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Model-Surface Method: An Infinite-Dimensional Version

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Abstract—The model-surface method is applied to the study of real analytic submanifolds of a complex Hilbert space. Generally, the results are analogous to those in the finite-dimensional case; however, there are some peculiarities and specific difficulties. One of these peculiarities is the existence of a model surface with the Levi–Tanaka algebra of infinite length.

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1. INTRODUCTION

In studying holomorphic maps of real submanifolds in complex spaces, just as in many other fields of geometry, the main problems are grouped into three directions. These are classification, invariants, and automorphisms. Among the approaches to these problems is a *model-surface method*. The structural features of the relevant theory have finally crystallized in recent years.

Let us list the main features:

• The approach is *local*; i.e., local analysis is primary to any global conclusions. The main object of study is a germ of a manifold M at a point, M_ξ, rather than the manifold itself.

• The approach is a *coordinate* one. It is assumed from the beginning that the surface representing a germ is embedded in a complex space, which can be identified with a standard linear finite-dimensional complex space $\mathbb{C}^N$ since the analysis is local. Then one can assume that the germ is defined by its local equation. The model surface itself is constructed as a normalized lowest-order term of the equation of the germ.

• The local technique is based on *calculations with power series*, either convergent or formal.

• The primary object in the analysis of automorphisms is the Lie algebra of germs of holomorphic vector fields tangent to a germ of a manifold, i.e., the *Lie algebra* of infinitesimal automorphisms of a germ. Conclusions concerning the family of holomorphic automorphisms of the germ that are generated by these fields are made later on the basis of information on the structure of the algebra.

This general view on the theory allows one to conclude that the specific features of the theory are the features of differential calculus adapted to the specific character of the problem; this differential calculus is definitely based on the coordinate analysis "in the small." Being aimed at solving geometric problems (holomorphic geometry of real submanifolds in a complex space), this theory represents an analytic alternative to the modern coordinate-free geometric approaches.

Let us briefly sketch the method:

• Each germ M_ξ is assigned its model surface $Q(M_\xi)$. The model surface is a holomorphically homogeneous real algebraic surface.

• One formulates and proves a simple criterion saying that $\operatorname{aut} Q$, the Lie algebra of infinitesimal automorphisms of the model surface Q, is finite-dimensional if and only if the surface is completely nondegenerate.

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• One describes the algebra of infinitesimal automorphisms of the model surface Q. The vector fields turn out to be polynomial, and the transformation group $\operatorname{Aut} Q$ corresponding to this Lie algebra turns out to be a Lie group acting in the whole space by birational transformations of bounded degree.

• One proves that a holomorphic equivalence of germs induces a linear equivalence of their model surfaces. In particular, the problem of classification of model surfaces is immersed in the algebraic context of invariant theory.

• One proves an embedding theorem; i.e., one constructs a canonical faithful representation of the stabilizer of a point as the Lie subalgebra of automorphisms $\operatorname{aut}_\xi M_\xi$ of a germ in the stabilizer of its model surface $\operatorname{aut}_0 Q(M_\xi)$.

Now, let us see how this theory is transformed under the substitution of some other space for the main structural element, the space $\mathbb{C}^N$. In this paper, we take a complex separable Hilbert space $\mathcal{H}$ as the basic space.

The main object of our study is a germ M_ξ of a real analytic Hilbert submanifold M of a complex Hilbert space at a point ξ. We begin with linear objects. In what follows, we assume that all subspaces under consideration are closed and linear, and polylinear forms are continuous; the case of finite-dimensional Hilbert spaces is not excluded.

Let H be a real Hilbert space and $\mathcal{H}$ be its complexification. There is an operator of complex structure in $\mathcal{H}$, the operator of multiplication by the imaginary unit, $v \to iv$. Therefore, we can write $\mathcal{H} = H \oplus iH$. Let $\mathcal{H}\mathbb{R}$ be the realification of $\mathcal{H}$; i.e., this is the same $\mathcal{H}$ but over the field of real constants. It can be identified with $H \oplus H$. Let L be a subspace in $\mathcal{H}\mathbb{R}$. The subspace L is said to be *generating* if $L + iL = \mathcal{H}\mathbb{R}$, i.e., if the closure of the linear span of L and iL coincides with $\mathcal{H}\mathbb{R}$. Denote $L \cap iL$ by L_{c}. It is clear that L_{c} is a complex subspace in $\mathcal{H}$. Let N be some direct complement of L_{c} to L. Then the fact that L is a generating subspace implies that the direct decomposition $\mathcal{H}\mathbb{R} = (L_{\mathrm{c}})\mathbb{R} \oplus N \oplus iN$ is valid, as well as $\mathcal{H} = L \oplus N\mathbb{C}$, where $N\mathbb{C} = N \oplus iN$. The multiplication by i establishes an isomorphism between the complement of L_{c} to L and the complement of L to $\mathcal{H}\mathbb{R}$. By the type of a generating subspace L we will mean a pair (n, K), where n is the dimension of L_{c} as a complex subspace and K is the codimension of L in $\mathcal{H}\mathbb{R}$. Each of the parameters may take both finite and infinite values.

