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Polynomial Model CR-Manifolds with the Rigidity Condition

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Abstract. In the present paper, the results recently obtained by the author for model manifolds with the Hörmander numbers (2,3) without the condition of complete nondegeneracy are extended to an arbitrary Bloom–Graham type. Here a simplifying assumption is made that the model surface is rigid.

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A CR-manifold in general position is completely nondegenerate [1]. The condition of complete nondegeneracy, in the spirit of the Bloom–Graham constructions [2], can be formulated both analytically and geometrically. In geometric terms, this means that the Levi–Tanaka algebra (the Lie algebra of vector fields generated by complex tangent fields) is a free graded Lie algebra. In analytical terms, this is a condition of the Bloom–Graham type. Namely, the multiplicity of every Hörmander number for a chosen dimension of a complex tangent is maximal. The complete nondegeneracy of the germ of a manifold at a point is equivalent to the complete nondegeneracy of the tangent model surface.

The consideration of completely nondegenerate model surfaces showed that the case of quadratic model surfaces studied earlier (in which the length l of the Levi–Tanaka algebra is equal to two) differs significantly from the general case $l > 2$. The Lie algebra of infinitesimal automorphisms of a completely nondegenerate germ of a CR-manifold is a finitely graded Lie algebra containing negative graded component g_-, the zero component g_0, and the positive component g_+. The components g_- and g_0 are always nontrivial. For $l = 2$, there is a lot of examples with $g_+ \neq 0$. At the same time, for $l > 2$, no example with $g_+ \neq 0$ is known. This enabled us to formulate the corresponding conjecture. This conjecture was proved for $l = 3$ in [4], and for the CR-dimension equal to one in [5].

In 2018, almost simultaneously, two independent proofs of this conjecture in full were published. The first was given by Sabzevari and Spiro [6] and the other, a day later, by Gregorovich [7]. The theorem proved by them has a lot of consequences. For example, for $l > 2$, the theorem implies directly that the biholomorphic mapping of the germ of a completely nondegenerate manifold on another such germ is uniquely determined by the restriction of the differential of the mapping at a single point to the complex tangent. This assertion is similar to Henry Cartan's theorem for domains of bounded form and to the Beloshapka–Loboda theorem for nonspherical real hypersurfaces. Another consequence concerns completely nondegenerate model manifolds. The corollary is that the stabilizer of a point in the group of holomorphic automorphisms of the model surface is some (explicitly described) subgroup of the complete linear group. In other words, the stabilizer of a point contains no nonlinear transformations.

Moreover, we note that, beyond the framework of the condition of complete nondegeneracy, there are many examples of model surfaces with nontrivial g_+-subalgebra and, respectively, with nonlinear automorphisms in the stabilizer of a point. Thus, the proof of the conjecture increases the interest in such model surfaces, which are the very subject of study of the present paper.

In [3], a theory of model surfaces with Hörmander numbers $(2, 3)$ without the condition of complete nondegeneracy was suggested. In the present paper, which can be regarded as a natural continuation of [3], a similar construction is carried out for arbitrary Bloom-Graham type. That is. an arbitrary set of Hörmander numbers of arbitrary multiplicities is admitted. In this case, however, it is assumed that the model surface is rigid (for the definition, see below).

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Let the coordinates in the complex space $\mathbf{C}^N$ be divided into groups

$$z \in \mathbf{C}^n, \quad w_1 \in \mathbf{C}^{k_1}, \ \ldots, \quad w_l \in \mathbf{C}^{k_l}, \quad n + k_1 + \cdots + k_l = N, \ w_j = u_j + i\, v_j, \ 1 \leqslant j \leqslant l.$$

Here a family of weights $2 \leqslant m_1 < m_2 < \cdots < m_l$ is given. A weight 1 is assigned to a variable z and the weight m_j to a variable w_j. Consider a germ M_ξ given in some coordinates with the origin at the point ξ by equations of the form

$$v_j = \Phi_j(z, \bar{z}) + o(m_j), \quad 1 \leqslant j \leqslant l, \tag{1}$$

where Φ_j is a real vector-valued form of homogeneous degree m_j containing no pluriharmonic terms and $o(\mu)$ is the sum of summands depending on $(z, \bar{z}, u)$ of weight exceeding μ. If the coordinates of all vector-valued forms Φ_j are linearly independent, then the given germ has the following type with respect to Bloom–Graham:

$$m = ((m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l)) = (m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots, m_l, \ldots, m_l).$$

In particular, it is a germ of finite type.

