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Algebraic Functions of Complexity One, a Weierstrass Theorem, and Three Arithmetic Operations

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Abstract. The Weierstrass theorem concerning the functions admitting an algebraic addition theorem enables us to give an explicit description of algebraic functions of two variables of analytical complexity one. Their description is divided into three cases: the general case, which is elliptic, and two special ones, a multiplicative and and additive. All cases have a unified description; these are the orbits of an action of the gauge pseudogroup. The first case is a 1-parameter family of orbits of "elliptic addition," the second is the orbit of multiplication, and the third of addition. Here the multiplication and addition can be derived from the "elliptic addition" by passages to a limit. On the other hand, the elliptic orbits correspond to complex structures on the torus, the multiplicative orbit corresponds to the complex structure on the cylinder, and the additive one to that on the complex plane.

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In [1], a method for measuring the complexity of analytic (possibly multivalued) functions of two variables was suggested. For every analytic function $z(x, y)$, its analytic complexity $N(z)$ is defined. This quantity can take the values $0, 1, 2, \dots, \infty$. The complexity is zero for the functions depending on only one variable, of the form $a(x)$ and $b(y)$. The has functions depending on both the variables have the complexity one provided that there is a local representation of the form $z(x, y) = c(a(x) + b(y))$, where (a, b, c) are analytic functions of one variable. The functions of complexity two are the functions, whose complexity is not equal to zero or one, which have a representation ;$z(x, y) = C(A_1(x, y) + B_1(x, y))$, where A_1 and B_1 are of complexity not exceeding one. And so on. If some function z does not enter any of the classes $Cl_n = \{z : \ N(z) \leqslant n\}$, then we set $N(z) = \infty$. The condition $N(z) \leqslant 1$ is equivalent to the condition that the germ locally representing z, satisfies the differential relation

$$\delta_1(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0 \tag{1}$$

The differential polynomial $\delta_1(z)$ is the numerator of the rational-differential expression $(\ln(z'_y/z'_x))''_{xy}$.

The functions of analytic complexity one are of special interest from this point of view. These are the simplest functions of two-variables. In [2], the simple (i.e. of complexity one) solutions of some equations of mathematical physics (Laplace, wave, thermal conductivity, ...) were described explicitly. Here we give a description of algebraic functions of analytic complexity one.

Let there be a function $P(x, y)$ of analytic complexity one. This means that P has a local representation in the form

$$P(x, y) = c(a(x) + b(y)), \tag{2}$$

where (a, b, c) are nonconstant analytic functions of one variable. We pose the following question. What families of analytic functions (a, b, c) give an algebraic functions as the result? One can immediately suggest an obvious version: (a, b, c) are algebraic functions of one variable. Further, since the addition, using the logarithm and the exponential function, can be transformed to the multiplication, namely,

$$xy = e^{\ln(x)+\ln(y)}, ;$$

it follows that the first (additive) variant can be completed by another (multiplicative):

$$c(t) = \gamma(e^t), \quad a(x) = \ln(\alpha(x)), \quad b(y) = \ln(\beta(y)), \quad P(x,y) = \gamma(\alpha(x)\beta(y)),$$

where (α, β, γ) are algebraic functions. Is there a third possible variant? This question becomes especially sharp in connection with a recent result of Stepanova [3]. She proved that, if the same question is posed with respect to rational functions or polynomials, then the additive and multiplicative variants give the entire list of possibilities. Every polynomial of analytic complexity one has either an additive or a multiplicative representation composed of polynomials. The same holds for the rational functions.

If an algebraic function is not constant both with respect to x and to y, then in the domain in which an element $c(a(x) + b(y))$ of the composition is defined, one can find a point (x_0, y_0) such that $\phi(x) = P(x, y_0)$ and $\psi(y) = P(x_0, y)$ are not constant. Transfer the origin on the plane to the point (x_0, y_0); in this case, in our representation, a and b have elements representing them in a neighborhood of zero, and $a(0) = b(0) = 0$ for these elements. In this case, substituting the zeros into (1) instead of x and instead of y and taking into account that c is nonconstant, we obtain

$$c(a(x) = \phi(x), \quad a(x) = c^{-1}(\phi(x)),$$
$$c(b(y)) = \psi(y), \quad b(y) = c^{-1}(\psi(y))$$

