ISSN 0001-4346, Mathematical Notes, 2015, Vol. 98, No. 6, pp. 867–871. © Pleiades Publishing, Ltd., 2015.
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A Seven-Dimensional Family of Simple Harmonic Functions

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Received June 25, 2015

Abstract—From the point of view of analytic complexity theory, all harmonic functions of two variables split into three classes: functions of complexity zero, one, and two. Only linear functions of one variable have complexity zero. This paper contains a complete description of simple harmonic functions, i.e., of functions of analytic complexity one. These functions constitute a seven-dimensional family expressible as integrals of elliptic functions. All other harmonic functions have complexity two and are, in this sense, of higher complexity. Solutions of the wave equation, the heat equation, and the Hopf equation are also studied.

DOI: 10.1134/S0001434615110188

Keywords: *analytical complexity, harmonic function, elliptic function.*

Almost each mathematical science possesses its own approach to the problem of measuring complexity of objects studied in it. Classical complexity theory is the theory of complexity of algorithms with its variations [1], [2]. Mathematical analysis is no exception [3], [4]. A method for measuring the complexity of analytic functions of two variables was proposed in [5]; this method was later applied in [6] and [7]. For any analytic function $z(x, y)$, its analytic complexity $N(z)$ was defined. This quantity can take the values $(0, 1, 2, \dots, \infty)$. Functions depending only on one variable, such as $a(x)$ or $b(y)$, have complexity zero. Functions depending on two variables expressible as $z(x, y) = c(a(x) + b(y))$, where (a, b, c) are analytic functions of one variable, have complexity one. It is easy to verify that the condition $N(z) = 1$ is equivalent to the fact that the germ locally expressing z satisfies the differential relation

$$\delta_1(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0. \qquad (1)$$

The differential polynomial $\delta_1(z)$ is the numerator of the differential rational expression $(ln(z'_y/z'_x))''_{xy}$. Functions of complexity two are functions whose complexity is neither zero nor one and which can be expressed in the form $z(x, y) = C(A_1(x, y) + B_1(x, y))$, where A_1 and B_1 have complexity at most one. And so forth. But if a function z does not belong to any one of the classes $\mathrm{Cl}_n = \{z : N(z) \le n\}$, then we set $N(z) = \infty$.

From this point of view, let us consider the class of real-valued (somewhere harmonic, possibly multivalued) functions of two variables, i.e., the class of the local solutions of the Laplace equation

$$\Delta z = z''_{xx} + z''_{yy} = 0.$$

Harmonic functions of complexity zero are linear functions of one variable, i.e., functions of the form $z = kx + b$ or $z = ky + b$. On the other hand, any harmonic function can be expressed locally as $z(x, y) = f(x + iy) + \overline{f}(x - iy)$ (recall that complex-valued functions can be used in the construction of complexity classes). Hence it immediately follows that its complexity $N(z)$ is at most two. Using this criterion, we can see that a generic harmonic function has complexity two. For example, the complexity of the real and imaginary parts of all power functions $z = (x + iy)^n$ for all $n \ge 3$ is two. Thus, all harmonic functions are divided into two groups: *simple* with complexity at most one and *complex* with complexity two. It is well known that the linear space of harmonic functions on the fixed disk is infinite-dimensional. In the present paper, it will be shown that the family of simple harmonic functions is seven-dimensional, i.e., it depends on seven independent parameters.

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Here are six simple functions whose complexity is one:

$$x^2 - y^2, \quad 2xy, \qquad e^x \cos(y), \quad e^x \sin(y), \qquad \frac{1}{2}\ln(x^2+y^2), \quad \arctan\left(\frac{y}{x}\right). \tag{2}$$

On the basis of each of these functions, we can construct a parametric family. For example, here are two four-parametric solutions:

$$z = c_1((x+c_2)^2 - (y+c_3)^2) + c_4, \qquad z = c_1 e^{c_2 x}\cos(c_2 y + c_3) + c_4. \tag{3}$$

Thus, let $z(x,y) = c(a(x)+b(y))$ be a simple harmonic function. For the derivatives of the functions a, b, c, we shall use the following simplified notation:

$$a'(x) = a_1, \quad a''(x) = a_2, \quad \dots, \qquad b'(y) = b_1, \quad b''(y) = b_2, \quad \dots,$$
$$c'(a(x)+b(y)) = c_1, \quad c''(a(x)+b(y)) = c_2, \quad \dots .$$

