ISSN 0001-4346, Mathematical Notes, 2016, Vol. 100, No. 6, pp. 774–780. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © V. K. Beloshapka, 2016, published in Matematicheskie Zametki, 2016, Vol. 100, No. 6, pp. 781–789.

Analytic Complexity of Functions of Several Variables

V. K. Beloshapka*

Lomonosov Moscow State University, Moscow, Russia

Received June 25, 2016

Abstract—An approach to estimating the complexity of analytic functions of arbitrarily many variables is proposed. A description of harmonic functions of complexity one of three variables and of algebraic functions of complexity one of arbitrarily many variables is given.

DOI: 10.1134/S000143461611016X

Keywords: *analytic complexity, harmonic functions, algebraic functions.*

1. SUPERPOSITIONS OF TYPE ($m \to 1$). DEFINITION OF COMPLEXITY

The theory of complexity of analytic functions whose fragments were presented by the author in [1] and [2] considered problems related to the possibility of representing functions of two variables using functions of a single variable (a superposition of type ($2 \to 1$)). When posing the problem of representing analytic functions of arbitrarily many ($m \geq 2$) variables $(x_1, \dots, x_m)$ by analytic functions of a single variable (superposition of type ($n \to 1$)), diverse inductive definitions are possible. Here, developing the approach of [1], we adopt the following point of view. Assume that Cl_0 are functions of a single (arbitrary) variable, i.e., all these functions are of complexity zero. The class Cl_1 is formed by functions of the form

$$z(x_1, \dots, x_m) = c(a_1(x_1) + \dots + a_m(x_m)),$$

where $(a_1, \dots, a_m, c)$ are analytic functions of a single variable. These functions, except for those belonging to Cl_0, are of complexity one. As in [1], the functions of complexity two are functions whose complexity is not equal to zero or one and which admit a representation

$$z(x, y) = C(A_1(x, y) + B_1(x, y)),$$

where A_1 and B_1 are of complexity not exceeding one. And so on. We finally obtain a complexity function $N(z)$ which is defined for all analytic functions z and takes the values $(0, 1, 2, \dots, \infty)$. We write $N(z) = n$ if $z \in \mathrm{Cl}_n \setminus \mathrm{Cl}_{n-1}$. If z does not belong to any class Cl_n, then $N(z) = \infty$. Although the membership condition for a class is formulated locally, one can show that the complexity of a germ is not modified under an analytic continuation. This follows from the fact that all classes Cl_n are differential-algebraic sets, i.e., are the families of analytic solutions of a finite set of differential-polynomial relations with complex coefficients.

The complexity defined in the above way is similar to the degree of a polynomial. If $P(x_1, \dots, x_m)$ is a polynomial in m variables of degree at most d, then, fixing one of the variables $x_m = a$, we obtain a polynomial in $m - 1$ variables of degree at most d. A similar assertion also holds for analytic complexity.

Proposition 1. *If $N(z(x_1, \dots, x_m)) \leq n$, then*

$$N(z(x_1, \dots, x_{m-1}, a)) \leq n.$$

*E-mail: vkb@strogino.ru

Thus, if we take a function of several variables in Cl_1 and fix all the variables except two, then we obtain a function of the first class in the sense of the old definition [1] (which was given for functions of two variables). An equation defining the first class for functions of variables (x_1, x_2) is of the form

$$\delta_{12}(z) = \left(\log\left(\frac{z'_{x_1}}{z'_{x_2}}\right)\right)''_{x_1x_2} = 0.$$

Therefore, for every $z \in \mathrm{Cl}_1$ and for every pair of variables (x_i, x_j), $i \neq j$, the equation $\delta_{ij}(z) = 0$ of the first class holds with respect to this pair of variables.

Example 2. (a) We have

$$z = x_1x_2\cdots x_m = e^{(\ln(x_1)+\cdots+\ln(x_m))},$$

and therefore $N(z) = 1$.

(b) It can readily be seen that the function of three variables $z = x_1x_2 + x_3$ is of complexity one with respect to every pair of variables; however, it is not representable in the form $c(a_1(x_1) + a_2(x_2) + a_3(x_3))$ and, therefore, $N(z) = 2$.

The following lemma holds.