Substitute this into (1) nd make the change

$$u = \phi(x), \qquad v = \psi(y).$$

We obtain

$$c(c^{-1}(u) + c^{-1}(v)) = P(\phi^{-1}(u), \psi^{-1}(v)) \tag{3}$$

The expression on the right-hand side is an algebraic function; denote it by $Q(u,v)$. Make the change

$$u = c(U), \qquad v = c(V)$$

in (3); we obtain

$$c(U + V) = Q(c(U), c(V)) \tag{4}$$

Relation (4) means precisely that the analytic function $c(t)$ *satisfies the algebraic addition theorem.* In other words, this means that, among $c(U)$, $c(V)$, and $c(U+V)$, there is an algebraic relation of the form

$$F(c(U), c(V), c(U+V)) = 0,$$

where F is a polynomial in three variables with complex coefficients. As an example of such a function, one can consider an arbitrary algebraic function. However, it is not hard to find nonalgebraic examples. For example, e^t, $sin(t)$, $cos(t)$. The following Weierstrass theorem [4, 5] provides a complete list of such functions.

Weierstrass theorem. *If a holomorphic element of an analytic function $c(t)$ satisfies relation (4) in some domain of the space (U,V) (i.e., the algebraic addition theorem), then this is possible in one and only one of the following three cases:*

(1) algebraic, $c(t) = \eta(t)$,

(2) periodic, $c(t) = \eta(e^{\lambda t})$,

(3) doubly periodic, $c(t) = \eta(\wp(t))$, where $\eta(s)$ is an arbitrary algebraic function and $\wp(t)$ is the Weierstrass $\wp$-function constructed from the lattice $L(\omega_1, \omega_2)$, where (ω_1, ω_2) are the generators of the lattice.

For the Weierstrass function itself, $w = \wp(U + V)$ is expressed using $u = \wp(U)$ and $v = \wp(V)$ by the function [4]

$$w = u \diamond v = \wp(\wp^{-1}(u) + \wp^{-1}(v)) = -(u + v) + \frac{(\sqrt{H(u)} - \sqrt{H(v)})^2}{4(u - v)}$$

where $H(t) = 4t^3 - g_2 t - g_3$ is a cubic polynomial without multiple roots in the Weierstrass form ($u \diamond v$ is the "elliptic addition"). It can readily be seen that a nonconstant algebraic function can occur in only one of the three classes.

Theorem 1. *If $P(x, y)$ is an algebraic function of two variables of analytic complexity one, then it has one and only one representation of the form*
(1)$P(x, y) = \gamma(\alpha(x) + \beta(y))$ (additive representation),
(2) $P(x, y) = \gamma(\alpha(x) \cdot \beta(y))$ (multiplicative representation), or
(3) $P(x, y) = \gamma(\alpha(x) \diamond \beta(y))$ (elliptic representation),
where (α, β, γ) are algebraic.

Proof. By (4), one can apply the Weierstrass theorem to $c(t)$. For the **first** case, c is algebraic, and thus so are a and b. We obtain

$$P(x, y) = \gamma(\alpha(x) + \beta(y)),$$

where (α, β, γ) are algebraic.

For the **second** case: $\tau = c(t) = \gamma(e^{\lambda t}), \quad t = c^{-1}(\tau) = 1/\lambda \ln(\gamma^{-1}(\tau)), \quad a(x) = 1/\lambda \ln(\gamma^{-1}(\phi(x))), \quad b(y) = 1/\lambda \ln(\gamma^{-1}(\psi(y)))$, and therefore

$$\begin{aligned} P(x, y) &= c(a(x) + b(y)) = \gamma(\exp \lambda(1/\lambda \ln(\gamma^{-1}(\phi(x))) + 1/\lambda \ln(\gamma^{-1}(\psi(y))))) \\ &= \gamma(\alpha(x)\beta(y)), \text{ where } \alpha(x) = \gamma^{-1}(\phi(x)), \quad \beta(y) = \gamma^{-1}(\psi(y)). \end{aligned} \tag{5}$$

For the **third** case: $\tau = c(t) = \gamma(\wp(t)), \quad t = c^{-1}(\tau) = \wp^{-1}(\gamma^{-1}(\tau)), \quad a(x) = \wp^{-1}(\gamma^{-1}(\phi(x))), \quad b(y) = \wp^{-1}(\gamma^{-1}(\psi(y)))$, and therefore

$$\begin{aligned} P(x, y) &= c(a(x) + b(y)) = \gamma(\wp(\wp^{-1}(\gamma^{-1}(\phi(x))) + \wp^{-1}(\gamma^{-1}(\psi(y)))) \\ &= \gamma(\wp(\wp^{-1}(\alpha(x)) + \wp^{-1}(\beta(y)) = \gamma(\wp(\wp^{-1}(\alpha(x)) \diamond \wp^{-1}(\beta(y)), \\ &\text{where } \alpha(x) = \gamma^{-1}(\phi(x)), \quad \beta(y) = \gamma^{-1}(\psi(y)). \end{aligned} \tag{6}$$

To complete the proof of the theorem, it remains to show that the classes of algebraic functions of complexity one (mentioned in the theorem) are disjoint. This follows from the uniqueness of the representation of an analytic function in the form (1) with nonconstant (a, b, c) (Lemma 2 and Proposition 3).