The condition for z to be harmonic yields

$$c_2 {a_1}^2 + c_1 a_2 + c_2 {b_1}^2 + c_1 b_2 = 0. \tag{4}$$

Differentiating this relation with respect to x and y, we obtain

$$c_3 {a_1}^3 + 3c_2 a_1 a_2 + c_1 a_3 + c_3 a_1 {b_1}^2 + c_2 a_1 b_2 = 0,$$
$$c_3 b_1 {a_1}^2 + c_2 b_1 a_2 + c_3 {b_1}^3 + 3c_2 b_1 b_2 + c_1 b_3 = 0.$$

The three relations given above are linear in the derivatives (c_1, c_2, c_3). If there exists a solution with a nonconstant function $c(t)$, then the determinant of this homogenous system must be zero; hence we have

$$({a_1}^2 + {b_1}^2)\big(b_3 {a_1}^3 - a_3 b_1 {a_1}^2 + 2a_1 {a_2}^2 b_1 + a_1 b_3 {b_1}^2 - 2a_1 {b_2}^2 b_1 - a_3 {b_1}^3\big) = 0.$$

If a and b are not constant, then $({a_1}^2 + {b_1}^2)$ is a nonzero multiplier, and we obtain the following relation for a and b:

$$b_3 {a_1}^3 - a_3 b_1 {a_1}^2 + 2a_1 {a_2}^2 b_1 + a_1 b_3 {b_1}^2 - 2a_1 {b_2}^2 b_1 - a_3 {b_1}^3 = 0. \tag{5}$$

This relation can be obtained in another way. If b is not constant, then, locally, we can pass from the variables (x,y) to the variables $(x, t = a(x)+b(y))$. Here the variable y becomes the function $y = y(x,t)$; furthermore,

$$\frac{\partial}{\partial x} y(x,t) = -\frac{da(x)/dx}{db(y)/dy}.$$

Using relation (4) in these variables, we can express $(ln(c'(t)))' = c''(t)/c'(t)$ in terms of a and b. And then Eq. (5) obtained above is a condition for the right-hand side of the expression for the logarithmic derivative to be independent of x.

Statement 1. (1) *The nonconstant functions a and b satisfy relation* (5) *if and only if there exists a nonconstant function $c(t)$ such that $z = c(a(x)+b(y))$ is harmonic, i.e., z is a simple harmonic function.*

(2) *If the solution of* (5) *is fixed, then the function $c(t)$ is uniquely defined up to real linear replacement $c(t) \to k_1(c(t) - k_2)$.*

Expressing b_3 from (5), we obtain

$$b_3 = \frac{b_1(2a_1{b_2}^2 + a_3{a_1}^2 - 2a_1{a_2}^2 + a_3{b_1}^2)}{a_1({a_1}^2 + {b_1}^2)}.$$

Differentiating this relation with respect to x, we can write

$$-4a_2{b_2}^2{a_1}^3 + a_4{a_1}^5 + 2a_4{a_1}^3{b_1}^2 - 5a_3{a_1}^4a_2$$
$$- 6a_3{a_1}^2a_2{b_1}^2 + 4{a_2}^3{a_1}^3 + a_1a_4{b_1}^4 - a_2a_3{b_1}^4 = 0.$$

Expressing b_2 from the above relation, we obtain

$$b_2 = \frac{\sqrt{a_1 a_2\left(-6a_3{a_1}^2 a_2{b_1}^2 + a_4{a_1}^5 + 2a_4{a_1}^3{b_1}^2 - 5a_3{a_1}^4 a_2 + 4{a_2}^3{a_1}^3 + a_1 a_4{b_1}^4 - a_2 a_3{b_1}^4\right)}}{2{a_1}^2 a_2}. \tag{6}$$

Differentiating this relation with respect to x, we can write

$${a_1}^2 a_2 a_5 - {a_1}^2 a_3 a_4 - 3{a_2}^2 a_4 a_1 + 3{a_2}^3 a_3 = 0. \tag{7}$$

It can be verified that if relations (7) and (6) hold, then the condition for the expressions for b_2 and b_3 to be consistent, which is the equality between the derivative of b_2 and b_3, holds automatically.