In this case, the *tangent model surface to the germ* M_ξ is the surface Q given by the equations

$$v_j = \Phi_j(z, \bar{z}), \quad 1 \leqslant j \leqslant l. \tag{2}$$

Here we say that the germs given by equations of the form (1) are germs *subordinated to the model surface* Q. It is clear that here the tangent space at the origin is $\{v = 0\}$, and its complex part is $\{w = 0\}$. That is, the CR-dimension is n, and z is the coordinate that parametrizes the complex tangent. Moreover, the codimension of the manifold is equal to $K = k_1 + \cdots + k_l$, and the dimension $(2n + K)$.

It is not true that any germ of finite type m has local equations of the form (1). According to the Bloom–Graham theorem, every germ of type m has equations similar to (1), but the forms Φ_j may depend on $u = \mathrm{Re} w$. If there is no dependence of the forms Φ_j on $u = \mathrm{Re} w$, then we say that the model surface Q is *rigid*. It can readily be seen that this statement is equivalent to the fact that, among the automorphisms of the surface Q, there are shifts of the form

$$(z \to z, \quad w \to w + b) \quad \text{for all} \quad b \in \mathbf{R}^K.$$

The term "rigidity" is used, even in CR-geometry, in several ways. Its use in the situation under consideration does not seem to be very successful; however, since it is customary, we shall use it. In what follows, in this paper, we assume that the model surface Q is rigid. We stress that this assumption is made only with respect to model surfaces and does not include the germs of CR-manifolds that are their perturbations.

The above definition of rigidity is related to a fixed system of local coordinates. One can suggest its coordinateless version. Namely, the rigidity of a germ means that, in the Lie algebra of infinitesimal automorphisms, there is a field that differs from zero at the center of the germ and is transversal there to the complex tangent.

Proposition 1.

(a) *The germ M_ξ and its model surface Q have finite type $m = ((m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$ if and only if there are no identical zeros among the coordinates of the forms Φ_j in the Bloom–Graham normal form.*

(b) *The germ M_ξ and its model surface Q have finite type $m = ((m_1, k_1), (m_2, k_2), \ldots, (m_l, k_l))$ if and only if the coordinates of all forms Φ_j are linearly independent.*

(c) *If the coordinate forms are linearly dependent, then Q has infinite type and is not minimal at the origin.*

Proof. The parts (a) and (b) follow from Corollary 8.3 of [2], and (c) is obvious.

Let two germs M_ξ and $\tilde{M}_{\tilde{\xi}}$ be holomorphically equivalent. The type of a germ is a holomorphic invariant, and therefore, the types of the germs coincide. Let the common type of the germs be $m = ((m_1, k_1), (m_2, k_2), \dots, (m_l, k_l))$, and let the surfaces Q and $\tilde{Q}$ be the corresponding tangent model surfaces each of which is defined by the families of forms $\{\Phi_j\}$ and $\{\tilde{\Phi}_j\}$. Let, further,

$$(z \to f(z,w),\ w_j \to g_j(z,w))$$

be a (holomorphic in a neighborhood of the origin) invertible mapping taking M_ξ to $\tilde{M}_{\tilde{\xi}}$ and preserving the origin. In what follows, we shall use expansions of the form

$$f(z,w) = \sum_1^\infty f_j(z,w),$$

where f_j stands for the jth weight component of the expansion f.

Proposition 2.