Lemma 2. *Let three nonconstant holomorphic functions $(a(t),\ b(t),\ c(t))$ satisfy a relation of the form*

$$c(a(x) + b(y)) = x + y$$

on an open set of the plane (x, y); then

$$a(x) = (x + p)/k,\ b(y) = (y + q)/k,\ c(t) = kt - (p + q)),$$

where $k \neq 0$.

Proof. Differentiate the identity with respect to x and to y. Since a and b are nonconstant, it follows that $c'' = 0$ and $c(t) = kt + l$, which implies in turn that a and b are also linear and the family (a, b, c) has the above form.

Lemma 2 implies an assertion concerning the uniqueness of the representation for the functions of the first class.

Proposition 3. *If*

$$c(a(x) + b(y)) = C(A(x) + B(y)),$$

then $A(x) = (a(x) + p)/k$, $B(y) = (b(y) + q)/k$, and $C(t) = c(kt - (p + q))$.

Proof. In the domain of a holomorphic element of this function, choose values $x = x_0$ and $y = y_0$ in such a way that a is invertible in a neighborhood of x_0, b is invertible in a neighborhood of y_0 and c is invertible in a neighborhood of $t_0 = a(x_0) + b(y_0)$. For the domain mentioned in

Lemma 2, we take a neighborhood of (x_0, y_0) and, for the functions, we take elements representing (a, b, c). Replacing x by $a^{-1}(x)$ and y by $b^{-1}(y)$, we obtain

$$c^{-1} \circ C(A \circ a^{-1}(x) + B \circ b^{-1}(y)) = x + y$$

We see from Lemma 2 that

$$A \circ a^{-1}(x) = (x + p)/k, \qquad B \circ b^{-1}(y) = (y + q)/k, \qquad c^{-1} \circ C(t) = kt - (p + q),$$

which implies our assertion.

The following pseudogroup (the gage semigroup) acts on the space of functions of two variables:

$$\mathcal{G} = \{z(x, y) \to \chi^{-1}(z(\varphi(x), \psi(y)))\},$$

where $(\varphi,\ \psi,\ \chi)$ are the germs of nonconstant analytic functions. It is clear that this action does not change the complexity and that the functions of complexity one form precisely the orbit of the function $(x + y)$. In other words, with respect to this action, all functions of complexity one are equivalent. If, to construct the action, one uses algebraic functions (φ, ψ, χ) only, then the number of equivalence classes with respect to this smaller pseudogroup $\mathcal{G}A$ is larger. Denote by Cl_1^{alg} the intersection of Cl_1 with the family of algebraic functions in two variables. It is clear that $\mathcal{G}A$ acts on Cl_1^{alg}. One can now reformulate Theorem 1 as follows.

Theorem 4. *With regard to the action of $\mathcal{G}A$, the family of algebraic functions of the first class, Cl_1^{alg}, is the family of orbits of*

$$x + y, \quad xy, \quad x \diamond y.$$

The functions $x + y$ and xy are specific functions in two variables, while the function $x \diamond y$, which is associated with the Weierstrass $\wp$-function, depends, together with $\wp$, on two free parameters, for example, on a pair of periods. Therefore, the following question is suitable. Let $x \diamond y$ correspond to $\wp(t, \omega_1, \omega_2)$, and $x \tilde{\diamond} y$ correspond to

$$\wp(t, \tilde{\omega_1}, \tilde{\omega_2}).$$

Under what relation between these couples of parameters, the functions $x \diamond y$ and $x \tilde{\diamond} y$ are equivalent up to a transformation in $\mathcal{G}A$? Here is the answer.

Proposition 5. *The functions $x \tilde{\diamond} y$ and $x \diamond y$ are equivalent, i.e., belong to the same orbit of $\mathcal{G}A$, if and only if he lattices of periods coincide up to multiplication by a nonzero complex constant, i.e.,*

$$\tilde{\omega}_1 = k\omega_1, \qquad \tilde{\omega}_2 = k\omega_2.$$

Here $\wp_2(t) = \wp_1(t/k)$.

