ISSN 0001-4346, Mathematical Notes, 2019, Vol. 105, No. 3, pp. 309–315. © Pleiades Publishing, Ltd., 2019.
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On the Complexity of the Differential-Algebraic Description of Analytic Complexity Classes

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Received April 30, 2018; in final form, September 4, 2018; accepted September 12, 2018

Abstract—The objective of this paper is to trace the increase in the complexity of the description of classes of analytic complexity (introduced by the author in previous works) under the passage from the class Cl_1 to the class Cl_2. To this end, two subclasses, Cl_1^+ and Cl_1^{++}, of Cl_2 that are not contained in Cl_1 are described from the point of view of the complexity of the differential equations determining these subclasses. It turns out that Cl_1^+ has fairly simple defining relations, namely, two differential polynomials of differential order 5 and algebraic degree 6 (Theorem 1), while a criterion for a function to belong to Cl_1^{++} obtained in the paper consists of one relation of order 6 and five relations of order 7, which have degree 435 (Theorem 2). The "complexity drop" phenomenon is discussed; in particular, those functions in the class Cl_1^+ which are contained in Cl_1 are explicitly described (Theorem 3).

DOI: 10.1134/S0001434619030015

Keywords: *superposition of analytic functions, analytic complexity, differential polynomials.*

1. INTRODUCTION

The possibility of representing analytic functions by superpositions of functions of fewer variables was discussed in a number of papers [1], [2].

Consider the strictly increasing hierarchy of the classes of complexity of analytic functions $z(x, y)$ of two variables defined recursively on the basis of the function $x + y$:

$$Cl_0 \subset Cl_1 \subset Cl_2 \subset Cl_n \subset \cdots .$$

The class Cl_0 consists of all analytic function of one variable (x or y), which are assigned the complexity $N(z) = 0$; Cl_1 consists of all functions z of the form $c(a(x) + b(y))$, which have complexity $N(z) \leq 1$; in general, for any nonnegative integer n, Cl_{n+1} consists of functions of the form

$$C(A_n(x, y) + B_n(x, y)),$$

where C is a function of one variable and A_n and B_n are functions in Cl_n; the functions $z \in Cl_{n+1}$ have complexity $N(z) \leq n + 1$. If a function z does not belong to any of the classes $Cl_n = \{z : N(z) \leq n\}$, then we set $N(z) = \infty$. All these classes are differential-algebraic sets; they have defining equations, which are differential polynomials with integer coefficients. Among the differential relations determining the class number n is a relation of differential order $2^{n+2} - 5$. The first class Cl_1 is determined by a relation of order 3, namely, $(ln(z'_y/z'_x))''_{xy} = 0$, or

$$\delta(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0. \tag{1}$$

The second class, which consists of functions of the form

$$z(x, y) = s\{c[a(x) + b(y)] + r[p(x) + q(y)]\},$$

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is determined by relations of order at least 11 (see [3]).

The equations of classes do not explicitly contain the coordinates x, y, and z for an obvious reason (shift invariance). Let V_k be a truncated k-jet, i.e., the affine space whose coordinates are the values of all derivatives up to order k of a function z at a point (x, y) (the coordinates of the point (x, y) and the value of z are not included). The dimension of this space equals $k(k+3)/2$.

Each differential-algebraic set M is determined by a system of differential polynomial equations. The complexity of this system gives an idea of the complexity of the set itself. The complexity of such a system (not to be confused with the complexity of a function) can be characterized by a triple (k, m, l) of positive integers. Here k is the maximum differential order of an equation in the system, m is the number of equations, and l is the maximum algebraic degree of an equation in the system. These characteristics are determined by the system; their relationship with the set M itself is not invariant. There is an invariant approach based on considering the characteristics of the differential ideal $I(M)$ corresponding to M. However, we find this approach hard to realize and do not discuss it here.