(a) *The lower terms of the mapping have the form*

$$f(z,w) = C\ z + o(1), \quad g_j(z,w) = \rho_j\ w_j + o(m_j), \tag{3}$$

where $C \in GL(n, \mathbf{C})$ *and* $\rho_j \in GL(k_j, \mathbf{R})$; *here*

$$\tilde{\Phi}_j(z, \bar{z}) = \rho_j^{-1}\ \Phi_j(Cz, \overline{Cz}). \tag{4}$$

(b) *The linear mapping* $(z \to C\ z,\ w_j \to \rho_j\, w_j)$ *takes* Q *onto* $\tilde{Q}$, *i.e., two model surfaces are holomorphically equivalent if and only if they are linearly equivalent.*

Proof. Let the equations of germs have the form

$$v_j = \Phi_j(z,\bar{z}) + \varphi_j(z,\bar{z},u), \quad v_j = \tilde{\Phi}_j(z,\bar{z}) + \tilde{\varphi}_j(z,\bar{z},u).$$

Then the relations expressing the fact that, if $(z,w) \in M_\xi$, then $(f, g_1, \dots, g_l) \in \tilde{M}_{\tilde{\xi}}$, have the form

$$\begin{aligned} \text{Im}\ g_1 &= \tilde{\Phi}_1(f,\bar{f}) + \tilde{\varphi}_1(f,\bar{f},\text{Re}\ g), \\ &\dots \\ \text{Im}\ g_l &= \tilde{\Phi}_l(f,\bar{f}) + \tilde{\varphi}_l(f,\bar{f},\text{Re}\ g), \\ \text{for}\ w_j &= u_j + i\ (\Phi_j(z,\bar{z}) + \varphi_j(z,\bar{z},u)). \end{aligned} \tag{5}$$

Separating the components of weight from 1 to m_1 in the first of these relations, the components from 1 to m_2 in the second one, etc., we obtain (a), which immediately implies (b). This completes the proof of the proposition.

Note that relation (4) defines an action of the direct product of linear groups

$$GL(n,\mathbf{C}) \times GL(k_1,\mathbf{R}) \times \dots \times GL(k_l,\mathbf{R})$$

on the space of forms $\Phi = (\Phi_1, \dots, \Phi_l)$ which define model surfaces of a given type. That is, two model surfaces are CR-equivalent if and only if the corresponding families of forms belong to the same orbit of the above action. By Proposition 2, any invariants of this actions are holomorphic invariants of a germ. This enables us to describe, using the Hilbert basis theorem, the moduli space of model surfaces of a fixed type, to construct a "Gaussian" mapping of a manifold of given type into its moduli space, and to give a definition of CR-characteristic classes. The corresponding constructions for completely nondegenerate surfaces were described in [9]. All of them are transferred to the present context without modifications.

To understand the CR-geometry of a germ, its holomorphic automorphisms are of great interest. Let $\mathrm{aut} M_\xi$ be the Lie algebra of infinitesimal automorphisms. These objects consist of germs of real holomorphic vector fields of the form

$$X = 2\,\mathrm{Re}\left(\alpha\,\frac{\partial}{\partial z} + \sum \beta_j\,\frac{\partial}{\partial w_j}\right), \tag{6}$$

where $(\alpha, \beta_1, \ldots, \beta_l)$ are holomorphic in a neighborhood of the point ξ, and the field X is tangent to M_ξ at the points of M_ξ. Every field of this kind generates a local one-parameter group of invertible holomorphic mappings of the germ into itself. The family of mappings generated in this way is the local automorphism group of M_ξ, which we denote by $\mathrm{Aut} M_\xi$. In $\mathrm{aut} M_\xi$ one can distinguish the Lie subalgebra $\mathrm{aut}_\xi M_\xi$ which consists of the vector fields vanishing at ξ. The fields in this subalgebra generate local one-parameter groups of holomorphic transformations fixing ξ. These transformations form the local subgroup $\mathrm{Aut}_\xi M_\xi$ of $\mathrm{Aut} M_\xi$.

