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Analytical Complexity: Development of the Topic

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Abstract. The complexity of analytic functions of two variables is studied in terms of the order of complexity suggested in [1]. This paper continues [1]. An estimate for the complexity of a polynomial using its degree is given. Examples of homogeneous and harmonic functions are treated. An estimate for the complexity of a power series in terms of the geometry of the support of the series is given. Differential equations defining classes of complexity are considered. For polynomial mappings of complexity one, the conjecture on the Jacobian is proved. In this connection, the complexity of mappings of the plane is discussed.

DOI

1. INTRODUCTION

For every analytic function of two variables $z(x, y)$, one can define the *order of complexity* of this function, $N(z)$. One can define the complexity by using every germ representing this function. The quantity $N(z)$ is preserved under any analytic continuation. If the germ depends on only one of the variables, we set $N(z) = 0$. If this is not the case, but the germ can be represented in the form $z = c(a(x)+b(y))$, where (a, b, c) are germs of analytic functions of one variable, then $N(z) = 1$, and so on. In other words, we write $N(z) = n$ if z can be represented in the form $C(A(x, y) + B(x, y))$, where C is the germ of a function of one variable, the complexity of A and B is less than n, and there is no representation of this kind with lesser value of n. An increasing system of classes of functions arises,

$$Cl_0 \subset Cl_1 \subset Cl_2 \dots$$

The class Cl_n contains the functions whose complexity does not exceed n. If a function does not belong to any of these classes, we write $N(z) = \infty$. If one chooses some domain in the two-dimensional space, then the subset of functions of finite complexity is a first category set in the Fréchet space of functions holomorphic in the domain (a meager set). Every class is defined by differential-algebraic relations. For example, Cl_0 is defined by the condition $z'_x z'_y = 0$, and Cl_1 by the condition

$$(\log(z'_x/z'_y))''_{xy} = 0.$$

The replacement of the basic binary operation $x + y$ by each of the three remaining arithmetic operations does not influence the order of complexity by the relation

$$xy = \exp(\log(x) + \log(y)).$$

It can be noted that the suggested approach is somewhat similar to the approaches suggested by A. Ostrowski [7] (the system of ranks) and by D. Grigoriev [5] (the additive complexity). In this connection, we note that there is a wide spectrum of complexity theories that are focused on mathematical objects of different nature. These are the algorithmic complexity, topological complexity, the Kolmogorov complexity, etc. Moreover, using the "complexity" point of view, one can consider a lot of mathematical properties, like the degree of a polynomial, the dimension of a space, the genus, the order of a differential equation, the transcendence degree, the ε-entropy, etc. Both the Galois theory and Cantor's theory of transfinite numbers can be regarded as some complexity theories. Among Hilbert's problems, a half of the list can be treated as complexity problems with no stretch. The thirteenth problem should be mentioned especially [3]. Despite a great diversity of mathematical objects and approaches to the evaluation of their complexity, it is possible to note some general features of these theories:

(1) anything complex is constructed from something simple by using basic operations,

(2) a large gap between upper bounds for the complexity, which are obtained readily, and the lower bounds obtained with great efforts,

(3) scales of complexities go to infinity; however, the main collection of mathematical objects fits within the "riverside" of objects whose complexity does not exceed 3. (For example, the transfinite scale goes very far, while "almost all" mathematics manages with sets which are finite, countable, or have the cardinality of continuum).

In a way, all these specific features are manifested in the *theory of analytic complexity* considered here.

2. POLYNOMIALS OF DEGREE TWO

It is clear that the order of complexity of any polynomial of degree one is equal to zero or one.

Proposition 1. *(a) All polynomials of degree two,*

$$z = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f,$$

have the complexity not exceeding two.

Such a polynomial has complexity not exceeding one in three cases:

(b.1) if $b = 0$; in this case, $z = a(x+\alpha)^2 + c(y+\beta)^2 + g$,

(b.2) if $a = c = 0$; in this case, $z = b(x+\alpha)(y+\beta) + g$,

(b.3) if $ac - b^2 = dc - eb = 0$; in this case, $z = (px + qy + r)^2 + s$.