Let f be a map of a domain D of a complex Hilbert space $\mathcal{H}_1$ into another complex Hilbert space $\mathcal{H}_2$. We say that f is *holomorphic* if, in some closed circular neighborhood of every point a, the map can be represented as the sum of a series,

$$f(z) = f_0 + f_1(z-a) + f_2(z-a, z-a) + \ldots + f_m(z-a, \ldots, z-a) + \ldots,$$

where f_m is a continuous symmetric $\mathcal{H}_2$-valued polylinear form of m vector variables in $\mathcal{H}_1$, and the series converges in the norm of the space $\mathcal{H}_2$. By definition, a neighborhood is circular if, together with any of its points z, it contains all points of the form $a + t(z-a)$ for all complex numbers t with $|t| \leq 1$. This definition is equivalent to the following: at every point of D, the map f has a total differential that is a continuous complex linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$. If we replace the spaces and series in the first definition by real spaces and series, then such a map is said to be *real analytic*. A map to the one-dimensional space $\mathbb{C}^1$ is called a *holomorphic function* [5].

The series involved in the definition of a holomorphic or real analytic map is a series in homogeneous components, with the homogeneity understood as follows: if $t \in \mathbb{C}$ and we make the transformation $z \to tz$, then f_m is multiplied by t^m. Suppose that the space on which the forms are defined is represented as a direct sum that generates a representation $z = z_1 + \ldots + z_s$ of the variable. Then each term can be assigned a certain weight $[z_j] = \mu_j$, and the homogeneity of f_m can be understood as follows: this component changes by a factor of t^m after the transformation $z_1 \to t^{\mu_1} z_1, \ldots, z_s \to t^{\mu_s} z_s$. In this case we speak of *weighted components*.

Let M_ξ be a germ of an embedded real analytic submanifold of the space $\mathcal{H}\mathbb{R}$. Let L be the tangent space to the germ at ξ. We will assume that L is a generating subspace of type (n,K); in this case, we say that M_ξ is a *generating germ of type* (n,K). The whole complex space is decomposed into a direct sum $\mathcal{H} = L \oplus N\mathbb{C}$, and, in accordance with this decomposition, every element of the space is represented as $z+w$. Let us change the notation: $L_c = H_z$ and $N\mathbb{C} = H_w$; now, $z \in H_z$, $w = u+iv \in H_w$, $u \in N$, and $v \in N$. We will assume that ξ is the origin (this can be achieved by a shift). Then, in a neighborhood of the origin of the space $\mathcal{H} = H_z \oplus H_w$, the germ can be defined by an equation

$$v = F(z,\bar{z},u),$$

where F is a real analytic map of a neighborhood of the origin of the space $(H_z)\mathbb{R} \oplus (H_z)\mathbb{R} \oplus N$ to the space N, with both F and dF vanishing at the origin. This is a direct corollary to the implicit map theorem. Here one can use either real variables (x,y), with $z = x+iy$, or formal variables $(z,\bar{z})$, bearing in mind that F is a real map, i.e., $\overline{F(z,\bar{z},u)} = F(\bar{z},z,u)$.

2. HYPERSURFACES, $K=1$

First, consider the case of codimension 1. So, $n \le \infty$ and $K=1$; i.e., we deal with a germ of type $(\infty,1)$ of a real analytic hypersurface Γ in a complex Hilbert space. The space is the direct sum $\mathcal{H} = H_z \oplus \mathbb{C}$, and by its coordinates we will mean a pair (z,w), where z is a vector of the space H_z and $w = u+iv$ is a complex number. The equation of a germ has the form $v = F(z,\bar{z},u)$, where F is a real-valued function that is real analytic in a neighborhood of the origin and is such that F and dF vanish at the origin. Let us assign weights to the variables as follows: $[z]=1$ and $[w]=[u]=2$. Then the equation of the germ can be written as

$$v = F_{2,0,0}(z,z) + F_{1,1,0}(z,\bar{z}) + F_{0,2,0}(\bar{z},\bar{z}) + O(3), \tag{1}$$

where the terms are polylinear forms of their variables and $O(m)$ stands for the sum of forms of weight m and higher. Since F is real, the form $F_{1,1,0}(z,\bar{z})$ is Hermitian; denote this form by $\langle z,\bar{z}\rangle$ and note that $F_{2,0,0}(z,z) + F_{0,2,0}(\bar{z},\bar{z}) = 2\operatorname{Re} F_{2,0,0}(z,z)$. Then, after the quadratic triangular change of coordinates $z \to z$, $w \to w + iF_{2,0,0}(z,z)$, the equation reduces to

$$v = \langle z,\bar{z}\rangle + F_3(z,\bar{z},u) + F_4(z,\bar{z},u) + \dots. \tag{2}$$