The weights of variables (which were introduced above) enable us to introduce a grading in the Lie algebra of vector fields in space $\mathbf{C}^N$. To this end, the agreement on the weights of variables must be extended to coordinate differentiations by setting

$$\left[\frac{\partial}{\partial z}\right] = -1, \quad \left[\frac{\partial}{\partial w_j}\right] = -m_j.$$

After this, $\mathrm{aut} M_\xi$ also obtains the structure of a graded Lie algebra that splits into a direct sum of graded components, beginning with a component of weight $(-l)$ and, generally speaking, going up to $+\infty$. Denote the component of weight j of the algebra $\mathrm{aut}\, Q$ by $\mathcal{G}_j$, the sum of the negative components of the algebra $\mathrm{aut}\, Q$ by $\mathcal{G}_-$, and that of the positive ones by $\mathcal{G}_+$. Here we have $\mathrm{aut}\; Q = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+$ and $\mathrm{aut}_0\; Q = \mathcal{G}_0 + \mathcal{G}_+$. The algebra of a model surface Q has several specific features.

Proposition 3.

(a) *If a field* $X = \sum_{-l}^{+\infty} X_j$ *belongs to* $\mathrm{aut}\; Q$, *then* $\forall j\; X_j \in \mathrm{aut}\; Q$.

(b) $\mathrm{aut}\; Q$ *contains a field of weight zero,*

$$X = 2\;\mathrm{Re}\left(z\,\frac{\partial}{\partial z} + \Sigma\, m_j\, w_j\,\frac{\partial}{\partial w_j}\right)$$

to which a one-parameter subgroup of stretchings corresponds,

$$z \to e^t\, z, \quad w_j \to e^{m_j\, t}\, w_j.$$

(c) *The subalgebra* $\mathcal{G}_- = \mathcal{G}_{-l} + \cdots + \mathcal{G}_{-2} + \mathcal{G}_{-1}$ *generates the subgroup* G_- *of holomorphic transformations of* Q, *and thus the orbit of the origin under this subgroup coincides wuth the orbit of the origin under the complete group of automorphisms.*

(d) *The Lie algebra is finitely graded* (*only finitely many components are nonzero in the expansion in the components*) *if and only if this algebra is finite-dimensional. In this case, the algebra consists of vector fields with polynomial coefficients.*

(e) *The condition that a field* (6) *belongs to* $\mathrm{aut}\; Q_0$ *is the relation*

$$2\,\mathrm{Re}\,(i\,\beta_j(z,w) + 2\;\mathrm{Re}\,(\partial\Phi_j(z,\bar z)(\alpha(z,w))) = 0 \quad \textit{for} \quad w_j = u_j + i\,\Phi_j(z,\bar z). \tag{7}$$

Proof. The tangent space to Q is defined by the relations

$$\mathrm{Im}\,(d\,w_j) = 2\;\mathrm{Re}\,(\partial\Phi_j(z,\bar z)(dz))$$

which implies (e). Every substitution does not change the weights w_j, and therefore, the linear relations (7) are relations for every graded component of a field X_j which is formed by homogeneous component of the coefficients, namely, from $(\alpha^{j+1}, \beta_1^{j+m_1}, \dots, \beta_l^{j+m_l})$. This proves (a). The parts (b) and (c) are obvious, and (d) follows from (a). This completes the proof of the proposition.

A criterion for the finite-dimensionality of the Lie algebra $\mathrm{aut} M_\xi$ of infinitesimal automorphisms of a germ of finite type is the holomorphic nondegeneracy ([11]). According to the definition, the holomorphic degeneracy of Q implies the existence of a nonzero holomorphic vector field, i.e., a field of the form

$$X = \alpha \frac{\partial}{\partial z} + \sum \beta_j \frac{\partial}{\partial w_j},$$

where $(\alpha, \beta_1, \dots, \beta_l)$ are holomorphic in a neighborhood of the origin, which is tangent to Q, i.e., satisfies the condition

$$\beta_j = 2\, i\, \partial \Phi_j(z, \bar z)(\alpha) \quad \text{for} \quad w_j = u_j + i\, \Phi_j(z, \bar z). \tag{8}$$

This criterion for finite-dimensionality is not constructive in itself. However, one can make this criterion quite constructive when Q is an algebraic surface of special form (as has been done in the cases of the Hërmader numbers (2) and $(2,3)$ considered earlier).