In this paper, we plan to study the equations of two subclasses of Cl_2, namely,

$$Cl_1^+ = \{z(x,y) = c(a(x) + b(y)) + p(x)\},$$
$$Cl_1^{++} = \{z(x,y) = c(a(x) + b(y)) + d(x+y)\}.$$

The problem of describing these classes was also considered in [4].

To facilitate tracing the increase in the complexity of the description of classes M, we collect the existing related results, including those obtained in this paper.

For $M = Cl_0 = \{z = a(x) \text{ or } z = b(y)\}$, the triple mentioned above is

$$k = 1, \qquad m = 1, \qquad l = 2;$$

for $M = Cl_1 = \{z = c(a(x) + b(y)\}$,

$$k = 3, \qquad m = 1, \qquad l = 4;$$

for $M = Cl_1^+ = \{z(x,y) = c(a(x) + b(y)) + p(x)\}$, according to Theorem 1, we have

$$k = 5, \qquad m = 2, \qquad l = 6;$$

and for $M = Cl_1^{++} = \{z(x,y) = c(a(x) + b(y)) + d(x+y)\}$, according to Theorem 2,

$$k = 7, \qquad m = 6, \qquad l \leq 435.$$

For $M = Cl_2$, we have no precise statement, but we can propose the following likely argument. Let us look at the image of the second class in the 11-jet. This is the image of a polynomial map from the space of 11-jets of seven functions of one variable to the space of 11-jets of functions of two variables. The coordinate functions of this map are homogeneous forms of degrees $3, 5, \dots, 23$. Estimating the degree of the image in terms of the product of these degrees, we obtain

$$3 \cdot 5 \cdot \dots \cdot 23 \approx 2{,}5 \cdot 10^{89}.$$

It is easy to show that the codimension of the image is at least 3. For this surface of codimension 3 to be the intersection of three hypersurfaces, the degree of at least one of these hypersurfaces must be not lower than

$$(2{,}5 \cdot 10^{89})^{1/3} \approx 1{,}4 \cdot 10^{30}.$$

A general polynomial of this degree in 77 variables is too large for our small Universe. The number of monomials in it exceeds 10^{2019}.

We proceed to our constructions.

2. THE CLASS Cl_1^+

We begin with Cl_1^+. The relation $z \in Cl_1^+$ is equivalent to the existence of a $p(x)$ such that $z(x,y)-p(x)$ belongs to Cl_1. Writing the relation determining the first class, we obtain (derivatives are denoted by subscripts)

$$\begin{aligned}&- z_{0,1}p_1^2z_{1,2} + z_{0,2}z_{1,1}p_1^2 - z_{2,1}z_{0,1}^2p_1 + 2z_{0,1}z_{1,0}p_1z_{1,2} - 2z_{0,2}z_{1,1}z_{1,0}p_1\\ &\quad + p_2z_{0,1}^2z_{1,1} + z_{0,1}^2z_{1,0}z_{2,1} - z_{0,1}^2z_{1,1}z_{2,0} - z_{0,1}z_{1,0}^2z_{1,2} + z_{0,2}z_{1,0}^2z_{1,1} = 0.\end{aligned}$$

Let us express p_2:

$$\begin{aligned}p_2 = \big(&p_1^2z_{0,1}z_{1,2} - p_1^2z_{0,2}z_{1,1} + p_1z_{0,1}^2z_{2,1} - 2p_1z_{0,1}z_{1,0}z_{1,2} + 2z_{0,2}z_{1,1}z_{1,0}p_1\\ &- z_{0,1}^2z_{1,0}z_{2,1} + z_{0,1}^2z_{1,1}z_{2,0} + z_{0,1}z_{1,0}^2z_{1,2} - z_{0,2}z_{1,0}^2z_{1,1}\big)/(z_{0,1}^2z_{1,1}).\end{aligned}\tag{2}$$

