a) \neq 0$ and $m(\xi) = (3)$ if $\mathrm{Re}(a) = 0$. Similarly, the highest degree is $\mu(\xi) = 2$ everywhere except for $\mathrm{Re}(a) = 0$ and $\mu(\xi) = 3$ on the plane.

One of the main properties of model surface is that the local group of its holomorphic automorphisms parameterizes the family of holomorphic mappings between the germs of the same type. In particular, the model surface is the most holomorphically symmetric surface, since the dimension of the stabilizer of the model surface dominates the dimension of the stabilizer of any nondegenerate germ.

Theorem 5 implies the following corollary.

Corollary 11. *Every holomorphic mapping χ of the germ* (4) *written out in the reduced form onto another germ* (5) *of this kind can be represented at the composition $\chi = \varphi \circ \psi$, where*

$$\varphi = (z \to z + o(1), \quad w_j \to w_j + o(m_j)) \tag{13}$$

and $\psi \in G_0$, i.e., an element of the group of quasilinear automorphisms of $\tilde Q$ that preserves the origin.

Let there be a mapping of (4) onto (5),

$$(z \to z + f_2 + \dots \quad w_j \to w_j + g_{m_j(m_j+1)} \dots)$$

Let us write out relation (7) for this mapping and single out in it the components of the following weights. In the coordinates of the group w_1, we single out the component of the weight $m_1 + \mu$, in the coordinates of w_2, we single out the component of the weight $m_2+\mu$, and so on. We obtain

$$\begin{gathered}-\operatorname{Im} g_{j(m_j+\mu)} + d\,\Phi_j(z,\bar z,u_1,\dots,u_{j-1})(f_{1+\mu},\overline{f_{1+\mu}},\operatorname{Re} g_{1(m_1+\mu)},\dots,\operatorname{Re} g_{(j-1)(m_{j-1}+\mu)}) + \dots = 0,\\ 1\leqslant j\leqslant l, \quad \text{and the arguments in } f \text{ and } g \text{ are } w_\nu=(u_\nu+i\,\Phi_\nu).\end{gathered} \tag{14}$$

Here the dots stand for the terms of the relation that depend on the families

$$h_\nu=(f_{1+\nu},g_{1,(m_j+\nu)},\dots,g_{j-1,(m_j+\nu)}) \quad \text{for} \quad \nu<\mu.$$

The resulting relation (the homological equation) can be used to calculate the family h_μ if all families with lower indices are known. To do this, one must solve the nonhomogeneous linear algebraic equation (14) with respect to h_μ, where the expression indicated by the dots is determined using the known values of h_ν for $\nu<\mu$. Since the dimension of the family of solutions of an inhomogeneous linear equation does not exceed the dimension of the space of solutions of the corresponding homogeneous equation, we see that the dimension of the family of mappings of (4) into (5) does not exceed the dimension of the family of solutions of the homogeneous equation

$$\begin{gathered}-\operatorname{Im} g_j + d\,\Phi_j(z,\bar z,u_1,\dots,u_{j-1})(f,\overline{f},\operatorname{Re} g_1,\dots,\operatorname{Re} g_{j-1})=0,\\ 1\leqslant j\leqslant l,\ \text{where the arguments in } f \text{ and } g \text{ are } w_\nu=u_\nu+i\,\Phi_\nu.\end{gathered} \tag{15}$$

This equation is obtained by adding the homogeneous parts of relations (14) over all μ. Note now that equation (15) coincides with the condition that the vector field of the form (1) is an element of $\operatorname{aut} Q_0$. Thus, we have proved the following theorem.

Theorem 12. *(a) The dimension of the family of mappings of* (4) *onto* (5) *preserving the origin does not exceed the dimension of* $\operatorname{aut}_0 Q_0$*; in particular,*

$$\dim \operatorname{aut}_\xi M_\xi \leqslant \dim \operatorname{aut}_0 Q_0$$

(b) Let Q be nondegenerate and let $\delta=\delta(Q)$ be the index of the highest with respect to the weight nonzero component of the algebra $\operatorname{aut} Q_0$*. Then, if there are two mappings of* (4) *onto* (5) *preserving the origin and having the same weight δ-jets at the origin, then these mappings coincide. In particular, an automorphism of M_ξ with the identity δ-jet is the identity mapping.*

This argument goes back to Poincaré [1] and is a version of the implicit mapping theorem in the class of formal power series. A general outline of the implicit mapping theorem is as follows: if the linear part of some relation is uniquely solvable with respect to some group of variables, then the same can be said about the initial nonlinear relation. In the present case, the linear part of the relation is the relation

$$\mathcal{L}(f,g)=\partial_z\Phi(z,\bar z,u)(f)+\partial_{\bar z}\Phi(z,\bar z,u)(\bar f)+\partial_u\Phi(z,\bar z,u)(Re\, g)-Im\, g,$$

where $w_j=u_j+i\,\Phi_j$, which, in turn, is precisely the condition that the field (f,g) is tangent to the model surface Q.

Remark 13. The recursive scheme described above for computing the components of the mapping and using the operator $\mathcal{L}$ can be further improved and turned into a recurrent process of computing the components of the mapping together with the reduction of equations of the germ to a normal form. This was done in [1, 2] for nondegenerate hypersurfaces. The convergence of the constructed normal form was also proved there. Moreover, the proof of convergence, even in this simplest situation, required a considerable effort. For arbitrary manifolds of the finite type, it is not difficult to describe the formal normal form corresponding to the operator $\mathcal{L}$. The procedure for its construction is reduced to the construction of a direct complement to the image of the operator $\mathcal{L}$ in a suitable space of formal series. However, one can hardly hope to find here a convergent normal form suitable for all situations in the general case.