Note that $X = 0$ if and only if $\alpha = 0$. Moreover, the validity of these relations on Q implies their validity in a neighborhood of the origin.

Theorem 4. *Each of the following conditions is a necessary and sufficient condition for the holomorphic degeneracy of* Q.

(a) *The existence of a homogeneous holomorphic* $\mathbf{C}^n$*-valued form* $a(z) \neq 0$ *on* $\mathbf{C}^n$ *of degree not exceeding* $(l-1)(n-1)$ *and such that, for all* $(z,\ \eta,\ \zeta)$ *in* $\mathbf{C}^n$,

$$\partial \overline{\partial} \Phi_j(z, \bar\zeta)(a(z), \bar\eta) = 0.$$

(b) *The existence of a homogeneous holomorphic* $\mathbf{C}^n$*-valued form* $a(z) \neq 0$ *on* $\mathbf{C}^n$ *of degree not exceeding* $(l-1)(n-1)$ *and such that, for all* z *in* $\mathbf{C}^n$ *and every* i_1 *and* I_1 *(i.e.* $1 \leqslant i_1 \leqslant n$, $1 \leqslant i_2 \leqslant n$),

$$\partial \overline{\partial} \Phi_j(z, \bar e_{i_1})(a(z), \bar e_{i_2}) = 0,$$

where e_i *is an element of the standard basis in* $\mathbf{C}^n$.

(c) *The rank of the system of* $K\, n^2$ *linear equations with respect to an unknown* $A \in \mathbf{C}^n$ *given by*

$$L(\Phi)(z)(A) = \partial \overline{\partial} \Phi_j(z, \bar e_{i_1})(A, \bar e_{i_2}) = 0$$

is less than n *for all* z *for all* j, i_1, i_2), *i.e.,* $1 \leqslant j \leqslant l$, $1 \leqslant i1 \leqslant n$, $1 \leqslant i2 \leqslant n$.

Proof. Let us choose a field $\alpha(z, w)$ holomorphic in a neighborhood of zero. A necessary and sufficient condition that the coefficients $(\beta_1, \dots, \beta_l)$ defined by (8) on Q are holomorphic in a neighborhood of the origin are the tangent Cauchy–Riemann equations. Let $\eta \in \mathbf{C}^n$ be an arbitrary vector. Consider the CR-vector field on Q of the form

$$\bar X_{\bar\eta} = \bar\eta \frac{\partial}{\partial \bar z} - 2\, i \sum \bar\partial \Phi_j(z, \bar z)(\bar\eta) \frac{\partial}{\partial \bar w_j}.$$

If $(\eta_1, \dots, \eta_n)$ is a basis in $\mathbf{C}^n$, then the family of fields $(\bar X_{\bar\eta_1}, \dots, \bar X_{\bar\eta_n})$ is a basis family of CR-fields on Q. Applying these fields to relations (8), we obtain relations that are a criterion for the existence of holomorphic coefficients β_j,

$$\partial \overline{\partial} \Phi_j(z, \bar z)(\alpha(z, w), \bar\eta) = 0, \quad \forall \eta \in \mathbf{C}^n, \quad 1 \leqslant j \leqslant l. \tag{9}$$

Let $\zeta \in \mathbf{C}^n$ be a new independent variable. Applying $\bar{X}_{\bar\zeta}$ to (9) sufficient many times and using the holomorphic property of α, we can write

$$\partial\overline{\partial}\Phi_j(z,\bar\zeta)(\alpha(z,w),\bar\eta) = 0, \quad \forall \eta,\zeta \in \mathbf{C}^n, \quad 1 \leqslant j \leqslant l. \tag{10}$$

Substitute $\bar z = 0$ into (10); we arrive at

$$\partial\overline{\partial}\Phi_j(z,\bar\zeta)(\alpha(z,u),\bar\eta) = 0. \tag{11}$$

