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Cubic Model CR-Manifolds Without the Assumption of Complete Nondegeneracy

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Abstract. In the paper, we study model surfaces with Hörmander numbers (2, 3) without satisfying the condition of complete nondegeneracy. Simple criteria are given for the finiteness of the type, holomorphic nondegeneracy, and holomorphic homogeneity. It is proved that the dimension of the automorphism group of the model surface is maximal in the class of germs subordinated to the surface, and also that this group is a Lie subgroup of the Cremona group. We consider the case of a surface with a unique Hörmander number 3 separately.

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Bloom and Graham [1] introduced the type of a germ of CR-varieties, and also the equivalence of two definitions of the type was proved (the geometric and coordinate ones). In the geometric definition, the type was associated with properties of the distribution of complex tangents, and, in coordinate one, with the form of equations defining the germ. In the coordinate definition, the type of germ coincides with the type of a quasihomogeneous surface which is given by the lower weight components of the germ equations. This surface is a canonical representative of germs of this type i.e., a model surface whose properties determine the properties of the germ in many respects. The model surfaces with a single Hörmander number $m_1 = 2$ of arbitrary multiplicity k_1 has been studied in a sufficiently detailed way. In the paper [2], completely nondegenerate model surfaces were described and studied. These are model surfaces corresponding to germs of arbitrary CR-dimensions and codimensions of general position. In particular, if the type of a surface of CR-dimension n are two Hörmander numbers, $m_1 = 2$ of multiplicity k_1 and $m_2 = 3$ of multiplicity k_2, then, provided that the condition of complete nondegeneracy is satisfied, the equation $k_2 = n^2$ and the inequality

$$k_3 \leqslant n^2\,(n+1)$$

must hold. Completely nondegenerate model surfaces of the type $((2, k_1), (3, k_2))$ were considered in [3] and [4].

In this paper, the technique of model surfaces is applied to the study of CR-manifolds of type $((2, k), (3, K))$, which, are not completely nondegenerate in the case of $k < n^2$.

Let the coordinates in a complex space C^N be divided into three groups,

$$z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_k),\ W = (W_1, \ldots, W_K),$$
$$w = u + i\,v,\ W = U + i\,V;$$

let the weight 1 be assigned to the variable z, the weight 2 be assigned to the variable w, and the weight 3 be assigned to the variable W. A germ M_ξ of type

$$((2, k), (3, K)) = (2, \ldots, 2, 3, \ldots, 3)$$

(k twos and K threes) is a germ given in some coordinates with the origin at the point ξ by equations of the form

$$v = \theta(z, \bar z) + o(2), \quad V = \Theta(z, \bar z) + o(3)$$

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where θ is a real vector-valued form which is homogeneous of degree 2, Θ is a real vector-valued form which is homogeneous of degree 3, and both the forms do not contain pluriharmonic terms, and $o(j)$ stands for the sum of summands depending on $(z, \bar{z}, u, U)$ of the weights exceeding j. The condition of absence of pluriharmonic terms enables us to represent these forms as follows:

$$\theta(z, \bar{z}) = \Phi(z, \bar{z}), \quad \Theta(z, \bar{z}) = 2\operatorname{Re}\Psi(z, z, \bar{z}) = \Psi(z, z, \bar{z}) + \bar{\Psi}(\bar{z}, \bar{z}, z)$$

where $\Phi(z, \bar{z})$ is a vector-valued Hermitian form and $\Psi(z, z, \bar{z})$ is a form of bidegree $(2, 1)$ which is multilinear and symmetric with respect to the first two arguments.

The Hermitian $\mathbf{R}^k$-valued form $\Phi(z, \bar{z})$, in the developed form, is

$$\Phi(z, \bar{z}) = (\varphi_1\, z \cdot \bar{z}, \dots, \varphi_k\, z \cdot \bar{z}),$$

where $z \cdot \bar{\zeta}$ stands for the standard Hermitian inner product in $\mathbf{C}^n$, i.e.,

$$z \cdot \bar{\zeta} = z_1 \cdot \bar{\zeta}_1 + \dots + z_n \cdot \bar{\zeta}_n,$$

and $\varphi_j\, z \cdot \bar{z}$ is a j-th scalar Hermitian form. Further, the $\mathbf{C}^K$-valued form $\Psi(z, z, \bar{z})$ is

$$\Psi(z, z, \bar{z}) = \psi(z, z) \cdot \bar{z} = (\psi_1(z, z) \cdot \bar{z}, \dots, \psi_K(z, \bar{z}) \cdot \bar{z}),$$

where $\psi_j(z, z)$ stands for a $\mathbf{C}^n$-valued quadratic form on $\mathbf{C}^n$.