Note that if z is not contained in Cl_1, then the denominator is not identically zero. Since p does not depend on y, differentiating the expression for p with respect to y and equating the numerator to zero, we obtain

$$\begin{aligned}(p_1 - z_{1,0})\big(&p_1z_{0,1}^2z_{1,1}z_{1,3} - p_1z_{0,1}^2z_{1,2}^2 - p_1z_{0,1}z_{0,2}z_{1,1}z_{1,2} - p_1z_{0,1}z_{0,3}z_{1,1}^2 + 2p_1z_{0,2}^2z_{1,1}^2\\ &+ z_{0,1}^3z_{1,1}z_{2,2} - z_{0,1}^3z_{1,2}z_{2,1} - z_{0,1}^2z_{1,0}z_{1,1}z_{1,3} + z_{0,1}^2z_{1,0}z_{1,2}^2 - 2z_{0,1}^2z_{1,1}^2z_{1,2}\\ &+ z_{0,1}z_{0,2}z_{1,0}z_{1,1}z_{1,2} + 2z_{0,1}z_{0,2}z_{1,1}^3 + z_{0,1}z_{0,3}z_{1,0}z_{1,1}^2 - 2z_{0,2}^2z_{1,0}z_{1,1}^2\big) = 0.\end{aligned}$$

If z is not contained in Cl_1, then the first factor does not identically vanish, and we can equate the second one to zero. Expressing p_1, we obtain

$$\begin{aligned}p_1 = -\big(&z_{0,1}^3z_{1,1}z_{2,2} - z_{0,1}^3z_{1,2}z_{2,1} - z_{0,1}^2z_{1,0}z_{1,1}z_{1,3} + z_{0,1}^2z_{1,0}z_{1,2}^2 - 2z_{0,1}^2z_{1,1}^2z_{1,2}\\ &+ z_{0,1}z_{0,2}z_{1,0}z_{1,1}z_{1,2} + 2z_{0,1}z_{0,2}z_{1,1}^3 + z_{0,1}z_{0,3}z_{1,0}z_{1,1}^2 - 2z_{0,2}^2z_{1,0}z_{1,1}^2\big)/\\ &(z_{0,1}^2z_{1,1}z_{1,3} - z_{0,1}^2z_{1,2}^2 - z_{0,1}z_{0,2}z_{1,1}z_{1,2} - z_{0,1}z_{0,3}z_{1,1}^2 + 2z_{0,2}^2z_{1,1}^2).\end{aligned}\tag{3}$$

We differentiate with respect to y, separate out the numerator, remove the nonzero factor $z_{01}^2z_{11}$, and obtain