Let us expand α in a power series with respect to u. Then (11) is decomposed into coefficientwise relations, which implies that the existence of a nonzero $\alpha(z,u)$ is equivalent to the existence of a nonzero coefficient $a(z)$ satisfying (11) and, further, to the existence of a homogeneous with respect to z holomorphic $\mathbf{C}^n$-valued form $a(z)$. We have

$$\partial\overline{\partial}\Phi_j(z,\bar\zeta)(a(z),\bar\eta) = 0. \tag{12}$$

This system is equivalent to a system of the form

$$\partial\overline{\partial}\Phi_j(z,\bar e_{i_1})(a(z),\bar e_{i_2}) = 0, \tag{13}$$

where e_i is an element of the standard basis in $\mathbf{C}^n$. If we replace $a(z)$ by a formal unknown $A \in \mathbf{C}^n$, then we obtain a system of linear equations $L(\Phi)(A) = 0$ for A, and the elements of the coefficient matrix are forms whose degrees are not higher than $(l-1)$ and which are composed of derivatives of the forms Φ. As is well kinown, the condition for the existence of a nonzero solution is that the rank r of the matrix is less than n. In this case, the solution, according to Kramer's rule, is a set of determinants of the matrix of order r. This implies that the form $A = a(z)$ we are interested in is a form of degree not exceeding $(l-1)(n-1)$. This completes the proof of the theorem.

Definition. If the coordinate forms of the vector-valued forms Φ_j are linearly independent (the finiteness of the type) and if Φ satisfies any of the conditions of Theorem 4 (the holomorphic nondegeneracy), then we will say that the model surface Q *is nondegenerate* (in contrast to the old condition of complete nondegeneracy). If a germ M_ξ of a manifold M is subordinate to a nondegenerate surface Q, then we will say that the manifold M is nondegenerate at the point ξ.

Asertions 1 and 4 imply the following corollary.

Corollary 5. *The Lie algebra* aut Q *of automorphisms of a model surface* Q *is finite-dimensional if and only if* Q *is nondegenerate.*

Proof. If Q is nondegenerate, then it follows from Theorems 1 and 4 that Q is minimal and holomorphically nondegenerate, i.e., is finite-dimensional. If any of these conditions is violated, then we immediately obtain an infinite-dimensional group of automorphisms. This completes the proof of the corollary.

Theorem 6. *Let two germs of the form* (1) M_ξ *and* $\tilde M_{\tilde\xi}$ *be holomorphically equivalent. Then*

(*a*) *the linear space* $\mathrm{aut}_0\, Q$ *parametrizes the family of mappings of the first germ to the other*;

(*b*) *if* Q *is nondegenerate, then this family has a dimension not exceeding* $\dim \mathrm{aut}_0\, Q$ *and, in particular,*

$$\dim\, \mathrm{aut}_\xi M_\xi \leqslant \dim\, \mathrm{aut}_0\, Q_0 < \infty.$$

Proof. It follows from the considerations carried out in the proof of Proposition 2 that every mapping of the germ M_ξ into $\tilde M_{\tilde\xi}$ can be presented as the composition of two mappings, namely, a mapping of the form

$$z \to z + f_2 + \cdots, \quad w_j \to w_j + g_j^{m_j+1} + \cdots \tag{14}$$

and

$$z \to C\, z, \quad w_j \to \rho_j\, w_j.$$

It is sufficient to prove the theorem for the first mapping, which does not change the equations of the tangent model surface Q, i.e., assuming that $\tilde{\Phi} = \Phi$. Substitute (14) into relations (5). Separating the $(m_1+\mu)$th weight component in the first of these relations and the $(m_2+\mu)$th weight component in the second relation, and so on, we obtain for every $\mu = 1, 2, \ldots$ and $j = 1, \ldots, l$ a relationship of the form

$$\mathrm{Re}\,(i\, g_j^{m_j+\mu}(z,w) + 2\, \Phi(f^{1+\mu}(z,w), \bar{z})) + \cdots = 0, \quad \text{where } w_j = u_j + i\, \Phi_j(z,\bar{z}) \tag{15}$$

and the dots stand for the sum of expressions depending on $(f^{1+\nu}, g_j^{m_j+\nu})$ for with $\nu < \mu$. For a chosen Φ, relation (15) can be used for a recursive computation of the successive families $(f^{1+\mu}, g_j^{m_j+\mu})$ consisting of components of the mapping. At every step of this calculation, to determine the next family, one must solve an algebraic system of linear equations whose right part is calculated at the previous step. The solution of the nonhomogeneous system of linear equations is uniquely determined up to a choice of a solution of a homogeneous system. By Proposition 3, part (e), the space of solutions of a homogeneous system coincides with aut Q_0. This completes the proof of the theorem.