Proof. The fact that complexity does not exceed two can readily be established by reducing the polynomial to the canonical form. The equation of the first class for z becomes

$$\Delta_1(z) = b(a^2c\, x^2 - ab^2\, x^2 - ac^2\, y^2 + b^2c\, y^2 - 2\,aeb\, x + 2\,acd\, x + 2\,bdc\, y - 2\,ace\, y + cd^2 - ae^2) = 0.$$

We equate the coefficients to zero, solve the system of equations, and obtain the answer.

Thus, the picture is as follows: in the six-dimensional space of polynomials of degree not exceeding two, the polynomials of the first class of complexity form an algebraic subset consisting of three irreducible components, namely, two linear subspaces (of codimension one and of codimension two) and a submanifold of codimension two, which is the intersection of two quadratic cones. Here the six-dimensional picture can be made four-dimensional. Exclude the constant term f from our consideration (this term is not reflected in the complexity) and go to the projective space $\mathbf{CP}^4$ with homogeneous coordinates $(a : b : c : d : e)$.

Note also that two cases described above for the representation of a quadratic polynomial by a formula of the first class of complexity use the procedure of selecting a full square and one case uses the factorization.

3. HOMOGENEOUS FUNCTIONS

Proposition 2. *Let $z(x,y)$ be a homogeneous function of degree k. Then*

(a) the complexity of z does not exceed two,

(b) if $z(x,y)$ is represented in the form $(xP(y/x))^k$, then the complexity of z does not exceed one if and only if P satisfies the ordinary differential equation

$$\begin{gathered} tP^{(2)}(t)P'(t)^2P(t) - tP^{(2)}(t)^2P(t)^2 + 2t^2P^{(2)}(t)^2P(t)P'(t) \\ +tP^{(3)}(t)P'(t)P(t)^2 - t^2P^{(3)}(t)P'(t)^2P(t) - P^{(2)}(t)P'(t)^2P(t)t \\ +P^{(2)}(t)P'(t)P(t)^2 - P'(t)^3P(t) + P'(t)^3P(t) - t^2P'(t)^3P^{(2)}(t) = 0. \end{gathered}$$

Proof. The bound $N(z) \leqslant 2$ follows immediately from the representation $(xP(y/x))^k$. We obtain the equation for P from the condition

$$(\ln(z'_x/z'_y))''_{xy} = 0,$$

which describes the first class.

If the homogeneity is understood in a generalized sense, namely, $z(t^\alpha x, t^\beta y) = t^\gamma z(x, y)$, then the bound $N(z) \leqslant 2$ certainly remains valid.

The computer algebra system MAPLE outputs the general solution of the above differential equation for P in the form of the following elementary function:

$$P(t) = \exp(-\frac{-C_3C_1 + \ln(\exp(-\ln(t)C_1) - \exp(C_1C_2)}{C_1}),$$

i.e., the general form of a homogeneous function of degree one and of the first class is

$$z = x\exp(-\frac{-C_3C_1 + \ln(\exp(-\ln(y/x)C_1) - \exp(C_1C_2)}{C_1}).$$

One can readily transform this three-parameter family to a more visible form,

$$z = (px^r + qy^r)^{1/r}.$$

One can write out another three-parameter family of functions of this kind,

$$z = R\,(x^P y^Q)^{1/(P+Q)}.$$

The latter family can be extracted from the solution of the differential equation by using the passage to the limit as $C_1 \to 0$. To these two families of homogeneous functions of degree $k = 1$ there correspond two three-parameter families of homogeneous functions of arbitrary degree,

$$z_+ = (px^r + qy^r)^{k/r} \quad \text{and} \quad z_\times = R\,(x^P\, y^Q)^{k/(P+Q)},$$

to which the following generating functions correspond:

$$P_+(t) = (p + q\,t^r)^{k/r} \quad \text{and} \quad P_\times(t) = R\,t^{\frac{k\,Q}{P+Q}},$$