Proof. Let $x \tilde{\diamond} y$ and $x \diamond y$ be equivalent, i.e.,

$$x \tilde{\diamond} y = c(a(x) \diamond b(y)),$$

where (a, b, c) are nonconstant algebraic functions. Let

$$\wp_1(t) = \wp(t, \omega_1, \omega_2), \qquad \wp_2(t) = \wp(t, \tilde{\omega_1}, \tilde{\omega_2}).$$

We have

$$c(\wp_1(\wp_1^{-1}(a(x)) + \wp_1^{-1}(b(y))) = \wp_2(\wp_2^{-1}(x) + \wp_2^{-1}(y))$$

Then Proposition 3 enables us to conclude that

$$\wp_1^{-1}(a(x) = \frac{\wp_2^{-1}(x) + p}{k}, \quad \wp_1^{-1}(b(y) = \frac{\wp_2^{-1}(y) + q}{k}, \quad c(\wp_1(t)) = \wp_2(kt - (p + q)),$$

whence, further,

$$a(x) = \wp_1(\frac{\wp_2^{-1}(x) + p}{k}), \qquad b(y) = \wp_1(\frac{\wp_2^{-1}(y) + q}{k}).$$

The elliptic integral $\wp_2^{-1}(t)$ is an infinitely-valued analytic function whose germs at a chosen point differ by constants of the form

$$\frac{m_1\tilde{\omega}_1 + m_2\tilde{\omega}_2}{k},$$

and therefore the functions a and b can be finitely valued only for the case in which the image of the second lattice

$$\frac{\tilde{L}}{k} = L(\frac{\tilde{\omega}_1}{k}, \frac{\tilde{\omega}_2}{k})$$

is contained in the first lattice, $L = (\omega_1, \omega_2)$. Since the action is invertible, we obtain the converse inclusion, that is, $\tilde{L} = kL$, and thus $\wp_2(t) = \wp_1(t/k)$. This completes the proof of the proposition.

Thus, all algebraic functions of analytic complexity one form the family of orbits of three binary operations (elliptic addition, multiplication, and the usual addition) under the action of the algebraic gauge group $\mathcal{G}A$. Here the elliptic addition is the base operation; the multiplication occurs in the passage to the limit as $\omega_2 \to \infty$, and the addition occurs, in turn, in the passage to the limit as $\omega_1 \to \infty, \omega_2 \to \infty$.

The possibility of factorization by the complex multiplication enables one to define "elliptic" orbits by a single parameter. By setting $k = 1/\omega_1$, we transform the lattice $L(\omega_1, \omega_2)$ to the lattice $L(1, \tau)$. The change $\tau \to -\tau$ does not modify the lattice, and we can therefore assume that $\mathrm{Im}\tau > 0$. The remaining discrete degrees of freedom are described by the action of the modular group [6]. After passing to the quotient of the upper half-plane by the action of the modular group, we obtain a one-dimensional complex manifold with singularities (an orbifold) homeomorphic to the two-dimensional real plane. The points of this orbifold are, on one hand, in a one-to-one correspondence with the elliptic orbits of $\mathcal{G}A$ and, on the other hand, with the classes of affinely equivalent lattices, i.e., with the complex structures on the two-dimensional torus. The quotients with respect to the degenerate lattices, the cylinder and plane, also, on one hand, correspond to unique operations, to the multiplication and addition, and, on the other hand, admit unique complex structures.

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REFERENCES

1. V. K. Beloshapka, "Analytic Complexity of Functions of Two Variables," Russ. J. Math. Phys. **14** (3), 243–249 (2007).
2. V. K. Beloshapka, "A seven-dimensional family of simple harmonic functions," Mat. Zametki **98** (6), 803–808 (2015) [Math. Notes **98** (6), 867–871 (2015)].
3. M. Stepanova, "On Rational Functions of First-Class Complexity," Russ. J. Math. Phys. **23** (2), 251–256 (2016).
4. V. V. Prasolov and Yu. P. Solov'ev, *Elliptic Functions and Algebraic Equations* (Faktorial, Moscow, 1997; V. Prasolov and Yu. Solovyev, *Elliptic Functions and Elliptic Integrals*, AMS, Providence, RI, 1997)..
5. M. B. Villarino, *Algebraic addition theorems* arXiv:12126471v2[math.CA] 7Jan2013.
6. A. Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* (Springer-Verlag, Berlin–New York, 1964; Nauka, Moscow, 1968).