It is the quadratic hypersurface (quadric) $Q = \{v = \langle z,\bar{z}\rangle\}$ that is a model surface of the type under consideration, and the form $\langle z,\bar{z}\rangle$ is called a *Levi form of the hypersurface* Γ *at the origin.* Such a form is said to be *nondegenerate* if it has no kernel, i.e., if the relation $\langle z,\bar{a}\rangle = 0$ holds for all z only when $a=0$. In this case, the germ of the hypersurface is also said to be *nondegenerate.*

Real vector fields on the space $\mathcal{H}$ can be expressed in the following symbolic coordinate form:

$$2\operatorname{Re}\left(f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}\right),$$

where the coefficients f and g take values in H_z and $\mathbb{C}$, respectively. Every such field generates a local one-parameter group of transformations of the space $\mathcal{H}$; moreover, if f and g are holomorphic, then the field generates a one-parameter group of holomorphic transformations of the space [6]. Such vector fields are said to be holomorphic. A real field is a sum of a holomorphic and an antiholomorphic term, each of which allows one to obtain the other by conjugation and thus to reconstruct the entire field. Therefore, when expressing fields, we will write for short only the holomorphic component. One can represent holomorphic fields as sums of terms of the form

$$f_\alpha(z,\dots,z)w^\beta\frac{\partial}{\partial z} \qquad \text{and} \qquad g_\gamma(z,\dots,z)w^\delta\frac{\partial}{\partial w},$$

where f is an α-form and g is a γ-form. The differentiations can be assigned weights in the following way:

$$\left[\frac{\partial}{\partial z}\right]=-1 \qquad \text{and} \qquad \left[\frac{\partial}{\partial w}\right]=-2.$$

Let us extend the weighted decomposition from series to vector fields; as a result, the Lie algebra of vector fields turns into a graded Lie algebra decomposed as $g=g_{-2}+g_{-1}+g_0+g_1+\dots$. Accordingly, every vector field can be decomposed into graded components: $X=X_{-2}+X_{-1}+X_0+X_1+\dots$.

Let us formulate five propositions.

(a) If a field $X=X_{-2}+X_{-1}+X_0+X_1+\dots$ belongs to $\operatorname{aut}Q$, then each of its components belongs to this algebra, $X_j\in\operatorname{aut}Q$.

(b) If the form $\langle z,\bar z\rangle$ is degenerate, then the decomposition $g_{-2}+g_{-1}+g_0+g_1+\dots$ of the algebra $\operatorname{aut}Q$ contains nonzero components of all weights. If the form $\langle z,\bar z\rangle$ is nondegenerate, then its decomposition is given by $\operatorname{aut}Q=g_{-2}+g_{-1}+g_0+g_1+g_2$.

(c) If the form $\langle z,\bar z\rangle$ is nondegenerate, then the components can be described explicitly:

$$\begin{aligned}
g_{-2}&=\left\{2\operatorname{Re}\left(q\frac{\partial}{\partial w}\right)\right\},\\
g_{-1}&=\left\{2\operatorname{Re}\left(p\frac{\partial}{\partial z}+2i\langle z,\bar p\rangle\frac{\partial}{\partial w}\right)\right\},\\
g_0&=\left\{2\operatorname{Re}\left(Cz\frac{\partial}{\partial z}+\rho w\frac{\partial}{\partial w}\right)\right\},\\
g_1&=\left\{2\operatorname{Re}\left(\big(aw+2i\langle z,\bar a\rangle z\big)\frac{\partial}{\partial z}+2i\langle z,\bar a\rangle w\frac{\partial}{\partial w}\right)\right\},\\
g_2&=\left\{2\operatorname{Re}\left(rwz\frac{\partial}{\partial z}+rw^2\frac{\partial}{\partial w}\right)\right\},
\end{aligned}$$

where $q,r\in\mathbb{R}$, $p,a\in H_z$, C is a continuous linear operator on H_z, and ρ is a real number related to C by the formula $2\operatorname{Re}\langle Cz,\bar z\rangle=\rho\langle z,\bar z\rangle$.

The subgroup

$$\operatorname{Aut}_- Q=\big\{z\to z+p,\ w\to w+2i\langle z,\bar p\rangle+(q+i\langle p,\bar p\rangle)\big\}$$

corresponding to the subalgebra $g_-=g_{-2}+g_{-1}$ acts transitively on Q by affine transformations of the space. In the finite-dimensional case, this subgroup is known as the Heisenberg group.