For a function $z_\times$ to be a polynomial, it is necessary that the values of r and k be nonnegative integers, where r is to be a divisor of k. Then $P_1(t) = (p + q\ t^r)^s$, where $k = r\ s$. Here $z = (p\ x^r + q\ y^r)^s$. For a function z_+ to be a polynomial, it is necessary that $m = \frac{k\,Q}{P+Q}$ and $n = \frac{k\,P}{P+Q}$ be nonnegative integers. Here $P_+(t) = R\ t^m$ and $z = R\ x^m\ y^n$. The complex roots of the polynomial P_+ belong to the circle of radius $|p/q|$ centered at the origin at the vertices of a regular r-gon, while the only root of $P_\times$ is $t = 0$. This configuration is obtained from the previous one by the passage to the limit as $p \to 0$, and one may assume that this is a regular polygon of zero size. If the zeros of the polynomial $P(t)$ do not belong to the vertices of a regular polygon centered at zero, then the homogeneous polynomial obtained from P is of complexity two.

4. HARMONIC FUNCTIONS

If $f(\zeta)$ is an analytic function of complex variable ζ, then it has the complexity zero, according to our definition. However, if we consider f as a function of real variables $f(x+iy)$, then its complexity turns out to be equal to one. Under this approach, the only holomorphic functions of complexity zero are constants. If we complexify the variables x and y, then the complexity of the expression $f(x + iy)$ remains equal to one. Suppose now that u is a harmonic function (possibly set-valued). Then, locally, we have the representation

$$u = f(z) + \bar{f}(\bar{z}) = f(x + iy) + \bar{f}(x - iy),$$

where $f(z)$ is holomorphic, which implies that $N(z) \leqslant 2$. The harmonic functions of the first class are functions satisfying not only the Laplace equation but also the equation of the first class. This condition written out for f looks rather enigmatically,

$$\Big(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2}\Big)\left(\frac{\log(f'(z) - \bar{f}'(\bar{z}))}{\log(f'(z) + \bar{f}'(\bar{z}))}\right) = 0.$$

However, it is clear from this relation that, if $f(z)$ is a solution, then $if(z)$ is also a solution, and therefore, if the harmonic function $u = \operatorname{Re} f$ is of complexity one, then so is the conjugate harmonic function $v = \operatorname{Im} f$. For examples of harmonic functions of first class, along with the linear functions, one can suggest real and imaginary parts of the following functions:

$$z^2 = (x^2 - y^2) + 2ixy, \qquad \log(z) = \frac{1}{2}\log(x^2 + y^2) + i\arctan(y/x),$$
$$\exp(z) = \exp(x)\cos(y) + i\exp(x)\sin(y).$$

One can use another approach. Let us write out the Laplace equation for $z = c(a(x) + b(y))$; we obtain

$$c''(a(x) + b(y))a'(x)^2 + c'(a(x) + b(y))a''(x) + c''(a(x) + b(y))b'(y)^2 + c'(a(x) + b(y))b''(y) = 0.$$

and this relation can be differentiated both with respect to x and with respect to y. After this, we shall have three equations linear with respect to (c', c'', c'''). Under the assumption that the function c is not constant, we may assume that the determinant of this system vanishes. This determinant is the product of the factor $(a'(x)^2 + b'(y)^2)$, which vanishes for constant (a, b) only, an of the expression

$$b'''(y)a'(x)^3 - a'''(x)b'(y)a'(x)^2 + 2a'(x)a''(x)^2b'(y)$$
$$+ a'(x)b'''(y))b'(y)^2 - 2a'(x)b''(y)^2b'(y) - a'''(x)b'(y)^3,$$

which is thus equal to zero. Let us express $b'''(y)$ from this equation, differentiate the equation thus obtained with respect to x, reduce the answer by $b(y)$, express $b''(y)^2$ from the relation thus obtained, differentiate the equation thus obtained with respect to x again, reduce the answer by $(a'(x)^2 + b'(y)^2)^2$, after which we obtain the desired relation for a,

$$F(a) = (a')^2a''a^{(5)} - (a')^2a'''a^{(4)} - 3(a'')^2a^{(4)}a' + 3(a'')^3a''' = 0.$$

By the symmetry between x and y, the function b must satisfy the same equation.