Lemma 3. *The general solution of the system of equations* $z''_{x_1x_2} = z'_{x_3} = \cdots = z'_{x_m} = 0$ *is of the form* $z = A(x_1) + B(x_2)$.

As a corollary, we obtain the following lemma.

Lemma 4. *The general solution of the system of equations* $\Delta_{12}(z) = 0$ *of the form*

$$\delta_{12}(z) = \left(\log\left(\frac{z'_{x_1}}{z'_{x_2}}\right)\right)''_{x_1x_2} = \left(\frac{z'_{x_1}}{z'_{x_2}}\right)'_{x_j} = 0, \qquad j = 3, \dots, m,$$

is the family of functions

$$z(x_1, \dots, x_m) = c(a_1(x_1) + a_2(x_2) + A(x_3, \dots, x_m)).$$

Proposition 5. *For an analytic function* z *to satisfy* $z(x_1, \dots, x_n) \in \mathrm{Cl}_1$, *it is necessary and sufficient that the relations* $\Delta_{j_1j_2}(z) = 0$ *be valid for all* $1 \leq j_1 \neq j_2 \leq m$.

Proof. The *necessity* is verified immediately. The condition that

$$z = c(t), \qquad \text{where} \quad t = a_1(x_1) + \cdots + a_m(x_m),$$

can be represented as the condition that z is constant on the level sets of t. In turn, this is equivalent to the condition $X_2z = \cdots = X_mz = 0$, where $X_2, \dots, X_m$ are vector fields generating a foliation on the level hypersurface of t,

$$X_j = a_j(x_j)\frac{\partial}{\partial x_1} - a_1(x_1)\frac{\partial}{\partial x_j}, \qquad j = 2, \dots, m.$$

The relation $X_jz = 0$ can be represented in the form $z'_{x_j}/z'_{x_1} = a'_j/a'_1$. Eliminating a_j and a_1 from this relation, we obtain an equivalent relation $\Delta_{j1}(z) = 0$ for all $j = 2, \dots, m$.

We obtain the *sufficiency* by a successive application of Lemma 3. This completes the proof of the proposition. □

The uniqueness problem for the representation of a function of complexity one can be solved exactly as in the case of functions of two variables. The following lemma holds.

Lemma 6. *Let* $(a_j(t), c(t))$, $j = 1, \dots, m$, *be nonconstant holomorphic functions, and let the relation*

$$c(a_1(x_1) + \dots + a_m(x_m)) = x_1 + \dots + x_m$$

hold on an open set of the variables $(x_1, \dots, x_m)$. *Then*

$$a_j(x_j) = \frac{x_j + p_j}{k}, \quad c(t) = kt - \sum p_j, \qquad \textit{where} \quad k \neq 0.$$

Proof. The proof is simple and does not differ from that in the two-dimensional case. □

The lemma proved above implies a uniqueness assertion for representations of functions of first class.

Proposition 7. *If a function* z *of complexity one has two representations*

$$z = c(a_1(x_1) + \dots + a_m(x_m)) = C(A_1(x_1) + \dots + A_m(x_m)),$$

then

$$A_j(x_j) = \frac{a_j(x_j) + p_j}{k} \qquad \textit{and} \qquad C(t) = c\Big(kt - \sum p_j\Big).$$

Proof. In the domain of a holomorphic element of this function, choose values $x_j = x_j^0$ in such a way that a_j is invertible in a neighborhood of x_j^0 and c is invertible in a neighborhood of $t^0 = \sum a_j(x_j^0)$. For the domain in Lemma 6, take a neighborhood $(x_0, \dots, x_m)$ and, for the functions, take the elements representing $(a_j, c))$. Replacing x_j by $a_j^{-1}(x_j)$, we obtain

$$c^{-1} \circ C(A_1 \circ a_1^{-1}(x_1) + \dots + A_m \circ a_m^{-1}(x_m)) = x_1 + \dots + x_m.$$

Now it follows from Lemma 6 that

$$A_j \circ a_j^{-1}(x_j) = \frac{x_j + p_j}{k}, \qquad c^{-1} \circ C(t) = kt - \sum p_j.$$

This implies Proposition 7. □

The following pseudogroup (gauge pseudogroup) acts on the space of functions of m variables:

$$\mathscr{G} = \{z(x_1, \dots, x_m) \to \psi^{-1}(z(\varphi_1(x_1), \dots, \varphi_1(x_m))\},$$

where (φ_j, ψ) are the germs of nonconstant analytic functions. It is clear that this action does not change the complexity and that the functions of complexity one form precisely the orbit of the function $(x_1 + \dots + x_m)$.