Let us fix some positive CR-dimension equal to n. There are types for which all nondegenerate model surfaces of the given type are equivalent and their automorphism groups are isomorphic. That is, in essence, there is only one model surface. However, this an exception. In general, the space of nondegenerate weight forms $\Omega = \{(\Phi_1, \dots, \Phi_l)\}$ splits into many orbits of the action of part (d) of Theorem 5; moreover, as a rule, the family of orbits depends on continuous parameters (has continuum many elements). In this connection, and also in connection with Theorem 12, we formulate a question. Is it true that, for a chosen CR-dimension and a Bloom–Graham type, there is a uniform bound, i.e., independent of the choice of a nondegenerate model surface, for the dimension of the group? For some types, uniform bounds of this kind, which are exact, are written out.

Let us offer here an argument showing that such a bound exists in the general case as well. Let us choose a CR-dimension n and some type m. Denote by $D(n,m)$ the maximum of dimensions of the automorphism groups over all nondegenerate model surfaces of the CR-dimension n and the Bloom–Graham type m and by $Ds(n,m)$ the maximum of dimensions of the stabilizers of the origin for these n and m. Denote by $D(n,\mu)$ the maximum of dimensions of the automorphism groups over all nondegenerate model surfaces of the highest weight $\mu = m_l$ and the CR-dimension n, and by $Ds(n,\mu)$, similarly, the maximum of dimensions of the stabilizers of the origin for these n and μ. Denote by $D(n,K)$ the maximum of dimensions of the automorphism groups over all nondegenerate model surfaces of CR-dimension n and of codimension K, and by $Ds(n,K)$ the maximum of dimensions of the stabilizers of the origin for these n and K.

Theorem 14. $Ds(n,m) \leqslant D(n,m) < \infty$, $\quad Ds(n,\mu) \leqslant D(n,\mu) < \infty$, *and* $Ds(n,K) \leqslant D(n,K) < \infty$.

Proof. Let us prove the first inequality. Let Q be an arbitrary nondegenerate model surface for the chosen n and m and let $\operatorname{aut} Q_\xi$ be the Lie algebra at the point ξ. To estimate the dimension of the algebra at zero, it suffices to estimate this dimension for a point close to zero outside a proper analytic subset (at a point in general position). The surface Q is holomorphically nondegenerate and has a finite type everywhere. Therefore, we can choose a point ξ close to zero at which Corollary 12.3.3 of [10] can be applied to Q; according to this corollary, a local automorphism of Q is uniquely determined by its jet at the point ξ of order $(K+1)\,n$, where K is the codimension corresponding to the type, i.e., the sum of the multiplicities for this type. Here the dimension of the ambient space is $n+K$. Therefore, we can estimate the dimensions of the algebras using the dimension of a jet. To prove the second inequality, we note that, for any chosen values of n and μ, there are only finitely many types for the possible types m. Let us prove (c). Let Q be a model surface. Consider the sequence of distributions D_ν generated by complex tangents using the operation of taking brackets. If there is a complete neighborhood of a point of Q where the dimension D_ν is not full and does not increase at every point by at least one, then this means that this surface has the infinite type at every point of this neighborhood, which contradicts Theorem 10. Thus, at a point in general position, the depth of successive operations of taking the bracket (and this is the very highest degree μ at this point) does not exceed $K+1$. Now the third inequality follows from the second one. The bounds for the stabilizers are obvious. The theorem is proved.

The bounds for dimensions arising in this argument are obviously far from being exact.

The holomorphic nondegeneracy of a real analytic manifold is a global characteristic. The holomorphic degeneracy at one point means the holomorphic degeneracy at all other points. The same holds for the holomorphic nondegeneracy. However, the finiteness of the type of a manifold at a point is a characteristic which can vary even for a real analytic manifold. A simple example is the hypersurface in $\mathbf{C}^2$ given by the equation $v = u\,|z|^2$. However, as proved in Theorem 10, this is impossible for any model surface. Theorems 10 and 12 imply the following corollary.

Corollary 15. *If a model surface Q is nondegenerate* (*Definition 8*) *at the origin, then it is nondegenerate at every point. In this case,* $\dim \operatorname{aut} Q_\xi < \infty$ *for every* ξ.

We note another circumstance. According to Definition 8, the condition of nondegeneracy for a manifold at a point is the condition of the nondegeneracy of its model surface at the point. In turn, the nondegeneracy condition of the model surface consists of two requirements. From the condition of the finiteness of the type and the holomorphic nondegeneracy. The condition of finiteness of the type for a germ and the same condition for its model surface is one and the same condition (see

[8]). How are the conditions of holomorphic nondegeneracy for a manifold and for the model surface related? Theorem 12 immediately implies the following assertion.

Corollary 16. *Let there be a germ of M_ξ and its model surface Q_0. If Q_0 is nondegenerate, then M_ξ is holomorphically nondegenerate.*

That is, the nondegeneracy of the model surface implies also the validity of both the conditions for the germ. Here the simple example of the hypersurface in $\mathbf{C}^3$ of the form $\{v = |z_1|^2 + |z_2|^4\}$ shows that the holomorphic nondegeneracy of a manifold certainly does not imply the holomorphic nondegeneracy of the model surface.

As proved in [7], for rigid model surfaces, i.e., for surfaces given by the equations $v_j = \Phi_j(z, \bar{z}),\ j = 1, \dots, l$, the necessary condition of holomorphic homogeneity, namely, the condition of the constancy of the Bloom–Graham type, is sufficient. The same criterion remains valid for an arbitrary model surface of finite type.