$$\begin{aligned}F_1(z) = &-z_{0,1}^3z_{1,1}z_{1,3}z_{2,3} + z_{0,1}^3z_{1,1}z_{1,4}z_{2,2} + z_{0,1}^3z_{1,2}^2z_{2,3} - z_{0,1}^3z_{1,2}z_{1,3}z_{2,2}\\ &- z_{0,1}^3z_{1,2}z_{1,4}z_{2,1} + z_{0,1}^3z_{1,3}^2z_{2,1} + z_{0,1}^2z_{0,2}z_{1,1}z_{1,2}z_{2,3} - 2z_{0,1}^2z_{0,2}z_{1,1}z_{1,3}z_{2,2}\\ &+ z_{0,1}^2z_{0,2}z_{1,2}z_{1,3}z_{2,1} + z_{0,1}^2z_{0,3}z_{1,1}^2z_{2,3} - 3z_{0,1}^2z_{0,3}z_{1,1}z_{1,2}z_{2,2}\\ &- z_{0,1}^2z_{0,3}z_{1,1}z_{1,3}z_{2,1} + 3z_{0,1}^2z_{0,3}z_{1,2}^2z_{2,1} - z_{0,1}^2z_{0,4}z_{1,1}^2z_{2,2} + z_{0,1}^2z_{0,4}z_{1,1}z_{1,2}z_{2,1}\\ &- 2z_{0,1}^2z_{1,1}^2z_{1,2}z_{1,4} + 3z_{0,1}^2z_{1,1}^2z_{1,3}^2 + 2z_{0,1}^2z_{1,1}z_{1,2}^2z_{1,3} - 3z_{0,1}^2z_{1,2}^4\\ &- 2z_{0,1}z_{0,2}^2z_{1,1}^2z_{2,3} + 6z_{0,1}z_{0,2}^2z_{1,1}z_{1,2}z_{2,2} + 2z_{0,1}z_{0,2}^2z_{1,1}z_{1,3}z_{2,1}\\ &- 6z_{0,1}z_{0,2}^2z_{1,2}^2z_{2,1} + 6z_{0,1}z_{0,2}z_{0,3}z_{1,1}^2z_{2,2} - 6z_{0,1}z_{0,2}z_{0,3}z_{1,1}z_{1,2}z_{2,1}\\ &+ 2z_{0,1}z_{0,2}z_{1,1}^3z_{1,4} - 10z_{0,1}z_{0,2}z_{1,1}^2z_{1,2}z_{1,3} + 6z_{0,1}z_{0,2}z_{1,1}z_{1,2}^3 - 6z_{0,1}z_{0,3}z_{1,1}^3z_{1,3}\\ &+ 6z_{0,1}z_{0,3}z_{1,1}^2z_{1,2}^2 + 2z_{0,1}z_{0,4}z_{1,1}^3z_{1,2} - 6z_{0,2}^3z_{1,1}^2z_{2,2} + 6z_{0,2}^3z_{1,1}z_{1,2}z_{2,1}\\ &+ 8z_{0,2}^2z_{1,1}^3z_{1,3} - 3z_{0,2}^2z_{1,1}^2z_{1,2}^2 - 6z_{0,2}z_{0,3}z_{1,1}^3z_{1,2} - 2z_{0,2}z_{0,4}z_{1,1}^4 + 3z_{0,3}^2z_{1,1}^4 = 0.\end{aligned}\tag{4}$$

Now, differentiating (3) with respect to x and equating the result to (2), we obtain a relation, which, after division by the nonzero factor z_{01}^3, takes the form

$$\begin{aligned}F_2(z) = &-z_{0,1}^2z_{1,1}^2z_{1,3}z_{3,2} + z_{0,1}^2z_{1,1}^2z_{2,2}z_{2,3} + z_{0,1}^2z_{1,1}z_{1,2}^2z_{3,2} + z_{0,1}^2z_{1,1}z_{1,2}z_{1,3}z_{3,1}\\ &- z_{0,1}^2z_{1,1}z_{1,2}z_{2,1}z_{2,3} - 3z_{0,1}^2z_{1,1}z_{1,2}z_{2,2}^2 + 2z_{0,1}^2z_{1,1}z_{1,3}z_{2,1}z_{2,2} - z_{0,1}^2z_{1,2}^3z_{3,1}\\ &+ 3z_{0,1}^2z_{1,2}^2z_{2,1}z_{2,2} - 2z_{0,1}^2z_{1,2}z_{1,3}z_{2,1}^2 + z_{0,1}z_{0,2}z_{1,1}^2z_{1,2}z_{3,2} - z_{0,1}z_{0,2}z_{1,1}z_{1,2}^2z_{3,1}\\ &- 3z_{0,1}z_{0,2}z_{1,1}z_{1,2}z_{2,1}z_{2,2} + 3z_{0,1}z_{0,2}z_{1,2}^2z_{2,1}^2 + z_{0,1}z_{0,3}z_{1,1}^3z_{3,2}\\ &- z_{0,1}z_{0,3}z_{1,1}^2z_{1,2}z_{3,1} - 3z_{0,1}z_{0,3}z_{1,1}^2z_{2,1}z_{2,2} + 3z_{0,1}z_{0,3}z_{1,1}z_{1,2}z_{2,1}^2\end{aligned}$$