If S is a family of analytic functions, then by $N(S)$ we denote the maximum (possibly infinite) of the complexities of functions entering the family.

Proposition 8. *Let* $m > 2$, *and let there be a single analytic partial differential relation of arbitrary order, with one unknown function* $u(x_1, \dots, x_m)$, $F(D, u, x) = 0$, *which satisfies the conditions of the Cauchy–Kovalevskaya theorem* (*resolved with respect to some highest derivative*). *Then the family of analytic solutions of this equation is of complexity equal to infinity.*

Proof. According to the Cauchy–Kovalevskaya theorem, analytic solutions of the Cauchy problem with data on a hyperplane are defined uniquely by a family of arbitrary analytic functions of the $(m-1)$th variable. If all solutions of this equation have finite complexity, then they can be expressed using a finite family of functions of a single variable; the same holds for the initial data of the Cauchy problem. However, if $m - 1 > 1$, then the computation of the number of parameters in the jets of sufficiently high order shows that this is impossible. A contradiction. □

2. HARMONIC FUNCTIONS OF THREE VARIABLES

The problem of the structure of harmonic functions of two variables of complexity one was considered in [2]. Consider this problem for functions of three variables. Since the problem concerning the harmonic functions of two variables was solved earlier, we can consider here only functions that are nonconstant with respect to any of the variables. Note that, if $u(x,y,z) = f(a(x)+b(y)+c(z))$ is a harmonic function of first class defined by the family of functions (a,b,c,f), then the function $\widetilde{u}(x,y,z)$ defined by the family

$$\widetilde{a}(x) = a(x-\alpha), \qquad \widetilde{b}(y) = b(y-\beta), \qquad \widetilde{c}(z) = c(z-\gamma), \qquad \widetilde{f}(t) = kf(lt-\delta),$$

(k and l are nonzero constants) is also harmonic of first class. The same can be said about every permutation of the variables. These transformations form a group G. For a complete description of harmonic functions of first class, it suffices to obtain such a description up to transformations in G.

Let a function

$$u(x,y,x) = f(a(x)+b(y)+c(z)),$$

where (a,b,c,f) are nonconstant analytic functions, be harmonic on some domain of the three-dimensional space. If $(a')^2+(b')^2+(c')^2$ is identically zero, then all its derivatives are constant, i.e., (a,b,c) are linear, and it follows from the Laplace equation that $f''=0$. That is, u is linear (case 0). If $(a')^2+(b')^2+(c')^2$ is not identically zero, then, writing out the Laplace equation, we obtain

$$\tau(x,y,x) = \frac{f''(a(x)+b(y)+c(z))}{f'(a(x)+b(y)+c(z))} = -\frac{a''(x)+b''(y)+c''(z)}{(a'(x))^2+(b'(y))^2+(c'(z))^2}.$$

The left-hand side is a function of $a+b+c$. This is equivalent to the condition that τ is constant on the level sets of this sum, which, in turn, can be represented as the vanishing condition for the derivatives along two vector fields,

$$\frac{1}{b'}\tau_y - \frac{1}{a'}\tau_x = 0, \qquad \frac{1}{c'}\tau_z - \frac{1}{a'}\tau_x = 0.$$

Let us write out the result using the subscript notation for the derivatives,

$$\begin{aligned} &-{a_1}^3b_3 + a_3{a_1}^2b_1 - 2a_1b_1{a_2}^2 - 2a_1b_1a_2c_2 - a_1{b_1}^2b_3 \\ &\quad + 2a_1b_1{b_2}^2 + 2a_1b_1b_2c_2 - a_1{c_1}^2b_3 + a_3{b_1}^3 + a_3b_1{c_1}^2 = 0, \\ &-c_3{a_1}^3 + a_3{a_1}^2c_1 - 2a_1c_1{a_2}^2 - 2a_1c_1a_2b_2 - c_3a_1{b_1}^2 \\ &\quad + 2a_1c_1b_2c_2 - c_3a_1{c_1}^2 + 2a_1c_1{c_2}^2 + a_3{b_1}^2c_1 + a_3{c_1}^3 = 0. \end{aligned} \tag{1}$$