Let a model surface Q be given by the equations

$$v_j = \Phi_j(z, \bar{z}, u_1, \dots, u_{j-1}), \quad j = 1, \dots, l,$$

where Φ_j is a form of weight m_j in the reduced form, where the coordinate forms are linearly independent, i.e., Q has the finite type m at the origin. Let $\xi = (a\ , b_1 + i\,\Phi_1(a, \bar{a}), \dots, b_j + i\,\Phi_l(a, \bar{a}, b_1, \dots, b_{l-1}))$ be some point of Q.

Theorem 17.

(a) For a model surface Q of finite type, a holomorphic automorphism that takes the origin to a point $\xi \in Q$ exists if and only if Q and ξ have the same Bloom–Graham type as that at the origin. In particular, this means that the condition of constancy for the type is a criterion for holomorphic homogeneity for the model surface Q.

(b) If the types at the origin and at ξ coincide, then one can translate the origin to ξ without changing the weights of the coordinates of the space by a triangular-polynomial mapping S_ξ of the form

$$z \to z + a, \quad w_j \to w_j + P_j(z, w_1, \dots, w_{j-1}; \xi), \quad j = 1, \dots, j,$$

where P_j is a holomorphic polynomial in $(z, w_1, \dots, w_{j-1})$ whose weight is strictly less than m_j; moreover, this "shift" S_ξ to the point ξ is defined uniquely.

Proof. Consider the transformation $(z \to z + a,\ u_j \to u_j + b_j)$, which transfers the origin to the point $(a, b_1, \dots b_l)$. For each vector-valued form Φ_j, we can write

$$\begin{aligned}&\Phi_j(z + a, \bar{z} + \bar{a}, u_1 + b_1, \dots, u_{j-1} + b_{j-1})\\ &= \Phi_j(z, \bar{z}, u_1, \dots, u_{j-1}) + \Delta\Phi_j\,(z, \bar{z}, u_1, \dots, u_{j-1},\, a, \bar{a}, b_1, \dots, b_{j-1}),\end{aligned}$$

where, for every chosen $(a, b_1, \dots b_{j-1})$, the expansion of $\Delta\Phi_j$ in the sum of weight components contains only weight components that are strictly less than m_j. In new coordinates, the equation of Q has neither a standard form nor a reduced form. The right-hand sides of the equations cease to be weighted homogeneous.

The procedure of reducing to the standard form, which is described in [8], is a step-by-step procedure indexed by the numbers of coordinates of the group w. That is, if we stop this process before its final completion, then the weights already assigned to the processed variables are not changed in what follows, as well as the form of the coordinates Φ obtained already in the standard form. Consider the first group of equations corresponding to the coordinate w_1. The variable z has the weight 1, regardless of what follows. Accordingly, the weight (which is equal here to the degree) of the form Φ_1 does not change and is equal to m_1. All possibilities for removing the occurring summands of weights that are less than m_1 are implemented by polynomially-triangular transformation using the holomorphic property of the monomials of the form z^α. In this way, all pluriharmonic terms are removed. If, after this, some coordinate of the group w_1 contains a term of weight less than m_1, then this means a type change. Namely, the occurrence of a new weight which is less than the minimum weight m_1. Thus, it follows from the condition of conservation of type that, after a triangular transformation of the form ($z \to z,\ w_1 \to w_1 + P_1(z; a, \bar{a})$), the first group of equations returns to its previous form, namely, $v_1 = \Phi_1(z, \bar{z})$. Here the polynomial P_1 is defined uniquely.

Further, let we have reduced all equations of the groups $(w_1, w_2, \dots, w_{j-1})$ at the point ξ to the standard form. The weights of all these variables remained the same, and the corresponding coordinate forms have not changed. Moreover, the weights of all summands standing on the right-hand sides of any equation of the j-th group are well defined. In particular, the highest term has the form $\Phi_j(z, \bar{z}, u_1, \dots u_{j-1})$ and the weight equal to m_j. All the possibilities for the next step use the holomorphic property of the monomials of the form $z^\alpha\ w_1^{\beta_1} \dots w_{j-1}^{\beta_{j-1}}$ of weights less than m_j, and these are implemented by polynomial-triangular transformations of the form $w_j \to w_j + P(z, w_1, \dots, w_{j-1})$ (z and all the lower w's are kept), where the weight of the polynomial P is strictly less than m_j and the polynomial is uniquely defined. If not all terms of lesser weight disappear, than this means a decrease of the multiplicity of the weight m_j and an increase of the multiplicity of one of the lesser weights. Thus, it follows from the condition of conservation of type that the equation of j-th group returns to the old form $v_j = \Phi_j(z, \bar{z}, u_1, \dots u_{j-1})$ and the variables of the group w_j preserve the old weight m_j. Completing this process, we obtain a proof of the theorem.

In the proof of Theorem 17, we have used the process of constructing standard or reduced forms of the surface equation. It makes sense to return once again to this process and to give it a more algebraic and algorithmic description.

Initially, we have equations of the germ of a manifold at the origin, of the form $v = F(z, \bar{z}, u)$, where F is a real vector-valued power series without the free and linear terms. Denote the space of all real scalar series in $(z, \bar{z}, u)$ by $\mathcal{F}$. The complex tangent at the origin is the plane of the variable z, i.e., $\{w = 0\}$. We assign the weight 1 to the variable z and the weight ∞ to the other variables. Below, the weights of the variables in the group w will obtain new finite values.