For the subalgebra $\operatorname{aut}_0 Q=g_0+g_1+g_2$, the corresponding subgroup consists of the automorphisms of Q that leave the origin fixed (the stabilizer of the origin). The subalgebra g_0 generates a subgroup in the stabilizer that consists of linear automorphisms and is a connected component of the group $\operatorname{AutL}Q$ of all linear automorphisms of Q preserving the origin. This group can be described as $z\to Cz$, $w\to\rho w$, where C is a continuous invertible linear operator on the space H_z, i.e., $C\in\mathrm{GL}(H_z)$, satisfying the relation $\langle Cz,\overline{Cz}\rangle=\rho\langle z,\bar z\rangle$ for some nonzero real factor ρ. The component g_0 is the Lie algebra of this linear group.

The subgroup $\operatorname{Aut}_+ Q$ corresponding to the subalgebra $g_+=g_1+g_2$ is composed of nonlinear automorphisms of Q that leave the origin fixed. The group $\operatorname{Aut}_+ Q$ consists of linear fractional transformations of $\mathcal{H}$ of the form

$$(z,w)\to\frac{(z+aw,w)}{1-\big(2i\langle z,\bar a\rangle+(r+i\langle a,\bar a\rangle)w\big)}.$$

Here we should make the following general remark. In the finite-dimensional situation, i.e., when the basic space is finite-dimensional, every graded component is also finite-dimensional. Therefore, the algebra is finite-dimensional if and only if the decomposition $\operatorname{aut} Q = \sum g_j$ contains only a finite number of nonzero components (finite grading). This assertion can be reformulated as follows: the elements of $\operatorname{aut} Q$ are fields with polynomial coefficients of bounded degree. In this case, the decomposition looks like $\operatorname{aut} Q = g_{-2} + g_{-1} + g_0 + g_1 + \ldots + g_d$, where d is the number of the leading nonzero component. When we pass from a finite-dimensional space to an infinite-dimensional one, the parameters remain formally the same but are not finite-dimensional any longer. For example, the parameters p and a, which were the vectors of a finite-dimensional space, become elements of H_z; and the parameter C, which was a linear operator on a finite-dimensional space, is now a linear operator on H_z. Therefore, when dealing with infinite-dimensional situations, it is natural to consider *finite grading* as analog of finite dimensionality.

(d) Let $\Gamma^1_{\xi^1}$ and $\Gamma^2_{\xi^2}$ be two germs and ϕ be a holomorphic invertible map of the former onto the latter, $\phi(\Gamma^1_{\xi^1}) = \Gamma^2_{\xi^2}$, $\phi(\xi) = \widetilde{\xi}$. Then the linear part (differential) of the map ϕ at the point ξ^1 has the form

$$z \to Cz + aw, \qquad w \to bz + \rho w.$$

Let us write the local equations of the germs as

$$v = \langle z, \bar{z}\rangle^1 + O(3) \qquad \text{and} \qquad v = \langle z, \bar{z}\rangle^2 + O(3).$$

The fact that the map sends the first surface to the second can be expressed as an identity. Separating the components of weights 1 and 2 in this identity, we find that $b = 0$ and

$$\langle z, \bar{z}\rangle^2 = \rho\langle C^{-1}z, \overline{C^{-1}z}\rangle^1. \tag{3}$$

This implies that the linear map $z \to Cz$, $w \to \rho w$ sends the first model surface $Q^1 = \{v = \langle z, \bar{z}\rangle^1\}$ to the second $Q^2 = \{v = \langle z, \bar{z}\rangle^2\}$; i.e., the holomorphic equivalence of germs generates the linear equivalence of their model surfaces.

(e) Let $X = X_0 + X_1 + X_2 + X_3 + \ldots$ be a field in $\operatorname{aut}_0 \Gamma_0$ for a nondegenerate hypersurface $\Gamma = \{v = \langle z, \bar{z}\rangle + O(3)\}$, and let $Q = \{v = \langle z, \bar{z}\rangle\}$ be its model surface. Then the map $\psi\colon X_0 + X_1 + X_2 + X_3 + \ldots \to X_0 + X_1 + X_2$ is a faithful representation of $\operatorname{aut}_0 \Gamma_0$ in $\operatorname{aut}_0 Q$. In particular, this allows us to argue that the maps of nondegenerate germs are uniquely defined by their 2-jets.

Thus, all the proofs are carried over without changes. Note that the only difference between the finite-dimensional case and the infinite-dimensional one is that, instead of finite dimensionality of the Lie algebra, we speak in the latter case of the finiteness of its graded decomposition.

3. QUADRICS, $l = 2$

Now, suppose that the complement H_w has an arbitrary finite or infinite dimension K. The scalar variable w becomes a finite- or infinite-dimensional vector variable. The Hermitian form $\langle z, \bar{z}\rangle$ now also takes values in the real Hilbert space N.

How do propositions (a)–(e) change?

Propositions (a) and (d) carry over without changes. Propositions (b) and (e) do not change either, i.e., can be repeated word for word; however, the finite dimensionality criterion is reformulated as follows.

A form is said to be nondegenerate if the following two generally independent conditions hold:

(1) *The kernel is trivial*; *i.e., the relation* $\langle z, \bar{a}\rangle = 0$ *is valid for all* z *only if* $a = 0$.