This consideration (the Poincaré construction, see [12]), which was applied in CR-geometry many times, is a version of the implicit mapping theorem for formal power series. Part (b) of the assertion thus obtained shows one of the main properties of model surfaces. A model surface is the most symmetric one in the class of germs of given type subordinated to the surface. Although the inequality proved above is not strict, in all known cases, the coincidence of dimensions means that the germ is equivalent to Q. Apparently, this holds also in the situation in question; however, the proof needs additional considerations.

All completely nondegenerate model surfaces Q are holomorphically homogeneous, i.e., the origin can be transferred by a holomorphic automorphism of Q to any other point of Q. Moreover, this automorphism is triangular-polynomial. Consider the problem of holomorphic homogeneity of a nondegenerate model surface which, generally speaking, is not completely nondegenerate. If the model surface Q, which is of the type m at the origin, is homogeneous, then it must have the same type also at all other points. For a model surface Q, this condition turns out to be also sufficient.

Theorem 7.

(a) *A model surface Q is holomorphically homogeneous if and only if t has a constant type at all points.*

(b) *In this case, the holomorphic homogeneity is ensured by the action of the subgroup G_- of* Aut Q*, consisting of triangular-polynomial mappings of degree less than l generated by $\mathcal{G}_-$.*

(c) *$\mathcal{G}_-$, as a Lie algebra, is generated by $\mathcal{G}_{-1}$, i.e., it is Tanaka fundamental.*

(d) *Here G_- admits the structure of a CR-manifold equivalent to Q.*

Proof. Let us re-expand the homogeneous forms $\Phi_j(z,\bar{z})$ defining Q at a point $a \in \mathbf{C}^n$. We obtain

$$\Phi_j(z+a, \bar{z}+\bar{a}) = \Phi_j(z,\bar{z}) + d\Phi_j(z,\bar{z})(a) + \frac{1}{2!} d^2\Phi_j(z,\bar{z})(a,a) + \cdots$$

Note that the second summand on the right-hand side has weight (= degree) equal to $(m_j - 1)$, that of the third one is $(m_j - 2)$ etc. If, for some J in this relation, there is at least one summand not reducible to zero in accordance with the process described in [2], then this means that the multiplicity of a Hörmander number m_j with $j < J$ has been increased. Thus, if the type is preserved, then all terms, starting from the second one, are reduced to zero, and this, in turn, means the presence of a triangular-polynomial mapping of Q into itself and moving the point a to the origin. This proves (a) and (b). Note that here that a mapping h from $\mathbf{C}^n$ to G_- arises, which assigns to a point a a triangular-polynomial automorphism h_a of Q such that $h_a(0) = a$. Using this correspondence, we can readily define a base family of vector fields in $\mathcal{G}_{-1}$ that generate the distribution of complex tangents at every point of Q. Further, in accordance with the definition of the Bloom–Graham type, these fields generate a Lie algebra, which thus turns out to be Tanaka fundamental [13]. After this, Q is identified with the standard model of the fundamental graded Lie algebra $\mathcal{G}_-$. This proves (c) and (d) and completes the proof of the theorem.

Taking into account the nontriviality of $\mathcal{G}_0$, the polynomiality of the vector fields forming $\operatorname{aut} Q$, and also using a technique going back to Kaup ([14, 15]), we obtain the following theorem.

Theorem 8. *The group of holomorphic automorphisms* $\operatorname{Aut} Q$ *of a nondegenerate model surface is a finite-dimensional Lie subgroup of the group of birational automorphisms* $\mathbf{C}^N$ (*Cremona group*) *of bounded degree. A uniform bound for the degree is a constant* $D = D(n, K)$ *depending on the* CR*-dimension and codimension.*

Using this result, we can show, as was done in [16], that Aut Q has the structure of a Lie group.