First step. Let $m_1 \geqslant 2$ be the lowest nonzero degree existing among the coordinates F, and let $\mathcal{F}_1$ be the finite-dimensional linear subspace of $\mathcal{F}$ generated by the components of the coordinates of F of weight m_1. Let $\mathcal{H}_1$ be the subspace of $\mathcal{F}_1$ generated by real and imaginary parts of the holomorphic monomials in z (the space of pluriharmonic polynomials of weight m_1), and the space $\mathcal{S}_1$ is the subspace of polynomials in the standard form. Then the space $\mathcal{F}_1$ is decomposed into the direct sum $\mathcal{H}_1 + \mathcal{S}_1$. Let $k_1 = \dim \mathcal{S}_1$ and let $\Phi_1 = (\Phi_1^1, \dots, \Phi_1^{k_1})$ be some basis of this space. Then the equations of the germ, after a triangular-polynomial transformation, can be represented in the form

$$v_1 = \Phi_1(z, \bar{z}) + o(m_1), \quad \tilde{v_1} = \tilde{\Phi}_1(z, \bar{z}, u_1, \tilde{u}_1)$$

where $w_1 = (w_1^1, \dots, w_1^{k_1})$ where is the part of variables of the group w that correspond to the coordinates Φ_1, the variables $\tilde{w}_1$ are the remaining variables of the group w, and $\tilde{\Phi}_1$ are the corresponding right-hand sides of the equations. Now the variable $w_1 \in \mathbf{C}^{k_1}$ obtains the weight m_1, and we go to the second step.

Second step. Let $m_2 > m_1$ be the lowest nonzero degree occurring among the coordinates $\tilde{\Phi}_1$, and let $\mathcal{F}_2$ be the finite-dimensional linear subspace of $\mathcal{F}$ generated by the components of the coordinates $\tilde{\Phi}_1$ of the weight m_2. Let $\mathcal{H}_2$ be the subspace of $\mathcal{F}_2$ generated by the real and imaginary parts of the holomorphic monomials in (z, w_1) (the space of pluriharmonic polynomials of weight m_2 on $Q(1)$) and the space $\mathcal{S}_2$ be the subspace of polynomials in the standard form. Then the space $\mathcal{F}_2$ decomposes into the direct sum $\mathcal{H}_2 + \mathcal{S}_2$. Then the equations of the germ, after a triangular-polynomial transformation, can be represented in the form

$$v_1 = \Phi_1(z, \bar{z}) + o(m_1),\ v_2 = \Phi_2(z, \bar{z}, u_1) + o(m_1), \quad \tilde{v_2} = \tilde{\Phi}_2(z, \bar{z}, u_1, u_2, \tilde{u}_2)$$

where $w_2 = (w_2^1, \dots, w_2^{k_2})$ is the part of variables of the group $\tilde{w}$ that correspond to the coordinates of Φ_2, the variables $\tilde{w}_2$ are the remaining variables of the group w, and $\tilde{\Phi}_2$ are the corresponding right-hand sides of the equation. The variable $w_2 \in \mathbf{C}^{k_2}$ obtains the weight m_2 and we go to the next step.

In the case of a germ of finite type, this process forms the equations of the germ in the reduced form in finitely many steps. Write $\mathcal{H} = \sum_1^l \mathcal{H}_j$ and $\mathcal{S} = \sum_1^l \mathcal{S}_j$. Denote by π the projection to the component $\mathcal{S}$. Now a condition ensuring that there is a "shift" S_ξ of the origin to a point ξ can be given in the following form.

Proposition 18. *An automorphism of Q taking the origin to the point*

$$\xi = (a\ , b_1 + i\,\Phi_1(a,\bar a), \ldots, b_j + i\,\Phi_l(a,\bar a, b_1, \ldots, b_{l-1}))$$

exists if and only if

$$\pi(\Delta\Phi_j(z,\bar z, u_1, \ldots, u_{j-1},\, a, \bar a, b_1, \ldots, b_{j-1})) = 0$$

for all $j = 1, \ldots, l$.

Proof. This follows from our argument in the proof of Theorem 16.

Theorem 19. *(a) If Q is holomorphically homogeneous, then the list of weights defining the type of Q can have only the form*

$$(m_1 = 2,\ m_2 = 3, \ldots, m_l = l+1).$$

(b) If Q is a homogeneous model surface of codimension K, then the highest weight $\mu(Q)$ is not greater than $K+1$.

Proof. If $m_1 > 2$, then, for $a \in \mathbf{C}^n$ in general position, $\Phi_1(z+a, \bar z + \bar a)$ contains nonzero not pluriharmonic summands. This means the reduction of the minimal weight m_1. Similarly, if there is a weight m_j for which the weight $m_j - 1$ is absent, it follows that, for a shift of Φ_j to a point in general position, $\Delta\Phi_j$ obtains nonzero summands of weight $m_j - 1$ that (Theorem 17) cannot be reduced to zero. This means the change of type and contradicts the homogeneity. We obtain (a); part (b) follows immediately from (a).

Remark 20. When moving to a point $\xi \in Q$, the type can change (remaining finite), while the highest weight m_l cannot increase. After making a triangular-polynomial change of coordinates taking the new components to the standard form, we obtain the equations of Q in the new coordinates with the origin at the point ξ. If the type had really changed, then the surface cannot be model in these new coordinates. In the coordinate at which the weight had reduced, there is an unremovable component of higher weight which was the lowest before the shift and which is not affected by the process of reduction to the standard form. However, this does not prevent the case in which the surface turns out to be locally biholomorphically equivalent to its new model surface. But, to implement this equivalence, one needs transformations going beyond the triangular-polynomial transformations that were used in the reduction.

The algebra aut Q can be represented in the form $g_- + g_0 + g_+$, where g_- is the sum of weight components of the negative weights, g_0 stands for the fields of weight zero, and g_+ for the fields of positive weights. Obviously, each of the three summands in a Lie subalgebra of aut Q. Let us give a description of the groups corresponding to these subalgebras.