Then the *tangent model surface* to the germ M_ξ is the surface Q given by the equations

$$\begin{aligned} v &= \Phi(z, \bar{z}) \\ V &= \Psi(z, z, \bar{z}) + \bar{\Psi}(\bar{z}, \bar{z}, z) \end{aligned} \tag{1}$$

We say here that the germs given by equations of the form

$$\begin{aligned} v &= \Phi(z, \bar{z}) + o(2) \\ V &= \Psi(z, z, \bar{z}) + \bar{\Psi}(\bar{z}, \bar{z}, z) + o(3) \end{aligned}$$

are the germs *subordinated* to the model surface Q. It is clear that here the tangent space at the origin is $\{v = 0,\ V = 0\}$, and its complex part is $\{w = 0,\ W = 0\}$. That is, the CR-dimension is equal to n and z is the coordinate parametrizing the complex tangent.

Proposition 1.

(a) The germ of the generating CR-manifold M at the point ξ and its tangent model surface Q at the origin coordinates have the type $((2, k), (3, K))$ if and only if the coordinates of the Hermitian form

$$\Phi(z, \bar{z}) = (\Phi_1(z, \bar{z}), \dots, \Phi_k(z, \bar{z}))$$

are really linearly independent and the coordinates of the trilinear form

$$\Psi(z, z, \bar{z}) = (\Psi_1(z, z, \bar{z}), \dots, \Psi_K(z, z, \bar{z}))$$

are really linearly independent.

(b) If the condition of linear independence in (a) is not satisfied, then Q has infinite type and is not minimal at the origin.

Proof. Part (a) follows from Corollary 8.3 of [1], and (b) is obvious.

Since the real dimension of the space of Hermitian forms on a space of dimension n is equal to n^2, and the real dimension of the space of forms represented by $2\operatorname{Re}\Psi(z, z, \bar{z})$ is equal to $n^2\,(n+1)$, it follows that the condition of linear independence implies the necessity of the validity of the inequalities

$$k \leqslant n^2, \quad K \leqslant n^2\,(n+1).$$

Suppose that two germs M_ξ and $\tilde{M}_{\tilde{\xi}}$ are holomorphically equivalent. The germ type is a holomorphic invariant, and therefore their types coincide. Let their common type be equal to $((2,k),(3,K))$, let the surfaces Q and $\tilde{Q}$ be their tangent model surfaces, each of which is given by a pair of forms (Φ,Ψ) and $(\tilde{\Phi},\tilde{\Psi})$. Let, further,

$$(z \to f(z,w,W),\ w \to g(z,w,W),\ W \to G(z,w,W))$$

be an invertible mapping holomorphic in a neighborhood of the origin is an which takes M_ξ to $\tilde{M}_{\tilde{\xi}}$ and preserves the origin. In what follows, we shall use expansions of the form

$$f(z,w,W) = \sum_1^\infty f_j(z,w,W),$$

where f_j stands for the j-th weighted component of the expansion of f.

Proposition 2.
(a) The lower terms of the mapping are of the form

$$f(z,w,W) = C\,z + o(1),$$
$$g(z,w,W) = \rho\,w + o(2),$$
$$G(z,w,W) = P\,W + o(3),$$

where $C \in GL(n,C),\ \rho \in GL(k,R),\ P \in GL(K,R)$*; moreover,*

$$\tilde{\Phi}(z,\bar{z}) = \rho^{-1}\,\Phi(Cz,\overline{Cz}), \quad \tilde{\Psi}(z,z,\bar{z}) = P^{-1}\,\Psi(Cz,Cz,\overline{Cz}). \tag{2}$$

(b) The linear mapping

$$(z \to C\,z,\ w \to \rho\,w,\ W \to P\,W)$$

takes Q *onto* $\tilde{Q}$*, i.e., the model surfaces are holomorphically equivalent if and only if they are linearly equivalent.*

Proof. Let the equations of the germs be

$$v = \Phi(z,\bar{z}) + \varphi(z,\bar{z},u,U), \quad V = 2\,\mathrm{Re}\,\Psi(z,z,\bar{z}) + \psi(z,\bar{z},u,U)$$
$$v = \tilde{\Phi}(z,\bar{z}) + \tilde{\varphi}(z,\bar{z},u,U), \quad V = 2\,\mathrm{Re}\,\tilde{\Psi}(z,z,\bar{z}) + \tilde{\psi}(z,\bar{z},u,U)$$