(2) *The image is full-dimensional*; *i.e., if α is a linear functional in N and $\alpha(\langle z,\bar z\rangle)=0$ for all z in H_z, then $\alpha=0$.*

If K is finite, then the second condition implies the linear independence of the coordinate Hermitian forms.

To obtain proposition (b), one can modify the finite-dimensional arguments as follows.

Proof of proposition (b). Writing the condition for the holomorphic tangent vector field

$$2\operatorname{Re}\biggl(f(z,w)\frac{\partial}{\partial z}+g(z,w)\frac{\partial}{\partial w}\biggr),$$

we obtain the relation

$$\operatorname{Im} g(z,u+i\langle z,\bar z\rangle)=2\operatorname{Re}\langle f(z,u+i\langle z,\bar z\rangle),\bar z\rangle. \tag{4}$$

We represent the coefficients of the fields as sums of components: $f(z,w)=\sum f_j(z,w)$ and $g(z,w)=\sum g_j(z,w)$. Equating the components of (4) of bidegrees $(j,0)$ in $(z,\bar z)$, we find that

$$f(z,u)=a(u)+C(u)z+A(u,z,z)\qquad\text{and}\qquad g(z,u)=b(u)+2i\langle z,\bar a(u)\rangle$$

since the Hermitian form is nondegenerate; here Cz is linear in z and A is a quadratic form of z. Substituting these expressions into (4), we conclude that

$$\begin{gathered}\langle A(u,z,z),\bar z\rangle=2i\langle z,\Delta\bar a(u)\rangle\langle z,\Delta^2\bar a(u)\rangle,\qquad \operatorname{Im} b(u)=0,\qquad \Delta^3 b(u)=0,\\ 2\operatorname{Re}\langle C(u)z,\bar z\rangle=\Delta b(u),\qquad 2\operatorname{Im}\langle C(u)z,\bar z\rangle=0,\end{gathered} \tag{5}$$

where Δ is the value of the differential with respect to u on the form $\langle z,\bar z\rangle$; i.e., $\Delta\phi(u)=\phi'_u(u)(\langle z,\bar z\rangle)$. From the relations obtained, we can easily derive

$$\langle\Delta^2C(u)z,\bar z\rangle=0. \tag{6}$$

Now, since the kernel of the form $\langle z,\bar z\rangle$ is trivial, we construct an expanding sequence of finite-dimensional subspaces of the space Z,

$$Z_1\subset Z_2\subset\dots Z_j\subset\dots,$$

such that the restriction of the Hermitian form $\langle z,\bar z\rangle$ to each of these subspaces is nondegenerate and the closure of these subspaces gives the whole Z. These subspaces correspond to an expanding sequence of finite-dimensional subspaces of the space N,

$$N_1\subset N_2\subset\dots N_j\subset\dots,$$

where N_j is the linear span of the image of the space Z_j under the map $z\mapsto\langle z,\bar z\rangle$. Due to the second nondegeneracy condition for the form, the closure of these subspaces gives the whole N. Standard arguments based on the exponential representation theorem allow one to find from (6) that the restriction of $C(u)z$ to the subspace $Z_j\oplus N_j$ depends linearly on u. Now, passing to the limit and using the continuity, we find that this is also valid for $Z\oplus N$. Then, standard reasoning shows that a is linear in u, A is independent of u, and b depends quadratically on u. $\square$

We should make corrections to the description of the Lie algebra of the model surface, i.e., to proposition (c); however, for the infinite-dimensional situation, these corrections formally coincide with those that arise in the finite-dimensional case for $K > 1$. Namely,

$$g_{-2} = \left\{2\operatorname{Re}\left(q\frac{\partial}{\partial w}\right)\right\},$$

$$g_{-1} = \left\{2\operatorname{Re}\left(p\frac{\partial}{\partial z} + 2i\langle z, \bar{p}\rangle\frac{\partial}{\partial w}\right)\right\},$$

$$g_0 = \left\{2\operatorname{Re}\left(Cz\frac{\partial}{\partial z} + \rho w\frac{\partial}{\partial w}\right)\right\}, \qquad \text{where} \quad 2\operatorname{Re}\langle Cz, \bar{z}\rangle = \rho\langle z, \bar{z}\rangle,$$

$$g_1 = \left\{2\operatorname{Re}\left((aw + A(z,z))\frac{\partial}{\partial z} + 2i\langle z, \bar{a}w\rangle\frac{\partial}{\partial w}\right)\right\}, \qquad \text{where} \quad \langle A(z,z), \bar{z}\rangle = 2i\langle z, \bar{a}\langle z, \bar{z}\rangle\rangle,$$

$$g_2 = \left\{2\operatorname{Re}\left(B(z,w)\frac{\partial}{\partial z} + r(w,w)\frac{\partial}{\partial w}\right)\right\}, \qquad \text{where} \quad \operatorname{Re}\langle B(z,u), \bar{z}\rangle = r(\langle z, \bar{z}\rangle, u)$$
$$\text{and} \quad \operatorname{Im}\langle B(z, \langle z, \bar{z}\rangle), \bar{z}\rangle = 0.$$

Here the parameters have an obvious meaning: $q \in N$, $p \in H_z$, $C \in \operatorname{gl}(H_z)$, $\rho \in \operatorname{gl}(N)$, the parameter A is an H_z-valued quadratic form on $H_z \oplus H_z$, the parameter B is an H_w-valued bilinear form on $H_z \oplus H_w$, and the parameter r is an N-valued quadratic form on $N \oplus N$.