Let G_- be the family of triangular-polynomial "shifts" S_ξ for all ξ belonging to the orbit Orb_0 of zero in the automorphism group of Q. Obviously, G_- is a group with respect to the composition operation. Here we obtain a one-to-one correspondence $(\xi \to S_\xi)$ between the points of Orb_0 and shifts in G_-.

Every automorphism of Q can be represented as a composition of a transformation preserving the origin, i.e., belonging to the stabilizer, and a transformation belonging to G_-. Therefore, the orbit of the origin Orb_0 with respect to the entire group Aut$Q$ coincides with the orbit with respect to the action of G_- only.

Let

$$\xi = (a, b_1 = br_1 + i\,bi_1,\ b_2 = br_2 + i\,bi_2, \ldots, b_l = br_l + i\,bi_l) \in Orb_0$$

then S_ξ, in accordance with the procedure for obtaining equations in the reduced form, has the form

$$\begin{aligned}
&z \to z + a,\\
&w_1 \to w_1 + \Delta_1(z, a, \bar a) + b_1,\\
&w_2 \to w_2 + \Delta_2(z, w_1, a, \bar a, br_1) + b_2,\\
&\ldots\\
&w_l \to w_l + \Delta_l(z, w_1, \ldots, w_{l-1}, a, \bar a, br_1, \ldots, br_{l-1}) + b_l,
\end{aligned} \tag{16}$$

Here the correcting summands $\Delta_j\,(z, w_1, \dots, w_{j-1}, a, \bar a, br_1, \dots, br_{j-1})$ are jointly homogeneous of weight m_j, are holomorphic with respect to $(z, w_1, \dots, w_{j-1})$, and vanish at $(a, b_1, \dots, b_{j-1}) = 0$. Thus, if we take the expansion of Δ_j in weights of (z, w) only, then the expansion contains components of the weights from 1 to $m_j - 1$. If $\xi = 0$, then $S_\xi = Id$, i.e., to the origin, there corresponds the identity of G_-.

As a manifold, G_- coincides with the orbit Orb_0 of the origin, i.e., with some submanifold of Q containing the origin. For the case in which Q is homogeneous, if we identify the space $\mathbf{C}^{n+K}$ of variables (z, w) and the space $\mathbf{C}^{n+K}$ of variables (a, b), then we identify Q and G_-. Here G_- becomes a CR-submanifold of $\mathbf{C}^{n+K}$ globally holomorphically equivalent to Q. After this, we can say that Q acts on itself by triangular-polynomial automorphisms of $\mathbf{C}^{n+K}$.

It follows from our description of the shifts S_ξ that the Lie algebra corresponding to G_- is contained in g_-. We claim that, if a field in g_- of the form

$$X = \mathrm{Re}(f\,\frac{\partial}{\partial\, z} + \sum g_j\,\frac{\partial}{\partial\, w_j})$$

vanishes at the origin, then it vanishes. Let us prove this by induction. The first step. The factor f has the weight zero, i.e., this is a constant, and it is equal to zero. Further, g_1 consists of components of the weight not higher than $m_1 - 1$. The first two weighted coordinates of the 1-parameter group of transformations generated by X have the form

$$z \to z, \quad w_1 \to w_1 + g_1(z)\,t + \text{terms of higher order with respect to } t$$

Substituting this into the first group of equations of Q, we obtain $\mathrm{Im}(g_1(z)) = 0$. This implies that $g_1 = 0$. Let $g_\nu = 0$ for $\nu < j$. In just the same way we see that the imaginary part of the holomorphic coefficient g_j is zero on the truncated generating model manifold Q^j. Since this coefficient is equal to zero at the origin, it is zero. Thus, g_- is the Lie algebra corresponding to G_-.

Let us now consider the subgroup G_0 of transformations belonging to the stabilizer of the origin of the form

$$(z \to C\,z,\ w_j \to \rho_j\,w_j + \theta_j(w_1, \dots, w_{j-1}),\ j = 1, \dots, l),$$

where $C \in GL(n, \mathbf{C})$, $\rho_j \in GL(k_j, \mathbf{R})$ and where $\theta_j(w_1, \dots, w_{j-1})$ is a homogeneous real form of weight m_j and, for all $j = 1, \dots, l$,

$$\begin{aligned}\rho_j\,\Phi_j(z, \bar z,\, u_1, \dots, u_{j-1}) + \mathrm{Im}\,\theta_j((u_1 + i\,\Phi_1(z, \bar z)), \dots, (u_{j-1} + i\,\Phi_{j-1}(z, \bar z, u_1, \dots, u_{j-2}))))\\ = \Phi_j(C\,z, \overline{C\,z}, \rho_1\,u_1, \dots, \rho_{j-1}\,u_{j-1})\end{aligned} \tag{17}$$

To obtain a description of the Lie algebra corresponding to G_0, we consider a 1-parameter subgroup of the form

$$C\,z = z + t\,\alpha\,z + \dots, \qquad \rho_j\,w_j = w_j + t\,\beta_j\,w_j + \dots,$$
$$\theta_j(w_1, \dots, w_{j-1}) = t\,\gamma_j(w_1, \dots, w_{j-1}) \dots,$$

where $t \in \mathbf{R}$, and the dots stand for higher-order terms with respect to t. Let us substitute this in (17), single out the part linear in t, and set $t = 0$. We see that the Lie algebra corresponding to this Lie group consists of the vector fields of the form

$$X = \mathrm{Re}(\alpha z\,\frac{\partial}{\partial\, z} + \sum_{j=1}^{l}(\beta_j\,w_j + \gamma_j(w_1, \dots, w_{j-1})\,\frac{\partial}{\partial\, w_j})) \tag{18}$$

and

$$\beta_j\,\Phi_j + \mathrm{Im}\gamma_j((u_1 + i\,\Phi_1), \dots, (u_{j-1} + i\,\Phi_{j-1}) = 2\,\mathrm{Re}(\partial_z\,\Phi_j(\alpha\,z)) + \sum_{\nu=1}^{j-1} \partial_{u_\nu}\,\Phi_j(\beta_{nu}\,u_\nu) \tag{19}$$

Here we see that (18), under condition (19), is the general form of a field of weight zero, i.e., g_0 is the Lie algebra corresponding to the Lie group G_0.