Then the relations expressing the fact that, if $(z,w,W) \in M_\xi$, then $(f,g,G) \in \tilde{M}_{\tilde{\xi}}$ are of the form

$$\begin{aligned} &\mathrm{Im}\,g = \tilde{\Phi}(f,\bar{f}) + \tilde{\varphi}(f,\bar{f},\mathrm{Re}\,g,\mathrm{Re}\,G),\\ &\mathrm{Im}\,G = 2\,\mathrm{Re}\,\tilde{\Psi}(f,f,\bar{f}) + \tilde{\psi}(f,\bar{f},\mathrm{Re}\,g,\mathrm{Re}\,G)\\ &\text{for } w = u + i\,\Phi(z,\bar{z}) + \varphi(z,\bar{z},u), \quad W = U + i\,2\,\mathrm{Re}\,\Psi(z,z,\bar{z}) + \psi(z,\bar{z},u)\end{aligned} \tag{3}$$

Separating the components of weights 1 and 2 in the first of the relations and the components of weights 1,2, and 3 in the other, we obtain (a), which immediately implies (b). This proves the proposition.

Note that relation (17) defines an action of the direct product of three linear groups

$$GL(n,C) \times GL(k,R) \times GL(K,R)$$

on the space of pairs of forms (Φ, Ψ) that define model surfaces of this type. By Proposition 2, every invariants of this action are holomorphic invariants of the germ.

To understand the CR-geometry of a germ, its holomorphic automorphisms are of great importance. Let $\operatorname{aut} M_\xi$ be the Lie algebra of infinitesimal automorphisms. It consists of the germs of real holomorphic vector fields of the form

$$X = 2\ \mathrm{Re}\left(\alpha\, \frac{\partial}{\partial z} + \beta\, \frac{\partial}{\partial w} + \gamma\, \frac{\partial}{\partial W}\right), \tag{4}$$

where (α, β, γ) are holomorphic in a neighborhood of the point ξ and the filed X is tangent to M_ξ at the points of M_ξ. Every field of this kind generates a local one-parameter group of invertible holomorphic mappings of the germ into itself. The family of mapping generated in this way is a local group of automorphisms of M_ξ; we denote this group by $\operatorname{Aut} M_\xi$. In $\operatorname{aut} M_\xi$ one can single out a Lie subalgebra $\operatorname{aut}_\xi M_\xi$ which consists of the vector fields vanishing at ξ. The fields in this subalgebra generate local one-parameter groups of holomorphic transformations keeping ξ. These transformations form a local subgroup $\operatorname{Aut}_\xi M_\xi$ in $\operatorname{Aut} M_\xi$.

The weights (of variables) introduced above enable one to introduce a grading in the Lie algebra of vector fields of the space C^N. To this end, one should extend the agreement concerning the weights of the variables to the coordinate differentiations by setting

$$\left[\frac{\partial}{\partial z}\right] = -1, \quad \left[\frac{\partial}{\partial w}\right] = -2, \quad \left[\frac{\partial}{\partial W}\right] = -3.$$

After this, $\operatorname{aut} M_\xi$ also obtains the structure of a graded Lie algebra, which decomposes into the direct sum of graded components from the component of the weight (-3) and, generally speaking, to $+\infty$. The algebra of the model surface Q has some specific features.

Proposition 3.

(a) If a field

$$X = \sum_{-3}^{+\infty} X_j$$

belongs to $\operatorname{aut} Q$*, then* $\forall j\ X_j \in \operatorname{aut} Q$*.*

(b) $\operatorname{aut} Q$ *contains a field of weight zero,*

$$X = 2\ \mathrm{Re}\left(z\, \frac{\partial}{\partial z} + 2\, w\, \frac{\partial}{\partial w} + 3\, W\, \frac{\partial}{\partial W}\right)$$

to which the one-parameter subgroup of extensions corresponds,

$$z \to e^{t}\, z, \quad w \to e^{2\,t}\, w, \quad W \to e^{3\,t}\, W$$

(c) The subalgebra $\operatorname{aut}_0\ Q_0$ *is*

$$\mathcal{G}_0 + \mathcal{G}_1 + \dots,$$

i.e., the sum of nonnegative weight components of the full algebra. The subalgebra

$$\mathcal{G}_- = \mathcal{G}_{-3} + \mathcal{G}_{-2} + \mathcal{G}_{-1}$$

generates the group of holomorphic transformations Q such that the subgroup orbit of the origin coincides with the orbit of the origin with respect to the full group of automorphisms.