$$
\begin{aligned}
&- 2z_{0,1}z_{1,1}^3 z_{1,2}z_{2,3} + 6z_{0,1}z_{1,1}^2 z_{1,2}^2 z_{2,2} + 2z_{0,1}z_{1,1}^2 z_{1,2}z_{1,3}z_{2,1} - 6z_{0,1}z_{1,1}z_{1,2}^3 z_{2,1} \\
&- 2z_{0,2}^2 z_{1,1}^3 z_{3,2} + 2z_{0,2}^2 z_{1,1}^2 z_{1,2}z_{3,1} + 6z_{0,2}^2 z_{1,1}^2 z_{2,1}z_{2,2} - 6z_{0,2}^2 z_{1,1}z_{1,2}z_{2,1}^2 \\
&+ 2z_{0,2}z_{1,1}^4 z_{2,3} - 6z_{0,2}z_{1,1}^3 z_{1,2}z_{2,2} - 2z_{0,2}z_{1,1}^3 z_{1,3}z_{2,1} + 6z_{0,2}z_{1,1}^2 z_{1,2}^2 z_{2,1} = 0. \qquad (5)
\end{aligned}
$$

Theorem 1. *A function $z(x,y)$ belongs to Cl_1^+ if and only if*

$$F_1(z) = F_2(z) = 0,$$

where F_1 and F_2 are two different irreducible homogeneous forms of algebraic degree 6 and differential order 5 (see (4) and (5)). If, in addition, $\delta(z) \neq 0$ (see (1)), then $z \in Cl_1^+ \setminus Cl_1$ and $N(z) = 2$.

Proof. Necessity was proved above. Sufficiency follows from fact that the condition $F_1(z) = F_2(z) = 0$ makes it possible to define a function $p(x)$ such that $z(x,y) - p(x)$ belongs to Cl_1, because our argument is reversible. □

3. THE CLASS Cl_1^{++}

Now suppose that $z(x,y) \in Cl_1^{++}$, i.e.,

$$z(x,y) = c(a(x) + b(y)) - d(x+y).$$

Writing an equation of the first class for $z(x,y) + d(x+y)$, we obtain

$$
\begin{aligned}
&(-z_{1,0}z_{0,1}^2 z_{2,1} + z_{1,1}z_{2,0}z_{0,1}^2 + z_{1,2}z_{1,0}^2 z_{0,1} - z_{1,0}^2 z_{1,1}z_{0,2}) \\
&\quad + (-z_{0,1}^2 z_{2,1} + 2z_{0,1}z_{1,0}z_{1,2} - 2z_{0,1}z_{1,0}z_{2,1} + 2z_{0,1}z_{1,1}z_{2,0} - 2z_{0,2}z_{1,0}z_{1,1} + z_{1,2}z_{1,0}^2)d_1 \\
&\quad + (z_{0,1}^2 z_{1,1} + z_{0,1}^2 z_{2,0} - z_{0,2}z_{1,0}^2 - z_{1,0}^2 z_{1,1})d_2 + (-z_{1,0}z_{0,1}^2 + z_{0,1}z_{1,0}^2)d_3 \\
&\quad + (z_{0,1}z_{1,2} - 2z_{0,1}z_{2,1} - z_{0,2}z_{1,1} + 2z_{1,0}z_{1,2} - z_{1,0}z_{2,1} + z_{1,1}z_{2,0})d_1^2 \\
&\quad + (2z_{0,1}z_{1,1} + 2z_{0,1}z_{2,0} - 2z_{0,2}z_{1,0} - 2z_{1,0}z_{1,1})d_1 d_2 + (-z_{0,1}^2 + z_{1,0}^2)d_1 d_3 \\
&\quad + (z_{0,1}^2 - z_{1,0}^2)d_2^2 + (z_{1,2} - z_{2,1})d_1^3 + (-z_{0,2} + z_{2,0})d_1^2 d_2 \\
&\quad + (-z_{0,1} + z_{1,0})d_1^2 d_3 + (2z_{0,1} - 2z_{1,0})d_1 d_2^2 = 0.
\end{aligned}
$$