Let, further, G_+ is the subgroup of automorphisms of Q having the form

$$z \to z + o(1), \quad w_j \to w_j + o(m_j), \quad j = 1, \dots, l;$$

it is clear that the fields generating transformations in G_+ can belong to g_+ only.

Let us sum up.

Theorem 21. *(a.1)* g_- *is the Lie algebra corresponding to the Lie group* G_-*. Here the correspondence* $\xi \to S_\xi$ *identifies the orbit* Orb_0 *of the origin and the subgroup of triangular-polynomial transformations* G_-*.*

(a.2) If Q *is holomorphically homogeneous, i.e.,* $Orb_0 = Q$*, then* G_- *turns out to be realized as the family of triangular-polynomial transformations of the form* (16) *parametrized by the system of parameters* $(a, b) \in \mathbf{C}^{n+K}$ *satisfying the same system of equations as the points of* Q*. This parametrization enables one to regard* G_- *as a real algebraic surface in* $\mathbf{C}^{n+K}$ *holomorphically equivalent to* Q *as a* CR*-manifold.*

(b) g_0 *is the Lie algebra corresponding to the Lie group* G_0*. Here nonlinear terms* γ_j *in* g_0*, as well as nonlinear terms* θ_j *in* G_0*, can be present only in the presence of resonances of the form* $m_j = \mu_1 m_1 + \cdots + \mu_{j-1} m_{j-1}$ *with nonnegative integers* μ_ν*.*

(c) g_+ *is the Lie algebra corresponding to the Lie group* G_+*. Here, if* Q *is nondegenerate, then* g_+ *is finite-dimensional, finitely graded, and consists of fields with polynomial coefficients.*

Proof. It remains to prove only the correspondence $G_+ \to g_+$. This follows from the fact that every element in the stabilizer of zero can be represented as a composition of a mapping in G_+ and a mapping in G_0 (Corollary 11) and from part (b).

4. OPEN QUESTIONS

As noted above, the subalgebra g_+, under the assumption that Q is nondegenerate, consists of finitely many weight components $g_1 + \cdots + g_\delta$. If Q is *completely* nondegenerate and $\mu \geqslant 3$, then, as was shown by Kossovskiy [11], Sabzevari and Spiro [5], and also Gregorovich [6], we have $g_+ = 0$, i.e., $\delta = 0$. If Q is nondegenerate only, then this is not the case. A question to estimate the highest positive weight $\delta(Q)$ of the subalgebra g_+ for an arbitrary nondegenerate Q arises.

Conjecture 1 (a new version of the g_+-conjecture):

There is a constant $C(n, m)$ such that, for all nondegenerate model surfaces Q with the CR-dimension n and the Bloom–Graham type m, the bound $\delta(Q) \leqslant C(n, m)$ holds. In other words, for chosen n and m, the value of δ cannot be arbitrarily large.

The question of connecting this model surface theory of finite type to the Tanaka theory [17] is closely related to these problems. In this connection, we can suggest the following conjecture.

Conjecture 2: If Q is nondegenerate and holomorphically homogeneous, then g_- is fundamental.

In this case, the problem concerning the bound for $\delta(Q)$ (the highest positive weight of g_+) can be treated in the context of Tanaka prolongation.

As was shown in Theorem 18, for a homogeneous model surface, the list of weights has a very definite form $m_j = j + 1$. What can be said about their multiples k_j? For the surface

$$v_j = (\mathrm{Im} z)^{j+1}, \quad j = 1, \ldots, l$$

of the type $(2, 3, \ldots, l + 1)$, we have $k_1 = \cdots = k_l = 1$. This is a surface of CR-dimension 1 and codimension l. This example was examined in [22]. On the other hand, for completely nondegenerate model surfaces, the multiplicities grow rapidly as the weight increases. However, it is clear that the multiplicities cannot be arbitrary.

Question 3: Describe the conditions satisfying by the multiplicities of weights $(k_1, \ldots, k_l)$ for a holomorphically homogeneous model surface.

As shown in Theorem 10 (part (d)), if $\mu_{\min}$ is the minimal value of the highest weight over all points of Q, then $Q^{\mu_{min}}$ is an open subset of Q. However, we do not claim that this is the only open stratum.

Question 4: Are there weights $\nu > \mu_{\min}$ for which Q^ν is open? That is, are there open strata that differ from the minimal one?

Let us choose a pair (n, K), i.e., the CR-dimension and the codimension. As shown in Theorem 14, there is a uniform bound for the dimension of automorphisms $D(n, K)$ for all nondegenerate model surfaces.

Conjecture 5 (a new version of the dimension conjecture):

Let M_ξ be an arbitrary generating real analytic germ of CR-dimension n and of codimension K. If $\dim \operatorname{aut} M_\xi$ is finite, then

(5.a) for the full dimension, $\dim \operatorname{aut} M_\xi \leqslant D(n, K)$,

(5.b) for the dimension of the stabilizer, $\dim \operatorname{aut}_\xi M_\xi \leqslant Ds(n, K)$,

That is, it is claimed that, if the dimension of the group of the germ is finite, then it does not exceed the maximum dimension over all nondegenerate model surfaces of the same CR-type (i.e., CR-dimension and codimension), and the dimension of the stabilizer does not exceed the maximum dimension of the stabilizers. Certainly, when formulating a hypothesis, one could choose the highest weight μ or the type m rather than the codimension K.