(d) The Lie algebra is finitely graded (only finitely many components are nonzero in the decomposition into the components) if and only if it is finite-dimensional. In this case, it consists of vector fields with polynomial coefficients.

(e) The condition or the fact that the field (4) *belongs to* aut M_ξ *is given by the relations*

$$\begin{aligned} 2\ \mathrm{Re}(i\,\beta + 2\,\Phi(\alpha,\bar z)) &= 0\\ 2\ \mathrm{Re}(i\,\gamma + 4\,\Psi(\alpha,z,\bar z) + 2\,\bar\Psi(\bar z,\bar z,\alpha)) &= 0\\ \textit{for } w = u + i\,\tilde\Phi(z,\bar z),\quad W &= U + i\,2\ \mathrm{Re}\,\tilde\Psi(z,z,\bar z) \end{aligned} \tag{5}$$

Proof. The tangent space to Q is given by the relations

$$\begin{aligned} \mathrm{Im}(d\,w) &= 2\ \mathrm{Re}\,\Phi(d\,z,\bar z)\\ \mathrm{Im}(d\,W) &= 2\ \mathrm{Re}(2\,\Psi(d\,z,z,\bar z) + \bar\Psi(\bar z,\bar z,d\,z)) \end{aligned}$$

which implies (e). This substitution does not change the weights of w and W, and therefore the linear relations (5) are relations for every graded component of the field X_j that is formed from homogeneous components of the coefficients, namely, $(\alpha_{j+1},\beta_{j+2},\gamma_{j+3})$. This proves (a). Parts (b) and (c) can be verified directly, and part (d) follows from (a). The proposition is proved.

A criterion for the finite-dimensionality of the Lie algebra aut M_ξ of infinitesimal automorphisms of a germ of finite type is the holomorphic nondegeneracy ([6, 7]). By definition, the holomorphic degeneracy of Q means the existence of a nonzero holomorphic vector field, i.e., a field of the form

$$X = \alpha\,\frac{\partial}{\partial z} + \beta\,\frac{\partial}{\partial w} + \gamma\,\frac{\partial}{\partial W}, \tag{6}$$

where (α,β,γ) are holomorphic in a neighborhood of the origin and the vector field is tangent to Q, i.e., satisfies the condition

$$\begin{aligned} \beta &= 2\,i\,\Phi(\alpha,\bar z))\\ \gamma &= 4\,i\,\Psi(\alpha,z,\bar z) + 2\,i\,\bar\Psi(\bar z,\bar z,\alpha))\\ \text{for } w = u + i\,\tilde\Phi(z,\bar z),&\quad W = U + i\,2\ \mathit{Re}\ \tilde\Psi(z,z,\bar z) \end{aligned} \tag{7}$$

Note that $(\alpha,\beta,\gamma) = 0$ if and only if $\alpha = 0$, and the validity of these relations on Q means their validity in a neighborhood of the origin.

Theorem 4. *A necessary and sufficient condition of holomorphic degeneracy of Q is the existence of a homogeneous holomorphic* C^n*-valued form on* C^n*, say,* $a(z) \neq 0$*, of degree not exceeding* $(n-1)$ *and such that the following three conditions hold for all* $(z,\ \zeta,\ \eta)$ *in* C^n*:*

$$(4.i)\qquad \Phi(\zeta,\bar a(\bar z)) = 0, \tag{8}$$

$$(4.ii)\qquad \Psi(\eta,\zeta,\bar a(\bar z)) = 0, \tag{9}$$

$$(4.iii)\qquad \bar\Psi(\bar a(\bar z),\bar z,\zeta) = 0 \tag{10}$$

Proof. Let us choose an $\alpha(z,w,W)$ holomorphic in a neighborhood of zero. Necessary and sufficient conditions of the fact that (β,γ) defined by relation (7) on Q are holomorphic in a neighborhood of the origin are the tangent Cauchy–Riemann equations. Let $\zeta \in \mathrm{C}^n$ be an arbitrary vector. Then an arbitrary CR-vector field on Q is of the form

$$\bar X_{\bar\zeta} = \zeta\,\frac{\partial}{\partial\bar z} - 2\,i\,\bar\Phi(\bar\zeta,z)\,\frac{\partial}{\partial\bar w} - 2\,i\,(2\bar\Psi(\bar\zeta,\bar z,z) + \Psi(z,z,\bar\zeta))\,\frac{\partial}{\partial\bar W}).$$