The propositions concerning the structure of the transformation groups $\operatorname{Aut}_- Q$ and $\operatorname{Aut}_0 Q$ remain the same. However, the group $\operatorname{Aut}_+ Q$ is no longer a subgroup of the group of projective transformations. One can show that any transformation in $\operatorname{Aut}_+ Q$ is expressible in terms of operators inverse to polynomial transformations of bounded degree. The arguments that allow one to show this seem to be due to W. Kaup [9], and the first who applied them to the description of automorphisms of quadratic model surfaces was Tumanov [8, 10]. If the codimension is finite, then, just as in the finite-dimensional case, this implies birationality with a bound on the degree; however, one cannot assert this if the codimension is infinite.

4. MODEL SURFACES FOR $l \geq 3$

In the finite-dimensional case, i.e., for finite n, there is a natural restriction on the application of such quadratic model surfaces. This is associated with the fact that the dimension of the space of Hermitian forms on a complex space of dimension n is n^2. If $K > n^2$, then there are no nondegenerate model surfaces for this type of germs. In the case of an infinite-dimensional variable z, such a formal obstacle to the existence of nondegenerate quadratic (i.e., described above) model surfaces disappears. However, a phenomenon analogous to the violation of the inequality $K \leq n^2$ may occur in this case as well. Namely, it may happen that the closure of *all* Hermitian forms is a proper subspace in the space of the real complement N. Then we denote this subspace by N_2, its complexification by H_{w_2}, and the corresponding variable by $w_2 = u_2 + iv_2$, and pass to the next step. The whole construction is thoroughly described in [2] and is of recursive character. The number of steps is described by a new parameter l. At each step, we increase l by one and

- consider the space of homogeneous real (scalar rather than vector-valued) polynomials of the next $(l+1)$th weight $\mathcal{F}_{l+1}$;
- construct a direct decomposition of this space, $\mathcal{F}_{l+1} = \mathcal{R}_{l+1} \oplus \mathcal{N}_{l+1}$, and introduce a new variable $w_{l+1} = u_{l+1} + iv_{l+1}$ which is a vector of the space $H_{w_{l+1}}$, the complexification of the space $\mathcal{N}_{l+1}$, and which is assigned the weight $l+1$, with $v_{l+1} \in \mathcal{N}_{l+1}$. In addition, we perform the next step of reducing the local equations of the germ and write equations corresponding to the new group of variables. As a result, the equations take the form

$$v_2 = F_2 + O(3), \quad \ldots, \quad v_{l+1} = F_{l+1} + O(l+2), \qquad \widetilde{v} = \widetilde{F},$$

where $\widetilde{v}$ is a coordinate of the space $\widetilde{N}$, a direct complement of $\mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_l \oplus \mathcal{N}_{l+1}$ to N;

- verify that we have not exhausted the entire space N. If $\widetilde{N} = 0$, then the process terminates, while if $\widetilde{N} \neq 0$, then the process continues further.

Let us write what is obtained for $l = 7$. Here, for brevity, we denote the forms of degree a in x and b in y by $\langle x^a y^b \rangle$:

$$\begin{aligned}
v_2 &= \langle z\bar{z} \rangle, \\
v_3 &= \langle z^2\bar{z} \rangle + \langle z\bar{z}^2 \rangle, \\
v_4 &= \langle z^3\bar{z} \rangle + \langle z^2\bar{z}^2 \rangle + \langle z\bar{z}^2 \rangle, \\
v_5 &= \langle z^4\bar{z} \rangle + \langle z^3\bar{z}^2 \rangle + \langle z^2\bar{z}^3 \rangle + \langle z\bar{z}^4 \rangle + \langle z^2\bar{z}u_2 \rangle + \langle z\bar{z}^2u_2 \rangle, \\
v_6 &= \langle z^5\bar{z} \rangle + \langle z^4\bar{z}^2 \rangle + \langle z^3\bar{z}^3 \rangle + \langle z^2\bar{z}^4 \rangle + \langle z\bar{z}^5 \rangle + \langle z^3\bar{z}u_2 \rangle + \langle z^2\bar{z}^2u_2 \rangle + \langle z\bar{z}^3u_2 \rangle, \\
v_7 &= \langle z^6\bar{z} \rangle + \langle z^5\bar{z}^2 \rangle + \langle z^4\bar{z}^3 \rangle + \langle z^3\bar{z}^4 \rangle + \langle z^2\bar{z}^5 \rangle + \langle z\bar{z}^6 \rangle + \langle z^4\bar{z}u_2 \rangle + \langle z^3\bar{z}^2u_2 \rangle \\
&\quad + \langle z^2\bar{z}^3u_2 \rangle + \langle z\bar{z}^4u_2 \rangle + \langle z^2\bar{z}u_2^2 \rangle + \langle z\bar{z}^2u_2^2 \rangle + \langle z^3\bar{z}u_3 \rangle + \langle z^2\bar{z}^2u_3 \rangle + \langle z\bar{z}^3u_3 \rangle.
\end{aligned}$$