The immediate computation shows that $F(x^2) = 0$. However, this does not contradict our ideas. Indeed, we have examples $x^2 - y^2$ and $\ln(x^2 + y^2)$. However, $F(x^3) = 3888x^3$, which means that the function $z = c(x^3 + b(y))$ is not harmonic for any nonconstant b and c. This statement can be suggested as a problem for a student Olympiad in analysis.

Thus, the following assertion holds.

Proposition 3. *(a) the complexity of any harmonic function $u(x, y)$ does not exceed two,*

(b) for a harmonic function $u(x, y) = f(x+iy)+\bar{f}(x-iy)$ to be of complexity one, it is necessary and sufficient that

$$\Big(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2}\Big)\left(\frac{\log(f'(z) - \bar{f}'(\bar{z}))}{\log(f'(z) + \bar{f}'(\bar{z}))}\right) = 0;$$

(c) for a function $z = c(a(x)+b(y))$ of the first class of complexity to be harmonic, it is necessary that the functions a and b be satisfying the ordinary differential equations

$$F(a(x)) = 0, \quad F(b(y)) = 0.$$

An example with harmonic functions can be generalized as follows. Let D be a second-order homogeneous differential operator with constant coefficients and let $u(x, y)$ be a solution of the equation $D(u) = 0$. Then, using a linear change of variables, one can reduce the operator to the form

$$D = \frac{\partial^2}{\partial x \partial y} \qquad \text{or} \qquad D = \frac{\partial^2}{\partial x^2},$$

which shows that the complexity of any solution does not exceed two. In particular, this bound holds for the solutions of the wave equation $u_{yy} = u_{xx}$ (d'Alembert's formula).

5. POWER GEOMETRY

Consider the problem of estimating the complexity of analytic functions in terms of power geometry. Let a function be given by a power series or a Laurent series $z(x, y) = \sum z_{ij} x^i y^j$. It can readily be seen that a generic series of this kind has infinite complexity. The genericity condition means that a countable family of algebraic relations on the coefficients fail to hold. Let $\mathbf{Z}$ be the set of integers, i.e., $\{\dots, -2, -1, 0, 1, 2, 3, \dots\}$; then the support $\mathrm{supp}(z)$ of the power series is a subset of the Cartesian square $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$ consisting of the pairs (i, j) for which the corresponding elements z_{ij} are nonzero. Let us present some conditions on the support of the series which enable us to estimate the complexity of $z(x, y)$. These estimates use two obvious observations.

(1) Let (p, q) be a nonzero vector in $\mathbf{Z}^2$. To this vector, a straight line corresponds for which this vector is directing and which passes through the origin. If the support of the series belongs to this line, then one can write $z = c(x^p y^q)$, which implies that $N(z) \leqslant 1$. If the support is placed on a line which does not pass through the origin, than one can write $z = c(x^p y^q) x^i y^j$, and we see that

$$N(z) \leqslant 2.$$

(2) If the support of a series can be represented as the union of two subsets $M_1 \cup M_2$ of the two-dimensional integer lattice in such a way that all series with supports on M_1 and on M_2 have complexity not exceeding n, then the complexity of the original series does not exceed $n + 1$.

The most economical way of encoding is the bisection. Sadykov [6] attracted the attention of the author to the fact that this method is appropriate in the present situation. Let $[x]$ be the least integer not exceeding a real x. If positive integers k and n are such that $2^{n-1} < k \leqslant 2^n$, then, if we begin dividing a collection of k objects into equal halves of parts differing from each other by one object, then we reach parts containing at most one object in at most n steps. Therefore, the number of divisions does not exceed $[\log_2(k)]$. This leads us to the following claim.

Proposition 4. *(a) If the support of a series z is contained in the union of k lines, then*

$$N(z) \leqslant [\log_2(k)] + 2.$$

(b) If a function z can be represented as the sum of k homogeneous summands, then

$$N(z) \leqslant [\log_2(k)] + 2.$$

(c) If a function z can be represented as the sum of k summands of the form $c_{ij}(x, y) x^i y^j$, where the complexity of all coefficients $c_{ij}(x, y)$ does not exceed $n \geqslant 1$, then

$$N(z) \leqslant [\log_2(k)] + 1 + n.$$

(d) If a function is a polynomial of degree not exceeding d, then

$$N(z) \leqslant [\log_2(d)] + 1.$$

Therefore, we have the following corollary.