Applying it to relations (7), we obtain two relations,

$$\Phi(\alpha,\bar\zeta) = 0 \tag{11}$$

$$\Psi(\alpha,z,\bar\zeta) + \bar\Psi(\bar z,\bar\zeta,\alpha) = 0 \tag{12}$$

Applying $\bar{X}_{\bar{\eta}}$ to the second relation, we obtain

$$\bar{\Psi}(\bar{\eta}, \bar{\zeta}, \alpha) = 0. \tag{13}$$

It follows now from (11) that

$$\Psi(\alpha, z, \bar{\zeta}) = 0 \tag{14}$$

Choosing now a nonzero value of $\alpha(z, w, W)$ satisfying (12), (13), and (14) and substituting $\bar{z} = 0$ into them, we obtain similar relations with $\alpha(z, u, U) \neq 0$. If we expand α in a power series in (u, U), then we can obtain a nonzero solution of (14), (12), (13) for $\alpha = A(z) \neq 0$. Similarly, decomposing $A(z)$ into a sum of homogeneous forms, we obtain a nonzero solution in the form of a homogeneous form $a(z)$. The conditions (14) on $a(z)$ form a homogeneous system of linear equations for the coordinates of the form $a(z)$ whose coefficients are linear in z. As is known, the presence of a nonzero solution is equivalent to the fact that the rank of this system does not exceed $(n-1)$. Moreover, using the Cramer formulas, one can obtain a general solution in the form of forms of degree not exceeding $(n-1)$. The remaining two relations (12), (13) are additional linear relations for the coefficients of the resulting forms, and it is clear that the presence of a solution with an $a(z) \neq 0$ of this kind implies the holomorphic degeneracy. This completes the proof of the theorem.

Propositions 1 and 4 imply the following assertion.

Theorem 5. *The group* aut Q *is finite-dimensional if and only if the following two solutions on the forms* (Φ, Ψ) *hold*:

(i) The coordinate forms Φ *and* Ψ *are linearly independent.*

(ii) The forms satisfy conditions (4.i), (4.2), and (4.iii) (*of Theorem 4*).

Proof. Condition (i) implies the minimality, and condition (ii) implies the holomorphic nondegeneracy. This implies the finite-dimensionality. If any of these conditions is violated, we immediately obtain an infinite-dimensional group of automorphisms. This completes the proof of the theorem.

For the case in which Q satisfies both the conditions, (i) and (ii), we shall say that the model surface Q and the germs subordinated to it are *nondegenerate*. As was noted above, condition (i) means that the surface Q and all germs subordinated to it have the type $((2,k),(3,K))$. Note also that conditions (4.i) and (4.ii) are completely analogous to the condition of holomorphic nondegeneracy for the quadratic model surface, which is a surface of type $(2,k)$ (see [8]). These conditions can be formulated as the existence condition for a nonzero constant vector satisfying appropriate conditions. On the contrary, condition (4.iii) is a condition of another type. Its solutions are nonzero homogeneous forms of degree not exceeding $(n-1)$. This is a specific feature of the cubic case as compared with the quadratic one. In the quadratic case, the holomorphic nondegeneracy is equivalent to the existence of a nonzero tangential holomorphic field having a constant z-coordinate. In the cubic case, which we are considering here, this field has a non-zero z-coordinate of degree less than n.

Theorem 6. *Let* M_ξ *and* $\tilde{M}_{\tilde{\xi}}$ *be two germs of type* $((2,k),(3,K))$, *i.e., condition (i) holds. Then*

(a) the linear space $\mathrm{aut}_0\ Q$ *parametrizes the family of mappings of the first germ into another*;

(b) if Q *is holomorphically nondegenerate, i.e., condition (ii) holds, then this family is finite-dimensional and, in particular,*

$$\dim \mathrm{aut}_\xi M_\xi \leqslant \dim \mathrm{aut}_0\ Q < \infty.$$

Proof. It follows from the considerations used in the proof of Proposition 2 that every mapping of one of the germs into another can be decomposed into a composition of two mappings, namely, mappings of the form

$$z \to z + f_2 + \dots, \quad w \to w + f_3 + \dots, \quad W \to W + G_4 + \dots \tag{15}$$

and

$$z \to C\,z, \quad w \to \rho\,w, \quad W \to P\,W$$

Note that the first of these mappings does not modify the forms (Φ, Ψ) determining the tangent model surface. Write $(\tilde{\Phi}, \tilde{\Psi}) = (\Phi, \Psi)$ and substitute (15) into relations (3). Distinguishing the $(\mu+1)$-th component in the first of relations (3) and the $(\mu+2)$-th component in the second of these relations and singling out the leading terms, we obtain two relations of the form