Thus, we have obtained the following algorithm for constructing simple harmonic functions.

Statement 2. *Any simple harmonic function $z = c(a(x) + b(y))$ can be constructed as follows*:

- *the function $a(x)$ is an arbitrary solution of the fifth-order equation* (7);
- *the function $b(y)$ is an arbitrary solution of the second-order equation* (6) *with $a(x)$ obtained above*;
- *the function $c(t)$ is an arbitrary solution of the second-order equation* (4) *with a and b constructed above.*

Obviously, in Statement 2, we can interchange a and b, i.e., b will satisfy Eq. (7), while the function a can be obtained from the fixed solution b.

Let us estimate the number of parameters on which the family of simple harmonic functions depends. The choice of $a(x)$ depends on five parameters, one of which is additive. The choice of $b(y)$ depends on two parameters, one of which is additive as well. These two additive parameters are added together with the parameter k_2 of arbitrary value from Statement 1. Thus,

$$(5-1) + (2-1) + 2 = 7.$$

We can decrease the order of Eq. (5) twice. First, in passing to $A(x) = a'(x)$, $B(y) = b'(y)$ and, second, in introducing

$$P(A) = A'(x) = a''(x), \qquad Q(B) = B'(y) = b''(y)$$

for the unknown functions. Here the following relation holds:

$$A''(x) = P'(A)P(A), \qquad B''(y) = Q'(B)Q(B).$$

We obtain

$$P'(A)P(A)(A^2B + B^3) - Q'(B)Q(B)(B^2A + A^3) + 2AB((Q(B))^2 - (P(A))^2) = 0. \tag{8}$$

If we pass to $p(A) = (P(A))^2$, $q(B) = (Q(B))^2$, then the equation becomes linear:

$$(p'(A)B - q'(B)A)(A^2 + B^2) + 4AB(q(B) - p(A)) = 0. \tag{9}$$

Let us express $q'(B)$ from Eq. 9 and differentiate the result with respect to A, obtaining

$$\begin{aligned} &p''(A)A^5 + 2p''(A)A^3B^2 - 5p'(A)A^4 - 6p'(A)A^2B^2 \\ &\quad + Ap''(A)B^4 - 8q(B)A^3 + 8p(A)A^3 - p(A)B^4 = 0. \end{aligned}$$

This yields

$$\begin{aligned} q(B) = {}& \frac{1}{8}A^2p''(A) + \frac{1}{4}p''(A)B^2 - \frac{5}{8}Ap'(A) - \frac{3}{4}p'(A)B^2A \\ &+ \frac{1}{8}p''(A)B^4A^2 + p(A) - \frac{1}{8}p'(A)B^4A^3. \end{aligned} \tag{10}$$

Differentiating with respect to A, we obtain

$$p'''(A)A^3 - 3p''(A)A^2 + 3p'(A)A = 0. \tag{11}$$

Let us search for solutions of the form $p = A^m$; we have the equation

$$m(m-1)(m-2) + 3m(m-1) + 3m = 0$$

The solutions are $m = 0, 2, 4$; therefore, the general solution of Eq. (11) is of the form

$$p(A) = C_1 + C_2A^2 + C_3A^4 \tag{12}$$

From (10), we then obtain

$$q(B) = C_1 - C_2B^2 + C_3B^4. \tag{13}$$

Substituting (12)) and (13)) into (9), we see that the resulting $(p(A), q(B))$ is the general solution of (9)). Returning to the previous notation, we obtain an equation with separating variables,

$$a''(x) = P(A) = \sqrt{C_1 + C_2(a'(x))^2 + C_3(a'(x))^4}\,.$$

Denote

$$\phi(A) = \int \frac{dA}{\sqrt{C_1 + C_2A^2 + C_3A^4}}\,.$$

This function can be expressed via the elliptic integral of the first kind

$$F(x,k) = \int_0^x \frac{d\zeta}{\sqrt{C_1 + C_2\zeta^2 + C_3\zeta^4}}\,;$$

namely, if

$$C_1 + C_2A^2 + C_3A^4 = \frac{-1}{\lambda^2}(A^2 + \mu^2)(A^2 + \nu^2),$$

then

$$\phi(A) = \frac{1}{\nu}F\left(\frac{iA}{\nu}, \frac{\nu}{\mu}\right).$$

Thus, under the condition $\mu^2 \neq \nu^2$, the derivative $a'(x)$ is an elliptic function, $a'(x) = \phi^{-1}(x + C_4)$, and the function