This relation is linear in d_3. Expressing d_3 in terms of d_1 and d_2 and stating that the function depends on $(x+y)$, we obtain a polynomial-differential relation of the form

$$
\begin{aligned}
P(z, d_1, d_2) &= d_2^2(k_2(z)\,d_1^2 + k_1(z)\,d_1 + k_0(z)) \\
&\quad + d_2(l_4(z)\,d_1^4 + l_3(z)\,d_1^3 + l_2(z)\,d_1^2 + l_1(z)\,d_1 + l_0(z)) \\
&\quad + (m_5(z)\,d_1^5 + m_4(z)\,d_1^4 + m_3(z)\,d_1^3 + m_2(z)\,d_1^2 + m_1(z)\,d_1 + m_0(z)) = 0.
\end{aligned}
$$

The denominator by which we multiplied equals

$$(z_{0,1} - z_{1,0})^2 (d_1 + z_{1,0})^2 (d_1 + z_{0,1})^2.$$

The obtained expression has differential order 4. Applying the jth degree of the differential operator

$$D = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

to the relation $P = 0$, we obtain the sequence of relations

$$
\begin{aligned}
&P_j(z, d_1, d_2) \\
&\quad = d_2^2(k_{2j}(z)\,d_1^2 + k_{1j}(z)\,d_1 + k_{0j}(z)) \\
&\qquad + d_2(l_{4j}(z)\,d_1^4 + l_{3j}(z)\,d_1^3 + l_{2j}(z)\,d_1^2 + l_{1j}(z)\,d_1 + l_{0j}(z)) \\
&\qquad + \big(m_{5j}(z)\,d_1^5 + m_{4j}(z)\,d_1^4 + m_{j3}(z)\,d_1^3 + m_{j2}(z)\,d_1^2 + m_{j1}(z)\,d_1 + m_{j0}(z)\big) = 0,
\end{aligned}
$$

where each coefficient with subscript j is obtained by applying D^j to the corresponding coefficient in P. This raises the differential order to $4+j$. The next step is eliminating d_2 from these relations. The resultant of the pair of polynomials at^2+bt+c and At^2+Bt+C has the form

$$A^2c^2 - ABbc - 2ACac + ACb^2 + B^2ac - BCab + C^2a^2.$$

The polynomials P, P_1, and P_2 are irreducible. Let $Q_1(d_1, z)$ and $Q_2(d_1, z)$ be the resultants of the pairs (P_1, P) and (P_2, P), respectively, with respect to the variable d_2. Each of these expressions is a polynomial in d_1 of degree 15. The differential orders of Q_1 and Q_2 are 5 and 6, respectively. Both polynomials are divisible by $(z_{10}-z_{01})^2$, i.e.,

$$Q_1(z, d_1) = (z_{10}-z_{01})^2 q_1(z, d_1), \qquad Q_2(z, d_1) = (z_{10}-z_{01})^2 q_2(z, d_1),$$

where q_1 and q_2 are irreducible polynomials. Both q_1 and q_2 are homogeneous forms in the derivatives of z and d_1 of degree 22. Thus, the coefficients free of d_1 in both polynomials q_1 and q_2 are homogeneous forms of degree 22, and as the degree of d_1 increases, the degrees of coefficients decrease, and d_1^{15} is multiplied by forms of degree 7.

To obtain a relation for z, we perform the last step, namely, eliminate d_1 from the system $q_1(z, d_1) = q_2(z, d_1) = 0$. This relation is

$$R(z) = 0,$$

where $R(z)$ is the resultant of $q_1(z, d_1)$ and $q_2(z, d_1)$ with respect to the variable q_1. This resultant is the determinant of the 30×30 matrix composed of the coefficients of q_1 and q_2. The function R is a homogeneous form in derivatives of z. To calculate the degree of this form, it suffices to sum the degrees of the forms on the diagonal of the corresponding square matrix. We obtain

$$l = 15 \times 7 + 15 \times 22 = 435.$$

The differential order of $R(z)$ equals 6.