$$\begin{aligned} \mathrm{Re}(i\,g_{\mu+1}(z,w,W) + 2\,\Phi(f_\mu(z,w,W),\bar{z})) + \ldots = 0 \\ \mathrm{Re}(i\,G_{\mu+2}(z,w,W) + 4\,\Psi(f_\mu(z,w,W),z,\bar{z}) + 2\,\bar{\Psi}(\bar{z},\bar{z},f_\mu(z,w,W))) + \ldots = 0 \\ \text{where } w = u + i\,\tilde{\Phi}(z,\bar{z}),\ W = U + i\,2\,\mathrm{Re}\,\tilde{\Psi}(z,z,\bar{z}) \end{aligned} \tag{16}$$

and the dots stand for the sum of expressions depending on $(f_\nu, g_{\nu+1}, G_{\nu+2})$ for $\nu < \mu$. For chosen $(\varphi, \psi, \tilde{\varphi}, \tilde{\psi})$, relation (16) can be used for the recurrent calculation of successive families $(f_\mu, g_{\mu+1}, G_{\mu+2})$ formed by the components of the mapping. At each step of this calculation, to determine the next triple, we must solve the algebraic system of linear equations whose right-hand side was computed at the previous step. The solution of the inhomogeneous system of linear equations is determined uniquely up to the choice of solutions of the homogeneous system. By Proposition 3, part (d), the solution space of the homogeneous system is contained in $\mathrm{aut}_0\ Q$. This completes the proof of the theorem.

This reasoning (the Poincaré construction, see [9]), which was used many times in CR-geometry, is a a certain version of the implicit mapping theorem for formal power series. Part (b) of the assertion proved above demonstrates one of the main properties of model surfaces. A model surface is the most symmetrical in the class of germs of the given type that are subordinate to the surface. Although the inequality is not strict, in all known situations, the coincidence of dimensions means that the germ is equivalent to Q. This seems to be true in this situation as well, but the proof requires additional reasonings.

All completely nondegenerate model surfaces are Q-holomorphically homogeneous, i.e., the origin can be transferred by a holomorphic automorphism of Q to any other point of Q. Consider the question concerning the holomorphic homogeneity of a model surface of type $((2,k),(3,K))$, which, generally speaking, are not completely nondegenerate. If the model surface Q, which is of the type $((2,k),(3,K))$, is homogeneous, then it must be of the same type also at all other points. For the model surface Q, this condition turns out to be also sufficient.

Let $\mathcal{H}$ be the space of Hermitian forms on $\mathbf{C}^n$, let $\mathcal{H}_\Phi$ be the subspace generated by the coordinate forms $\Phi(z,\bar{z})$, let $\mathcal{H}'_\Phi$ be a direct complement, i.e., $\mathcal{H} = \mathcal{H}_\Phi + \mathcal{H}'_\Phi$, and let Π_Φ be the projection onto $\mathcal{H}'_\Phi$ along $\mathcal{H}_\Phi$.

Theorem 7.

(a) The Lie subalgebra $\mathcal{G}_- = \mathcal{G}_{-3} + \mathcal{G}_{-2} + \mathcal{G}_{-1}$ *of the Lie algebra* $\mathrm{aut}\,Q$ *has the following form:*

$$a\,\frac{\partial}{\partial z} + (b + 2\,i\,\Phi(z,\bar{a}))\,\frac{\partial}{\partial w} + (B + 2\,i\,\Psi(z,z,\bar{a}) + \delta'(w,a) + \delta''(w,\bar{a}))\,\frac{\partial}{\partial W}$$

where $\Pi_\Phi(\mathrm{Re}\ \Psi(a,z,\bar{z})) = 0$ *and the parameters* δ' *and* δ'' *are defined uniquely from* (19).

(b) The orbit of the origin, for the group $\mathrm{Aut}\,Q_0$ *of holomorphic automorphisms of the germ of the surface* Q *at the origin,* Orb_0, *is the family of points* $\xi = (a,b,B) \in Q$ *such that*

$$\mathrm{Orb}_0 = \{(a,b,B) \in Q : \Pi_\Phi(\mathrm{Re};\Psi(a,z,\bar{z})) = 0,\ \forall z\}.$$