As was noted in the introduction, if Q is completely nondegenerate, then $\mathcal{G}_+ = 0$. If Q is only nondegenerate, then the problem of estimating d (the index of the highest nonzero component of $\mathcal{G}_+$) for a fixed type m is open. If such an estimate will be obtained, then, from this bound for d and from the Kaup construction, a bound for $D(n, K)$ (the degree of birational automorphisms of Q) would readily be obtained. The existing data enable us to formulate the following conjecture.

Conjecture. If a model surface Q is nondegenerate, then $\mathcal{G}_j = 0$ for every $j > l$. In other words, the length of $\mathcal{G}_+$ does not exceed that of $\mathcal{G}_-$.

REFERENCES

1. V. K. Beloshapka, "Universal Models for Real Submanifolds," Mat. Zametki **75** (4), 507–522 (2004) [Math. Notes **75** (4), 475–488 (2004)].
2. Th. Bloom and I. Graham, "On Type Conditions for Generic Real Submanifolds of $\mathbf{C}^n$," Invent. Math. **40**, 217–243 (1977).
3. V. K. Beloshapka, "A Cubic Model of a Real Manifold," Mat. Zametki **70** (4), 503–519 (2001) [Math. Notes **70** (4), 457–470 (2001)].
4. R. V. Gammel and I. G. Kossovskii, "The Envelope of Holomorphy of a Model Third-Degree Surface and the Rigidity Phenomenon," Tr. Mat. Inst. Steklova **253**, 30–45 (2006) [Proc. Steklov Inst. Math. **253**, 22-36 (2006)].
5. M. Sabzevari, "Biholomorphic Equivalence to Totally Nondegenerate Model CR Manifolds," Ann. Mat. Pura Appl., (2018), doi: 10.1007/s10231-018-0812-2.
6. M. Sabzevari and A. Spiro, "On the Geometric Order of Totally Nondegenerate CR-manifolds," arXiv: 1807.03076v1 [mathCV], 9 Jul 2018.
7. J. Gregorovich, "On the Beloshapka's Rigidity Conjecture for Real Submanifolds in Complex Space," arXiv: 1807.03502v1 [mathCV], 10 Jul 2018.
8. V. K. Beloshapka, "Cubic Model CR-manifolds without the Assumption of Complete Nondegeneracy," Russ. J. Math. Phys. **25** (2), 148–157 (2018).
9. V. K. Beloshapka, "Moduli Space of Model Real Submanifolds," Russ. J. Math. Phys. **13** (3), 245–252 (2006).
10. N. Stanton, "Infinitesimal CR-Automorphisms," Amer. J. Math. **118**, 209–233 (1996).
11. M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "CR Automorphisms of Real Analytic CR in Complex Space," Comm. Anal. Geom. **6**, 291–315 (1998).
12. H. Poincaré, "Les fonctions analytiques de deux variables et la représentation conforme," Rend. Circ. Mat. Palermo **23**, 185–220 (1907).
13. N. Tanaka, "On the Pseudo-Conformal Geometry of Hypersurfaces of the Space of N Complex Variables," J. Math. Soc. Japan **14**, 397–429 (1962).
14. W. Kaup, "Einige Bemerkungen uber polynomiale Vektorfelder, Jordanalgebren und die Automorphismen von Siegelschen Gebieten," Math. Ann. **204**, 131–144 (1973).
15. A. E. Tumanov, "Finite-Dimensionality of the Group of CR Automorphisms of a Standard CR Manifold, and Proper Holomorphic Mappings of Siegel Domains," Izv. Akad. Nauk SSSR Ser. Mat. **52** (3), 651–659 (1988) [Math. USSR Izv. **32** (3), 655–662 (1989)].
16. A. Huckleberry and D. Zaitsev, "Actions of Groups of Birationally Extendible Automorphisms," Geometric complex analysis (Hayama, 1995), 261–285, World Sci. Publ., River Edge, NJ, (1996).