$$a(x) = \int \phi^{-1}(x + C_4)\,dx + \widetilde{C}_1$$

is its antiderivative. But if $\mu^2 = \nu^2$, then $\phi(A)$ is an elementary function and can be expressed via the logarithm. Further, if

$$\psi(B) = \int \frac{dB}{\sqrt{C_1 - C_2B^2 + C_3B^4}} = \frac{1}{i\nu}F\left(\frac{B}{\nu}, \frac{\nu}{\mu}\right),$$

then

$$b'(y) = \psi^{-1}(y + C_5), \qquad b(y) = \int \psi^{-1}(y + C_5)\,dy + \widetilde{C}_2$$

where b' is elliptic under the condition $\mu^2 \neq \nu^2$. Further, as described above (see Statement 1), we recover the function $c(t)$, which is uniquely defined up to the choice of (k_1, k_2). Here the sum $\widetilde{C}_1 + \widetilde{C}_2$ is absorbed by the constant k_2.

Recall that a function is said to be *expressed by quadratures* if it is presented as the composition of elementary functions, their antiderivatives, and their inverses. Above, we have proved the following statement.

Statement 3. (a) *The functions* $(a(x), b(y), c(t))$ *appearing in the representation of simple harmonic functions can be expressed by quadratures.*

(b) *The functions* $a'(x), b'(y)$ *are elementary or elliptic.*

(c) *The general solution depends on the family of seven constants* $(C_1, C_2, C_3, C_4, C_5, k_1, k_2)$.

In conclusion, note that it is easy to adapt our construction to the analysis of the complexity of the analytic solutions of other equations, such as those of the *wave equation* $z''_{xx} - z''_{yy} = 0$. The general solution is of the form

$$z = f(x+y) + g(x-y) \qquad \text{(d'Alembert's formula)};$$

therefore, the complexity of the analytic solutions of the wave equation is also at most two and this leads to the problem of describing simple solutions, i.e., solutions of complexity one. Our approach can be used to solve this problem. The answer is similar. The simple functions form a seven-dimensional family expressible as integrals of elliptic and elementary functions.

But if we turn to the solutions of the *heat equation* $z'_y = z''_{xx}$, then the situation is different. We have no upper bound for the complexity of its analytic solutions. But we can study the problem of describing simple solutions $z = c(a(x) + b(y))$, i.e., those of complexity one. Applying the procedure described above and eliminating the functions $c(t)$ and $b(y)$, we obtain the following equation for $a(x)$:

$$2a_2^2a_3^2a_1 - 2a_2^4a_3 - 5a_1^2a_3^3 + 5a_1^2a_2a_4a_3 + 2a_1a_2^3a_4 + a_1^3a_5a_3 - 2a_1^2a_5a_2^2 - a_1^3a_4^2 = 0, \qquad (14)$$

which, after decreasing the order twice, ($a'(x) = A$, $a''(x) = P(A)$) takes the form

$$\begin{aligned} &4P(A)P'(A)^2A - 2P(A)^2P'(A) - 2A^2P'(A)^3 - 3P(A)A^2P'(A)P''(A) \\ &\quad + 2P(A)^2AP''(A) + P(A)A^3P'(A)P'''(A) + 2A^3P'(A)^2P''(A) \\ &\quad - 2P(A)^2A^2P'''(A) - P(A)A^3P''(A)^2 = 0. \end{aligned}$$

This equation has a solution by quadratures, just as the equations for b and c. As a result, we also obtain a seven-dimensional family of simple solutions to the heat equation.

We are unaware of any estimate of the complexity of the analytic solutions of the Hopf equation $z'_y = zz'_x$. However, the simple solutions $z = c(a(x) + b(y))$ are of the form

$$z = \frac{\alpha - mx}{\beta + my},$$

where a and b are expressed via the logarithm and c via the exponential.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research under grants 14-01-00709-a and 13-01-12417-ofi-m2.

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