(c) The subalgebra $\mathcal{G}_0$ *is the Lie algebra of the group* $\mathbf{G}_0$, *which is the subgroup of the group* $GL(n,\mathrm{C}) \times GL(k,\mathrm{R}) \times GL(K,\mathrm{R})$ *given by the relation*

$$\Phi(z,\bar{z}) = \rho^{-1}\,\Phi(Cz,\overline{Cz}), \quad \Psi(z,z,\bar{z}) = P^{-1}\,\Psi(Cz,Cz,\overline{Cz}). \tag{17}$$

(d) The group $\mathbf{G}_-$ generated by $\mathcal{G}_- = \mathcal{G}_{-3} + \mathcal{G}_{-2} + \mathcal{G}_{-1}$ is the Lie group consisting of the triangular-quadratic transformations of C^N*.*

Proof. As was noted above, the coordinates of a field of weight j in $\operatorname{aut} Q$ are of the form $(f_{j+1}, g_{j+2}, G_{j+3})$ and satisfy relations (5).

For $j = -3$, we obtain $X_{-3} = (0, 0, B)$, where $B \in \mathbf{R}^K$ is a constant real vector. For $j = -2$, we obtain $X_{-2} = (0, b, Az)$, where $b \in \mathbf{R}^k$ is a constant real vector, and $Az = 0$. For $j = -1$, we obtain $X_{-1} = (a, \beta\, z, \gamma(z, z) + \delta(w))$, where $a \in \mathbf{C}^n$ is a constant vector. it follows from the first relation of (5) that $\beta z = 2\, i\, \Phi(z, \bar{a})$. It follows from the second relation that $\gamma(z, z) = 2\, i\, \Psi(z, z, \bar{a})$, and also that the following relation holds:

$$\delta(\Phi(z, \bar{z})) = 4 \operatorname{Re} \Psi(a, z, \bar{z}). \tag{18}$$

The criterion for the solvability of relation (18) with respect to δ is the condition

$$\Pi_\Phi(\operatorname{Re}\, \Psi(a, z, \bar{z})) = 0.$$

If this condition holds, then, since the coordinate forms of Φ are linearly independent, it follows that the linear operator δ can be recovered uniquely. Here the dependence of the solution on a is real, and we can write $\delta(w) = \delta'(w, a) + \delta''(w, \bar{a})$, where

$$\delta'(\Phi(z, \bar{z}), a) = 2\ \Psi(z, \bar{z}, a)\delta''(\Phi(z, \bar{z}), \bar{a}) = 2\ \bar{\Psi}(z, \bar{z}, \bar{a}) \tag{19}$$

This proves (a).

Let a point $\xi = (a, b, B) \in Q$ be such that that the germ Q_ξ is equivalent to Q_0. Let us transfer the origin to this point by the change

$$z \to a + z, \quad w \to b + w, \quad W \to c + W.$$

In the new coordinates, the equations Q become

$$\operatorname{Im}(w + b) = \Phi(z + a, \bar{z} + \bar{a}),$$
$$\operatorname{Im}(W + B) = 2 \operatorname{Re} \Psi(z + a, z + a, \bar{z} + \bar{a}).$$

The last equations can be reduced by simple triangular-quadratic changes to the form

$$\begin{aligned} v &= \Phi(z, \bar{z}), \\ V &= 2 \operatorname{Re}(2\Psi(z, a, \bar{z}) + \Psi(z, z, \bar{z})). \end{aligned} \tag{20}$$

If for at least one a and one of the coordinate forms, the Hermitian form 2 Re $(\Psi_j(z, a, \bar{z}))$ is not contained in $\mathcal{H}$, then this changes the type at the corresponding point. Indeed (see Corollary 8.3 of [1]), this increases the multiplicity of the number 2 and, accordingly, reduces the multiplicity of the number 3. If, however, for all coordinates, identically with respect to a, these forms are contained in $\mathcal{H}$, then by adding a linear term dependent on w to W, we reduce equations (20) to the original form (1). This proves parts (b) and (d). Part (c) follows from Part (b) of Proposition 2. This completes the proof of the theorem.

Theorem 8.

(a) A surface Q is holomorphically homogeneous if and only if its type does not depend on a point.

(b) A surface Q is holomorphically homogeneous if and only if $\Pi_\Phi(4\ \operatorname{Re}\ \Psi(a, z, \bar{z})) = 0$ *for all* $a \in \mathbf{C}^n$*. In this case, Q can naturally be identified with $\mathbf{G}_-$.*

Proof. Let the type does not depend on a point; then, as was proved in the proof of Theorem 7, the condition of part (b) holds and Q is homogeneous. This completes the proof of the theorem.

We note that for a completely nondegenerate surface with the Hörmander numbers (2.3), the homogeneity condition in Theorem 7 is satisfied automatically, since, in this case, the space $\mathcal{H}$ coincides with the entire space of Hermitian forms.