We lower the order in this equation by taking $a_1 = a'(x) = A$ as a new independent variable, $a_2 = a''(x) = P(A)$ as a new unknown function, and, correspondingly, $a_3 = P'P$, and by similarly setting $b_1 = b'(y) = B$, $b_2 = b''(y) = Q(B)$, $b_3 = Q'Q$, and $c_1 = c'(z) = C$, $c_2 = c''(z) = R(C)$, and $c_3 = R'R$. We obtain the relations

$$\begin{aligned} &-A^3\left(\frac{d}{dB}Q(B)\right)Q(B) + \left(\frac{d}{dA}P(A)\right)P(A)A^2B - 2AB(P(A))^2 - 2ABP(A)R(C) \\ &\quad - AB^2\left(\frac{d}{dB}Q(B)\right)Q(B) + 2AB(Q(B))^2 + 2ABQ(B)R(C) \\ &\quad - AC^2\left(\frac{d}{dB}Q(B)\right)Q(B) + \left(\frac{d}{dA}P(A)\right)P(A)B^3 + \left(\frac{d}{dA}P(A)\right)P(A)BC^2 = 0, \\ &-\left(\frac{d}{dC}R(C)\right)R(C)A^3 + \left(\frac{d}{dA}P(A)\right)P(A)A^2C - 2AC(P(A))^2 - 2ACP(A)Q(B) \\ &\quad - \left(\frac{d}{dC}R(C)\right)R(C)AB^2 + 2ACQ(B)R(C) - \left(\frac{d}{dC}R(C)\right)R(C)AC^2 \\ &\quad + 2AC(R(C))^2 + \left(\frac{d}{dA}P(A)\right)P(A)B^2C + \left(\frac{d}{dA}P(A)\right)P(A)C^3 = 0. \end{aligned} \tag{2}$$

Let us express $R(C)$ using the first relation. This is a fraction whose denominator becomes identically zero only if $P(A)$ and $Q(B)$ are equal constants. Substituting the expression for $R(C)$ into the second relation, we obtain the equation

$$A^3\frac{dQ}{dB}-A^2B\frac{dP}{dA}+AB^2\frac{dQ}{dB}+AC^2\frac{dQ}{dB}-B^3\frac{dP}{dA}-BC^2\frac{dP}{dA}Q(B)CP(A)=0,$$

which is decomposed into two ones (the coefficient at C^2 and the equation without C),

$$\begin{aligned}\Big(A^3\frac{d}{dB}Q(B)-A^2B\frac{d}{dA}P(A)+AB^2\frac{d}{dB}Q(B)-B^3\frac{d}{dA}P(A)\Big)Q(B)P(A)=0,\\ -\Big(A\frac{d}{dB}Q(B)-B\frac{d}{dA}P(A)\Big)Q(B)P(A)=0.\end{aligned}\tag{3}$$

Separating the variables in the second relation, we see that $P(A)$ and $Q(B)$ (and also $R(C)$, by the symmetry of the variables) are linear and

$$P(A)=\lambda A+p,\qquad Q(B)=\lambda B+q,\qquad R(C)=\lambda C+r.$$

Substituting this into (1) and distinguishing the coefficient at A^3B in the first equation, we see that $\lambda=0$. After this, both the relations (1) become

$$AB(p-q)(p+r+q)=AC(p-r)(p+r+q)=0.$$

Two cases are possible:

case 1: $p=q=r=2m$;

case 2: $p+q+r=0$.

Consider the first case. We obtain

$$a(x)=mx^2+\alpha_1x+\alpha_0,\qquad b(y)=my^2+\beta_1y+\beta_0,\qquad c(z)=mz^2+\gamma_1z+\gamma_0.$$