If $R(z) = 0$, then the equations $q_1(d_1, z) = 0$ and $q_2(d_1, z) = 0$ have a common root. Using Cramer's rule, we express this root at a generic point (see, e.g., [6, Corollary 1, p. 20]) as a rational function in the coefficients of the equations $d_1 = M_1(z)$, i.e., as a rational function in derivatives of z. Next, we obtain rational expressions for $d_2 = M_2(z)$ and $d_3 = M_3(z)$ in a similar way. To obtain a criterion for the existence of a function d, it remains to write down two conditions, the condition that M_1, M_2, and M_3 are functions of $(x+y)$, that is,

$$D(M_1(z)) = D(M_2(z)) = D(M_3(z)) = 0,$$

and the condition

$$(M_1(z))'_x - M_2(z) = 0, \qquad (M_2(z))'_x - M_3(z) = 0$$

that (d_1, d_2, d_3) are consecutive derivatives. All these conditions consist in the vanishing of differential-rational relations. Let $(m_1(z), m_2(z), m_3(z))$ be the numerators of $D(M_1(z))$, $D(M_2(z))$, and $D(M_3(z))$), and let $(m_4(z),\ m_5(z))$ be the numerators of $((M_1(z))'_x - M_2(z))$ and $((M_2(z))'_x - M_3(z))$, which are differential polynomials. As a result, we obtain the following theorem.

Theorem 2. *A function z belongs to Cl_1^{++} if and only if it satisfies the differential-polynomial relations*

$$R(z) = m_1(z) = m_2(z) = m_3(z) = m_4(z) = m_5(z) = 0,$$

where the algebraic degrees of all polynomials are at most 435*, R has differential order* 6*, and $(m_1(z), m_2(z), m_3(z), m_4(z), m_5(z))$ have differential order* 7.

Note that the subclass Cl_1^{++} of the class Cl_2 considered above has the following quite obvious property: any function $z(x,y) \in Cl_2$ is equivalent to a function in Cl_1^{++} up to a transformation from the gauge pseudogroup

$$G = \{x \to \alpha(x),\ y \to \beta(y),\ z \to \gamma(z)\}.$$

4. THE COMPLEXITY DROP PHENOMENON

Now consider the following question related to Theorem 1. For what sets (a, b, f, g) of nonconstant functions does a function $z(x, y) = f(a(x) + b(y)) + g(x)$ in the class Cl_1^+ belong to Cl_1?

The first obvious reason is the linearity of f, i.e., $f'' = 0$ (this is the $1st$ *case*). In what follows, we assume that $a'b'g'f'' \neq 0$. Since the class Cl_1 is invariant with respect to the changes $x \to \alpha(x)$, $y \to \beta(y)$, the function $z(x, y) = f(a(x) + b(y)) + g(x)$ belongs to the first class if and only if so does $w(x, y) = f(x + y) + d(a^{-1}(x))$. Therefore it suffices to answer the question of for which r and p the function $z = r(x + y) + p(x)$ is contained in Cl_1.

Writing an equation of the first class for z, we obtain

$$-r_3r_1p_1^2 + r_2^2p_1^2 - r_3r_1^2p_1 + 2r_1r_2^2p_1 - r_1^2r_2p_2 = 0. \qquad (6)$$

This relation can be reduced to the form (with nonzero denominator)

$$\frac{p_2}{p_1} = -\frac{p_1r_1r_3 - p_1r_2^2 + r_1^2r_3 - 2r_1r_2^2}{r_1^2r_2}.$$

Differentiating with respect to y and separating out the numerator, we obtain

$$-(r_1^2r_2r_4 - r_1^2r_3^2 - 2r_1r_2^2r_3 + 2r_2^4)(p_1 + r_1) = 0.$$

The second factor cannot vanish identically; therefore,

$$r_1^2r_2r_4 - r_1^2r_3^2 - 2r_1r_2^2r_3 + 2r_2^4 = 0.$$

Replacing $r(x + y)$ by $r(t)$, we obtain a fourth-order ordinary differential equation. Treating $r'(t) = s$ as an independent variable and $r''(t) = R(s)$ as a new unknown function, we obtain

$$\left(\frac{d^2}{ds^2}R(s)\right)t^2 - 2\left(\frac{d}{ds}R(s)\right)t + 2R(s) = 0.$$