For a hypersurface Γ in $\mathbf{C}^2$, i.e., for $n = K = 1$, part (a) of the conjecture is rather obvious. Indeed, if the Levi form of this hypersurface is identically zero, then, locally, this is a hyperplane, and the dimension is infinite. Let this condition fail to hold. In a neighborhood of $\xi \in \Gamma$, take nine fields belonging to $\operatorname{aut} \Gamma_\xi$. We consider these fields in a neighborhood of a close point at which Γ is Levi nondegenerate. The dimension of the algebra does not exceed there the dimension of the algebra of the model sphere, i.e., eight. Thus, the fields are linearly dependent, and, by analyticity, this dependence continues to the point ξ. However, even in this situation, an estimate for the dimension of the stabilizer, i.e., part (b), is a much more subtle problem requiring more subtle reasoning [18]. For a hypersurface in $\mathbf{C}^3$, i.e., for $K = 1$ and $n = 2$, Conjecture (5.a) was proved in [19] and Conjecture (5.b) in [20]. For a hypersurface in $\mathbf{C}^4$, i.e., for $K = 1$ and $n = 3$, the question is open for now.

In [21], using an example of a surface of CR-type (2,5), it was shown that, if the class of model surfaces is treated as that of completely nondegenerate model surfaces, then the conjecture fails to hold. Namely, an example of a surface of this type was given, which is not completely nondegenerate, and whose automorphisms have the dimension that is larger than that for all completely nondegenerate model surfaces of this type. In the new terminology, the surface from this example is a nondegenerate model surface of CR-dimension $n = 2$, its Bloom–Graham type is $m = (2, 2, 3, 3, 4)$, and the codimension is $K = 5$.

In [7], for rigid model surfaces Q, it was shown that the group $\operatorname{Aut} Q$ consists of birational mappings of the space $\mathbf{C}^N$ whose degrees are bounded by a constant depending only on n and K. The proof is based on the trick of W. Kaup [15]. Apparently, this remains true for the general model surfaces of finite type.

Conjecture 5: Let Q be a nondegenerate holomorphically homogeneous model surface of CR-dimension n and the Bloom–Graham type m; then the group $\operatorname{Aut} Q$ consists of birational transformations of $\mathbf{C}^N$ whose degrees are bounded by a constant depending on n and m only.

Any generating real analytic submanifold of M that has a finite type at a point ξ, as we have seen above, can locally be regarded as a perturbation of the model surface. It is appropriate to ask a question about the conditions of holomorphic equivalence of the germ M_ξ and its model surface Q. By analogy with hypersurfaces, we call such germs *spherical* at the point ξ. It is clear that, if a germ M_ξ is equivalent to Q_0, then the stabilizer of ξ in $\operatorname{aut} M_\xi$ contains a field $X_0 \neq 0$, namely, the field corresponding to (12). We can suggest a criterion of sphericity at a point, which is close to tautology.

Proposition 22. *A generating germ M_ξ is spherical if and only if the equations of M_ξ in some coordinates have the form* (3) *and the field X_0 has the form* (12) .

Proof. Writing the equation of M_ξ in these coordinates in the form (2) and using the extensions (11) generated by X_0, we obtain (3).

Note that the condition of finiteness of type is not used here. Using the field X_0, we can naturally introduce a grading on $\operatorname{aut} M_\xi$. After this, we can propose several necessary conditions for sphericity related to the fact that, in the coordinates (3), the field (12) generates the adjoint action on every weight component $\operatorname{aut} Q_0$,

$$X \to ad_{X_0}(X) = [X, X_0]$$

The equivalence of Q means that the field X_0 can be linearized, and the adjoint operator generated by it is diagonalized with the known family of eigenvalues, namely, $(1, m_1, \dots, m_l)$, with the multiplicities of these values k_j corresponding to the type.

Question 7: Suggest a constructive criterion for the sphericity of a manifold of a chosen Bloom–Graham type.

For some CR-types, in particular, for hypersurfaces, an answer to Question 7 is known.

Let M_ξ be the germ of a nondegenerate holomorphically homogeneous submanifold of a complex space and let Q be its model surface at the point ξ. Let M be a real analytic embedded submanifold, in some domain, which is a representative of our germ. It immediately follows from the local homogeneity of M that the model surfaces at all points of M are equivalent to Q, i.e., M has in essence the same model surface everywhere. However, there is no reason for the model surface Q itself to be holomorphically homogeneous. Here is the related question.

Question 8: Is there a locally homogeneous real manifold M such that its unique model surface Q is not homogeneous?