In the finite-dimensional case, i.e., when both n and K are finite, this process will necessarily terminate after a finite number of steps. The point is that the dimensions of the spaces N_j increase rapidly. For instance, $\dim N_2 = n^2$, $\dim N_3 = n^2(n+1)$, etc. As for the case $K = \infty$, the process may not terminate at all. This means that the right-hand sides of the normalized equations of the germ contain polynomials of all weights. In this case, we assume that $l = \infty$, and if n is finite, then the dimensions of all complements N_j are also finite, and the parameter l for an infinite value of K cannot be finite.

Now, let us see into what the main propositions are transformed. Proposition (a) carries over without changes.

(a) If a field $X = X_{-2} + X_{-1} + X_0 + X_1 + \ldots$ belongs to $\operatorname{aut} Q$, then each of its components belongs to this algebra, $X_j \in \operatorname{aut} Q$.

For $l > 2$, the nondegeneracy condition is simplified. In this case, the equations of the model surface begin with an equation $v_2 = \langle z, \bar{z} \rangle$, where the form $\langle z, \bar{z} \rangle$ takes values in a space $\mathcal{N}_2$ isomorphic to the space of all scalar Hermitian forms; the presence of a nontrivial kernel with respect to the forms of all weights implies, in particular, that the condition $\langle z, \bar{a} \rangle = 0$ holds for some nonzero $a \in H_z$. Let us represent the space H_z as a direct sum of the one-dimensional space generated by a and some complement; accordingly, any vector in H_z can be expressed as $z = z_1 + z_2$; in this case, the form is decomposed as $\langle z, \bar{z} \rangle = \langle z_1, \bar{z}_1 \rangle + \langle z_1, \bar{z}_2 \rangle + \langle z_2, \bar{z}_1 \rangle + \langle z_2, \bar{z}_2 \rangle$. Now, on N_2 consider a nonzero linear functional α such that $\alpha(\langle z_2, \bar{z}_2 \rangle) = 0$; then $\alpha(\langle z, \bar{z} \rangle) = 0$, which violates the second nondegeneracy condition. Thus, for $l > 2$, the first nondegeneracy condition (the absence of a kernel) is absorbed by the second condition (the full dimensionality of the image). So, now the nondegeneracy condition for the model surface $v = \Phi(z, \bar{z}, u)$ is that the image is full-dimensional.

A germ $v = F(z, \bar{z}, u)$ is said to be *nondegenerate* if the model surface $v = \Phi(z, \bar{z}, u)$ obtained from this germ by the above-described procedure is nondegenerate. *The nondegeneracy of a model surface means the full dimensionality of the image of the map* Φ; *i.e., if* $\alpha(\Phi(z, \bar{z}, u)) = 0$, *then this linear functional on* N *vanishes.* If K is finite, then this is just the condition of linear independence of the coordinate forms.

The weighted decomposition obtained in constructing a model surface induces weighted decompositions for all fields. Here one should distinguish between the cases of finite and infinite l. If l is finite, then the fields are decomposed as

$$X = X_{-l} + X_{-l+1} + \ldots + X_0 + X_1 + \ldots,$$

which corresponds to a decomposition

$$\operatorname{aut} Q = g_{-l} + g_{-l+1} + \ldots + g_0 + g_1 + \ldots$$

of the Lie algebra of the model surface. If $l = \infty$, then this decomposition is unbounded on the negative side as well.

Now let us turn to analogs of propositions (b) and (c).

Let g_- be the sum of all negative components of g. The group $\operatorname{Aut}_- Q$ corresponding to this Lie subalgebra is a group of triangular polynomial transformations of the space that acts transitively on Q, has no fixed points, and can be identified with Q. If l is finite, then $g_- = g_{-l} + \ldots + g_{-1}$, and the weight of the leading component of the polynomials in the corresponding group is $l - 1$. If $l = \infty$, then the degrees of the polynomials in the blocks indefinitely increase. This group is a natural analog of the Heisenberg group.