A general solution of this equation has the form $R(s) = ms + ns^2$. Substituting it into (6), we obtain

$$\left(\frac{d}{dx}p(x)\right)^2 n - \left(\frac{d}{dx}p(x)\right)m + \frac{d^2}{dx^2}p(x) = 0.$$

Solving this, we finally obtain

$$2nd\ case{:}\quad p(x) = \frac{\ln(C_1\mathrm{e}^{mx} + C_2)}{n} \qquad \text{for} \quad mn \neq 0,$$

$$3rd\ case{:}\quad p(x) = C_1\mathrm{e}^{mx} + C_2 \qquad \text{for} \quad n = 0, \quad m \neq 0,$$

$$4th\ case{:}\quad p(x) = \frac{\ln(C_1x + C_2)}{n} \qquad \text{for} \quad m = 0, \quad n \neq 0.$$

The last possibility $m = n = 0$ corresponds to $r'' = R = 0$, and r is linear, so that this possibility is absorbed by the 1st case. Solving the equation for the last three cases, we obtain

$$2nd\ case{:}\quad r(t) = -\frac{1}{n}\ln(C_3 - n\mathrm{e}^{mt}) + C_4,$$

$$3rd\ case{:}\quad r(t) = C_3\mathrm{e}^{mt} + C_4,$$

$$4th\ case{:}\quad r(t) = -\frac{1}{n}\ln(n(x + y) + C_3) + C_4.$$

Let us summarize.

Theorem 3. *Consider pairs of functions*

$$z_1 = f(a(x) + b(y)), \qquad z_2 = g(x),$$

where a, b, f, and g are nonconstant functions and $z_1 + z_2 \in Cl_1$. Up to the transformations

$$x \to \alpha(x), \qquad y \to \beta(y), \qquad f(t) \to f(t) + C_1, \qquad g(x) \to g(\alpha(x)) + C_2,$$

all such pairs are contained in the following list:

$$\begin{array}{lll}
(1) & z_1 = x + y, & z_2 = p(x), \\
(2) & z_1 = xy, & z_2 = Cx, \\
(3) & z_1 = C_1 \ln(xy + C_2), & z_2 = -C_1 \ln(x + C_2), \\
(4) & z_1 = C \ln(x + y), & z_2 = -C \ln(x).
\end{array}$$

This result can be interpreted as follows. Suppose that a function z belongs to the zeroth class but has a representation in the second class, for example,

$$z = s\{c[a(x) + b(y)] + C[A(x) + B(y)]\} = q(x),$$

or, equivalently,

$$c[a(x) + b(y)] + p(x) \in Cl_1^+,$$

where $p(x) = -s^{-1} \circ q(x)$ belongs to Cl_1. Then Theorem 2 gives a description of all (s, a, b, c, A, B, C) for which the corresponding composition depends only on x. Interchanging x and y, we obtain a description of all superpositions in the second class which represent functions depending only on y.

We can ask also the following similar question: When does the complexity of a function in the second class equal 1? As above, this question reduces to the question of what functions in Cl_1^{++} are contained in Cl_1, or to the problem of describing pairs of functions in the first class whose sum is again a function in the first class:

$$c(a(x) + b(y)) + C(A(x) + B(y)) = \gamma(\alpha(x) + \beta(y)).$$

We cannot give a constructive answer to this question, because we have no analog of Theorem 3 for Cl_1^{++}. However, pairs of functions in the first class subject to additional constraints, namely, such that any linear combination of these functions remains in the first class (such pairs of functions are called L-pairs), admit a simple description (see [5]).

The calculations described in this paper were implemented in *Maple*.

FUNDING

This work was supported by the Russian Foundation for Basic Research under grant 17-01-00592 a and grant 18-51-41011 Uzb t.

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