5. WHAT REMAINS BEHIND THE SCENES

Consider the example of a hypersurface from [23]. This is a hypersurface in the space $\mathbf{C}^{n+2}$ with the coordinates $(z_1, \dots, z_n, \zeta, w = u + i\, v)$ which is given by the equation

$$v = 2\,\mathrm{Re}(z_1\bar{\zeta} + \dots + z_n\bar{\zeta}^n).$$

This hypersurface is of the Bloom–Graham type $m = (2)$, and the model surface $Q = \{v = 2\ \mathrm{Re}(z_1\bar{\zeta})\}$ is holomorphically degenerate; the dimension of its automorphism group is infinite, while the automorphisms of the hypersurface itself are finite-dimensional (Theorem 12 becomes meaningless). We obtain these facts using the standard approach in which the weights of all coordinates z and ζ are the same and equal to 1. However, if we arrange the weights differently, namely: $([\zeta] = 1,\ [z_j] = 1 + n - j,\ [w] = n + 1)$, then the surface becomes weighted homogeneous (of weight $n+1$). Moreover, this hypersurface is holomorphically homogeneous. Using the "weighted" version of the Poincaré construction, we can obtain a bound for dimension of the automorphisms for the germ of the perturbed hypersurface

$$v = 2\mathrm{Re}\,(z_1\bar{\zeta} + \dots + z_n\bar{\zeta}^n) + o(n + 1).$$

It is clear that a **"weighted" theory of model surface finite type** is behind this example. Moreover, the first step in constructing such a theory, namely, the proof of a "weighted" analog of the Bloom–Graham theorem, has already been made by M. Stepanova [13]. Such a theory, when it will be constructed, will significantly expand the scope of applicability of the model surface method.

Problem 9: Construct a "weighted" theory of model surfaces of finite type that includes the possibility to introduce different weights of the coordinates of the complex tangent and based on the Bloom–Graham–Stepanova type of a manifold.

Global theory of singularities. A nondegenerate model surface is a globally defined object (algebraicity, polynomial nature of the automorphism algebra, etc.); however, at the same time it is connected with the original manifold *locally*, because this is a characterization of the germ at a point. Let there be a smooth compact real submanifold M of the complex space $\mathbf{C}^N$ of real dimension greater than N. Under this condition, the complex part of the tangent space at every point must have positive dimension. By a small deformation in a neighborhood of some point of M, one can remove all degenerations, namely, make the germ generating, completely nondegenerate, and nonspherical. If we assume that we are in a situation where $g_+ = 0$, then, adding to the nonsphericity some simple additional condition of nondegeneracy (let us call this condition the additional asymmetry), we may assume that the CR-structure in this neighborhood reduces to the $\{e\}$-structure, and we have a CR-invariant frame of the tangent bundle over this neighborhood (see [25]). However, there are global conditions that are formulated in terms of characteristic classes of the tangent bundle that prohibit the trivialization of the tangent bundle over the whole manifold. If M is such a manifold, then, after any deformation, there should be points on M at which some of the listed conditions (generation property, complete nondegeneracy, nonsphericity, additional asymmetry) are not satisfied.

Here is an example of such a situation. Let M^4 be a smooth (or even real analytic) 4-dimensional compact submanifold of $\mathbf{C}^3$, i.e., the codimension is $K = 2$. Locally, after a small deformation, we can always assume that M^4 is a generating completely nondegenerate manifold of CR-dimension $n = 1$ and its Bloom–Graham type is $m = (2, 3)$. Let M^4 be topologically simple, for example, diffeomorphic to a 4-dimensional sphere, which, as is known, is not parallelizable. What kinds of singularities are possible for such a manifold? Consider $T_\xi M$, the tangent to M at some point ξ. There are two possibilities. Either this plane contains a one-dimensional complex tangent, or it is itself a two-dimensional complex plane. In the first case, M is in a neighborhood of ξ a generating CR-manifold of CR-dimension one and of codimension two, and the Bloom–Graham type is well defined for M. In the other case, we are dealing with an RC-singular point. The set of these points is obviously closed. In terms of complex gradients of local defining functions, $M = \{\rho_1 = \rho_2 = 0\}$, the condition that a point is RC-singular reduces to the fact that the gradients $(\mathrm{grad}\rho_1, \mathrm{grad}\rho_2)$ are collinear. The collinearity of two three-dimensional vectors gives two complex conditions or four real conditions. We can expect that four relations on a 4-dimensional compact manifold in general position have only a finite, possibly empty, set of solutions. At each point of the complement to this set, the Bloom–Graham type is well defined. Moreover, after a small deformation, we may assume that, at the point in general position (outside a proper analytic subset), this is the base type $(2, 3)$. We call the points of the complement to this set the *BG-singular points*. Thus, the problem is reduced to the description of the set RC and of BG-singular points.

Degenerations at the level of 1-jet (RC-singular points) were studied in a series of papers (see [27]). However, it cannot be excluded that unremovable degenerations touch also the Bloom–Graham type of a point.

Problem 10: Describe generic (i.e., unremovable by a small deformation) degenerations and their invariants.

Surfaces of infinite type. Here is a simple example of such a hypersurface in $\mathbf{C}^2$:

$$v = u\,|z|^2.$$

This hypersurface has infinite type at the origin, and also at all points of the complex line $w = 0$. At the other points, this hypersurface has finite type(2) and Levi nondegenerate. For real analytic hypersurfaces, the finiteness of the type at a point in general position and the holomorphic nondegeneracy are equivalent [10]. Therefore, to prove Conjecture (5.a), one can ignore points of infinite type lying on a proper analytic submanifold.

However, for manifolds of greater codimension. a situation is possible in which a holomorphically nondegenerate manifold has infinite type *everywhere*. Moreover, the dimension of the algebra of local automorphisms can be either infinite or finite. Thus, for such manifolds, the holomorphic nondegeneracy is only a necessary condition, and the question concerning a criterion of finite-dimensionality of the automorphisms for the germs of such manifolds is open. At the same time, for the surface of a finite type on a dense open set, the holomorphic nondegeneracy is a criterion for finite-dimensionality in accordance with the Stanton–Ebenfelt theorem [10].

Problem 11: Let there be a holomorphically nondegenerate real analytic submanifold whose type at all points is infinite. Find a criterion for the finite-dimensionality of the holomorphic automorphisms of the germ of such a submanifold.

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