If $m=0$, then $f''=0$ and, therefore, $u(x,y,x)$ is a linear function and, up to an action of the group G, the function is of the form

$$u(x,y,x)=x+y+z\qquad (\textit{case}\ 1.1).$$

If $m\neq 0$ (*case* 1.2), then, using changes in G, one can reduce the sum $a+b+c$ to the form $x^2+y^2+z^2$, and the expression for f''/f' becomes

$$\frac{f''(t)}{f'(t)}=-\frac{3}{2t}.$$

Whence, integrating, we obtain $f(t)=d_1(1/\sqrt{t})+d_2$. Deleting the constants by an action of G, we obtain, in case (1.2), the well-known fundamental solution of the Laplace equation

$$u(x,y,x)=\frac{1}{\sqrt{x^2+y^2+z^2}}\qquad (\textit{case}\ 1.2).$$

In case 2, it follows from the condition $p+q+r=0$ that $a''+b''+c''=0$. This implies that $f''=0$, i.e., $f(t)$ is linear and, after an application of a transformation in G, we may assume that $f(t)=t$. Finally,

$$u(x,y,z)=px^2+qy^2+rz^2+\text{linear terms},\qquad \text{where}\quad p+q+r=0\qquad (\textit{case}\ 2).$$

It can readily be seen that, up to transformations in G, such a function (if it is not a function of two variables) can be represented in the form $u=x^2-y^2+z$ (*case* 2.1) or $u=x^2+qy^2-(q+1)z^2$ (*case* 2.2). Thus, we have proved the following theorem.

Theorem 9. *Every harmonic function of three variables of complexity one belongs to one of the following three classes*:

- *it depends on at most two variables*;
- *it is linear*;
- *it can be reduced, using transformations in G, to one of the following forms*:

$$u_1 = x^2 - y^2 + z, \qquad u_2 = x^2 + qy^2 - (q+1)z^2, \qquad u_3 = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

The theorem proved above has a curious corollary which shows that the case of two variables differs from that of three variables.

Corollary 10. *Every harmonic function of three variables of complexity one is algebraic up to transformations in G.*

The problem on the structure of harmonic functions of complexity one depending on $m > 3$ variables remains open. However, it is clear that all types distinguished in the theorem admit a generalization to arbitrarily many variables. In particular, for every m, the fundamental solution of the Laplace equation has complexity one. We also note that, using the theorem, one can obtain a description of solutions of complexity one of the wave equation.

3. ALGEBRAIC FUNCTIONS

The problem of algebraic functions of complexity one of two variables was considered in [2] (see also [3]). In these papers, a description using the Weierstrass theorem on the functions with algebraic addition theorem was suggested. Here, using similar considerations, we obtain a description of algebraic functions of complexity one of arbitrarily many variables.

Let an algebraic function $P(x_1,\dots,x_m)$ have analytic complexity one as a function of m variables, i.e.,

$$P(x_1,\dots,x_m) = c(a_1(x_1) + \dots + a_m(x_m)),$$

where $(a_1,\dots,a_m,c)$ are nonconstant analytic functions of a single variable. After a translation of the origin, we may assume that the origin belongs to the domain of an analytic element defining the right-hand side of the relation and that $a_j(0) = 0$ for all j. Since P is nonconstant with respect to all variables, we can assume here that the translation of the origin was made in such a way that the algebraic functions $\phi_j(x_j) = P(0,\dots,0,x_j,0,\dots,0)$ are not constant. It now follows from the relation for P that

$$a_j(x_j) = c^{-1}(\phi_j(x_j)) \qquad \text{for all} \quad j.$$

As a result, after the change $u_j = \phi_j(x_j)$, we obtain

$$c(c^{-1}(u_1) + \dots + c^{-1}(u_m)) = P(\phi_1^{-1}(u_1),\dots,\phi_m^{-1}(u_m)) = Q(u_1,\dots,u_m)$$

or, after the change $u_j = c(U_j)$,

$$c(U_1 + \dots + U_m) = Q(\phi_1(U_1),\dots,\phi_n(U_m)). \tag{4}$$

Diverse formulations of the Weierstrass theorem on functions admitting an algebraic addition theorem can be found in the beautiful survey [4] (see also [5]). The following theorem is sufficient here for our purposes.

Theorem (Weierstrass). *Let $c(t)$ be a nonconstant analytic function of a single variable, and let R be a nonzero polynomial in three variables with complex coefficients such that the following relation holds on some domain*:

$$R(c(U), c(V), c(U+V)) \equiv 0 \qquad (\textit{an algebraic addition theorem}).$$

This is possible in one and only one of the following three cases:

(1) *algebraic case*: $c(t) = \eta(t)$;

(2) *periodic case*: $c(t) = \eta(e^{\lambda t})$;

(3) *doubly periodic case*: $c(t) = \eta(\wp(t))$, *where* $\eta(s)$ *is an algebraic function and* $\wp(t)$ *is a Weierstrass* $\wp(t)$*-function constructed from a lattice* $L(\omega_1, \omega_2)$.