The subalgebra g_0 generates an AutL-subgroup of the group $\mathrm{GL}(H_z) \oplus \mathrm{GL}(N)$; this subgroup is distinguished by the condition $\Phi(Cz, \overline{Cz}, \rho u) = \rho\Phi(z, \overline{z}, u)$, which implies that ρ has a block structure corresponding to the decomposition $N = N_2 \oplus \ldots \oplus N_l$.

The situation with g_+ is rather paradoxical. There is no example for $l \geq 3$ with $g_+ \neq 0$. This pertains to both finite-dimensional and infinite-dimensional situations. In some finite-dimensional situations, the triviality of g_+ has been proved. For example, it was proved by Kossovskii [7] for all finite-dimensional model surfaces in the case of $l = 3$. The fact that g_+ is finitely graded, i.e., that it consists of a finite number of components, has been proved under the nondegeneracy assumption for all finite-dimensional situations. This is a corollary to Zaitsev's theorem [11]. Neither Kossovskii's nor Zaitsev's theorem extends directly to the infinite-dimensional situation. However, the direct analysis performed in [3] and [4] for the defining relations allows one to show, using the technique described above for $l = 2$, the following in the infinite-dimensional situation. For $l = 3$ the subalgebra g_+ is limited to $g_1 + \ldots + g_6$, and for $l = 4$ we have $g_+ = g_1$. In fact, the assertion $g_+ = g_1$ was proved in [4] for an arbitrary $l \geq 4$, but this was done for "rigid" surfaces, i.e., for those surfaces whose equations do not contain $u = \operatorname{Re} w$ on the right-hand side. For such surfaces, the assertion carries over to the infinite-dimensional situation. However, for $l \geq 5$, a general model surface does not satisfy this condition. Zaitsev's theorem says nothing about graded components. This is a theorem on unique dependence of a germ of a holomorphic map on its finite jet. The main assumption is that the type of a germ of a surface is finite. The main tools are the technique of Segre surfaces and the implicit function theorem. If an analog of Zaitsev's theorem holds in the infinite-dimensional situation, then, by virtue of (a), it immediately implies the finite grading. Zaitsev's theorem gives an estimate for the number of the jet for which such a uniqueness theorem holds. However, this estimate, proved for a finite-dimensional space, increases with the dimension. This fact casts doubts on the possibility of adapting Zaitsev's proof to the infinite-dimensional situation.

So, we can formulate two conjectures for a nondegenerate model surface: the minimum and maximum conjectures. The minimum conjecture has been proved for the finite-dimensional case, while the maximum conjecture remains open in both situations.

Minimum conjecture. *If l is finite, then g_+ is finitely graded.*

Maximum conjecture. *If $3 \leq l < \infty$ is finite, then $g_+ = 0$.*

(d) This proposition remains almost unchanged. Namely, if there is a *holomorphic* equivalence of germs, then, from the linear part of this map, one somehow derives a set of invertible linear operators

$C, \rho_2, \dots, \rho_l$ that act with respect to the corresponding variables and give a *linear* equivalence of the model surfaces.

(e) In the case of finite grading, a faithful representation of the stabilizer of a nondegenerate germ, $\mathrm{aut}_0 M_0$, in the stabilizer of its model surface, $\mathrm{aut}_0 Q$, is constructed in the same way,

$$\psi\colon\ X_0 + X_1 + \ldots + X_d + \ldots \to X_0 + \ldots + X_d.$$

All these propositions for infinite l have the same formulations as in the finite-dimensional situation. However, in the finite-dimensional situation, the infinite value of l is impossible. Model surfaces with infinite values of the parameter l are a new object that demands attention. The simplest situation with $l = \infty$ is given by nondegenerate surfaces with one-dimensional complex tangent $n = 1$ and infinite codimension $K = \infty$.

In this paper, the nondegeneracy condition is presented in a purely coordinate form. However, this condition can also be formulated in a coordinate-free form. The correspondence between our nondegeneracy condition (complete nondegeneracy) and some coordinate-free conditions is pointed out in [2]; the infinite-dimensional specificity has no effect on this correspondence. In particular, the parameter l in these terms is the length of the Levi–Tanaka algebra.

In the present paper, we have discussed only the main structural components of the theory. There are a plenty of more special subjects and constructions that can also be brought to the infinite-dimensional situation [1]. Here are a couple of issues:

• If the spaces in which the surfaces are situated have a certain additional algebraic structure, then such spaces may admit a more special theory. For example, if the basic real Hilbert space H possesses the structure of a real commutative algebra, then a specific class of quadrics arises; in the finite-dimensional case, such quadrics were considered in [12].

• In the finite-dimensional case, quadratic model surfaces of codimension 2 were classified and studied in [13]. What is the case when $n = \infty$?

The basis of this theory is the interplay between real and complex structures, as well as the application of power series. Therefore, there is a plenty of possibilities for generalizations: transition from Hilbert spaces to Banach spaces and, further, to Fréchet spaces; analysis of noncommutative situations, for example, submanifolds in a superspace; etc.

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