Taking into account the structure of $\mathbf{G}_-$ (Theorem 7, part (d)) and the polynomial property of the vector fields forming $\operatorname{aut} Q$ (Proposition 3 (d)) and using a technique that goes back to Kaup [10] and applied in a similar situation by Tumanov [11], we obtain the following assertion.

Theorem 9. *The group of holomorphic automorphisms of a cubic which is nondegenerate in the sense of satisfying requirements (i) and (ii) is a finite-dimensional Lie subgroup of the group of birational automorphisms* C^N *of bounded degree. Uniform estimation for the degree is given by the constant* $D = D(n, k, K)$.

As noted above, the quadratic model surfaces (with a unique Hörmander number equal to 2) are formally a special case of the type $((2, k), (3, K))$ under consideration for $K = 0$. On the other hand, setting $k = 0$, we obtain the second extremal case, i.e., a surface $\mathcal{Q}$ of type $(3, K)$ with a single Hörmander number equal to 3 (a cubic model surface). In this case, the coordinates of the space are divided into two groups of variables, z and W, the group of the variables w disappears, as well as the Hermitian form Φ:

$$V = \Psi(z, z, \bar{z}) + \bar{\Psi}(\bar{z}, \bar{z}, z) \tag{21}$$

How the eight statements obtained above will be realized in this case?

The answer is as follows: all statements from the 1st to the 6th inclusive are edited in an obvious way: one should remove w and Φ.

The argument in Theorem 7 shows that a cubic model surface can be homogeneous if and only if $\Psi = 0$. However, this is an infinite type. That is, a surface of type $(3, K))$ *cannot be homogeneous.* However, one can extract a more subtle information from this reasoning. Namely, we obtain the following analog of Theorems 7 and 8.

Proposition 10.

(a) The orbit of the origin with respect to the group $\mathrm{Aut}\,\mathcal{Q}_0$ *of holomorphic automorphisms of the surface* $\mathcal{Q}$ *is the family of points*

$$\mathrm{Orb}_0 = \{(a, B) \in \mathcal{Q} : 2\,\mathrm{Re}\,\Psi(z, a, \bar{z}) = 0,\ \forall z\},$$

(b) The subgroup $\mathbf{G}_-$ *of the group of automorphisms that corresponds to* $\mathcal{G}_{-3}+\mathcal{G}_{-2}+\mathcal{G}_{-1}$ *consists of triangular-quadratic transformations of* C^N.

(c) $\mathbf{G}_-$ *can naturally be identified with* $\mathrm{Orb}_0 \subset \mathcal{Q}$.

(d) The group $\mathrm{Aut}\,\mathcal{Q}_0$ *of holomorphic automorphisms of a cubic nondegenerate in the sense of the validity of requirements (i) and (ii) is a finite-dimensional Lie subgroup of the group of birational automorphisms of* C^N *of bounded degree.*

Let us return to the general case of the type $((2, k), (3, K))$. There are questions that remained outside the scope of our study. Denote by $\mathcal{G}_+$ the Lie subalgebra of aut Q_0 generated by fields of positive weight. The nondegeneracy conditions (i) and (ii) ensure that $\mathcal{G}_+$ is finite-dimensional and consists of a finitely many graded summands, i.e.,

$$\mathcal{G}_+ = \mathcal{G}_1 + \cdots + \mathcal{G}_d.$$

The question concerning the bound for d *remains open.*

If Q is completely nondegenerate, then, as was proved by Gammel and Kossovski [4], $\mathcal{G}_+ = 0$. If we reject the condition of complete nondegeneracy, then this is not the case.

Example 11. Let a model surface Q in C^4 with the coordinates

$$(z_1\,, z_2,\ w = u + i\,v,\ W = U + i\,V),$$

of type $((2, 1), (3, 1))$, be given by the equations

$$v = |z_1|^2, \quad V = 2\,\mathrm{Re} z_2^2\,\bar{z}_2.$$

This is a direct product of two hypersurfaces: a sphere Q^1 in C^2, whose automorphisms are well known, and a cubic hypersurface Q^2 in C^2 with a degeneracy at the origin (this example was considered in [12]). The algebra of the hypersurface Q^2 contains no fields of positive weight. However, the algebra of automorphisms of Q is the direct sum of the algebras Q^1 and Q^2. Since the algebra of the sphere contains fields of weights 1 and 2, it follows that this is also true for the algebra Q. That is, in this case, $d = 2$.

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