For the Weierstrass function itself, $w = \wp(U+V)$ is expressed in terms of $u = \wp(U)$ and $v = \wp(V)$ using the function [5]

$$w = u \diamond v = \wp(\wp^{-1}(u) + \wp^{-1}(v)) = -(u+v) + \left(\frac{\sqrt{H(u)} - \sqrt{H(v)}}{2(u-v)}\right)^2,$$

where $H(t) = 4t^3 - g_2 t - g_3$ is a cubic polynomial without multiple roots in Weierstrass form ($u \diamond v$ is the "elliptic addition"). It can readily be seen that a nonconstant algebraic function can enter only one of the three classes.

The following description of algebraic functions of complexity one of two variables was given in [6] using the above theorem.

Theorem A. *If* $P(x, y)$ *is an algebraic function of analytic complexity one of two variables, then it has one and only one representation of the following form*:

(1) $P(x, y) = \gamma(\alpha(x) + \beta(y))$ (*additive representation*);

(2) $P(x, y) = \gamma(\alpha(x) \cdot \beta(y))$ (*multiplicative representation*);

(3) $P(x, y) = \gamma(\alpha(x) \diamond \beta(y))$ (*elliptic representation*),

where α, β, *and* γ *are algebraic.*

Using this theorem, one can give a similar description of algebraic functions of complexity one of arbitrarily many variables.

Theorem 11. *If* $P(x_1, \ldots, x_m)$ *is an algebraic function of analytic complexity one of* m *variables, then it has one and only one representation of the following form*:

(1) $P(x_1, \ldots, x_m) = \gamma(\alpha_1(x_1) + \cdots + \alpha_m(x_m))$ (*additive representation*);

(2) $P(x_1, \ldots, x_m) = \gamma(\alpha_1(x_1) \cdots \alpha_m(x_m))$ (*multiplicative representation*);

(3) $P(x_1, \ldots, x_m) = \gamma(\alpha_1(x_1) \diamond \cdots \diamond \alpha_m(x_m))$ (*elliptic representation*),

where α_j *and* γ *are algebraic functions of a single variable.*

Proof. Let us fix all variables except x_1 and x_2. Relation (4) means that the function $c(t)$ satisfies the conditions of the Weierstrass theorem, and thus enters one of the three listed cases.

Let the first case hold, i.e., let $c(t)$ be algebraic. This implies that $a_1(x_1)$ and $a_1(x_2)$ are algebraic. Applying now our theorem to the pair of variables (x_1, x_j), we see that $a_j(x_j)$ is algebraic.

The second and third cases are treated in a similar way. This completes the proof of the theorem. □

ACKNOWLEDGMENTS

This work was financially supported by the Russian Foundation for Basic Research under grants 11-01-00495-a and 11-01-12033-ofi-m-2011.

REFERENCES

1. V. K. Beloshapka, "Analytic complexity of functions of two variables," Russ. J. Math. Phys. **14** (3), 243–249 (2007).
2. V. K. Beloshapka, "A seven-dimensional family of simple harmonic functions," Mat. Zametki **98** (6), 803–808 (2015) [Math. Notes **98** (5–6), 867–871 (2015)].
3. M. Stepanova, "On rational functions of first-class complexity," Russ. J. Math. Phys. **23** (2), 251–256 (2016).
4. M. B. Villarino, *Algebraic Addition Theorems*, http://arxiv.org/abs/1212.6471 (2013).
5. V. V. Prasolov and Yu. P. Solov'ev, *Elliptic Functions and Algebraic Equations* (Faktorial, Moscow, 1997; V. Prasolov and Yu. Solovyev, *Elliptic Functions and Elliptic Integrals*, Translations of Mathematical Monographs, 170. AMS, Providence, RI, 1997).
6. V. K. Beloshapka, "Algebraic functions of complexity one, a Weierstrass theorem, and three arithmetic operations," Russ. J. Math. Phys. **23** (3), 343–347 (2016).