Corollary. *If u is a solution of the equation*

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}(u) = 0,$$

then the complexity $N(u)$ of u satisfies the inequality

$$N(u) \leqslant [\log_2(m+n)] + 2.$$

One can pass from the variables (x, y) to the formal variables $(z = x+iy, \bar{z} = x-iy)$ and consider the complexity with respect to these variables. Proposition 4 gives a bound for the complexity of polyanalytic functions, i.e., solutions of the equation $\frac{\partial^n}{\partial \bar{z}^n}(f) = 0$ regarded as functions of $(z, \bar{z})$, namely, $N(f) \leqslant [\log_2(n)] + 2$.

As the simplest example of a series whose support is not contained in a union of lines and whose complexity is possibly infinite, one can suggest $z = \sum z_n x^n y^{n^2}$ and, in particular, $z = \sum x^n y^{n^2}$. Note that $u(x, y) = z(e^x, e^y)$ is a solution of the heat equation $u_y = u_{xx}$, which thus has the same complexity.

6. HOLOMORPHIC COMPLEXITY

In [1], we distinguished between the analytic complexity and the polynomial complexity. Let us introduce another kind of complexity, namely, the *holomorphic complexity*. Let $z(x, y)$ be a function holomorphic in a neighborhood of the origin which vanishes at the origin. Then its holomorphic complexity is defined by the same inductive scheme as the analytic complexity; however, the functions of one variable entering superpositions are functions of one variable that are holomorphic in a neighborhood of the origin and vanish at the origin. If $N(z)$, $N_{hol}(z)$, and $N_{pol}(z)$ are the analytic, holomorphic, and polynomial complexities of z, respectively, then one can write out the inequality $N(z) \leqslant N_{hol}(z) \leqslant N_{pol}(z)$.

Let $z = xy$. Then $N(z) = 1$ by the relation $xy = \exp(\log(x) + \log(y))$. However, a representation of xy in the form of $c(a(x) + b(y))$ with a, b, and c holomorphic at the origin is impossible. Indeed, let $a(x) = a_1 x + a_2 x^2 + O(3)$, and let similar relations hold for b and c. In this case, comparing the coefficients in the expansion, we obtain

$$c_1 a_1 = c_1 b_1 = (c_2 b_1^2 + c_1 b_2) = (c_2 a_1^2 + c_1 a_2) = 0, \qquad 2c_2 a_1 b_1 = 1,$$

and it can readily be seen that this system has no solution (another proof of this assertion was given by Sadykov [6]). However, on the other hand,

$$xy = \frac{(x+y)^2 - (x^2 + y^2)}{2}$$

Thus, $N_{hol}(xy) = N_{pol}(xy) = 2$. This means that, forbidding to place a stability at some point, one can change the order of complexity. Seemingly, the holomorphic and the polynomial complexity can also differ.

What is the structure of polynomials whose analytic complexity is equal to one? We suggest the following conjecture.

Conjecture. *The analytic complexity of a polynomial is equal to one if and only if the polynomial is of the form $R(P(x) + Q(y))$ or of the form $R(P(x)Q(y))$, where (P, Q, R) are nonconstant polynomials in one variable.*

Note that such a polynomial is of polynomial complexity one in the first case and of polynomial complexity two in the other.

7. CLASSES OF COMPLEXITY AS DIFFERENTIAL-ALGEBRAIC SETS

Let us discuss some specific features of differential-algebraic relations defining classes of complexity.

Proposition 5. *The equations $\Delta_n(z) = 0$ defining Cl_n are relations in which the independent variables x and y and also the unknown function z by itself do not occur explicitly, i.e., these are relations involving derivatives of the function z only.*

Proof. This follows from the invariance of the classes under all translations

$$(x \to x + a), \qquad (y \to y + b), \qquad (z \to z + c).$$

In this connection, when speaking of a k-jet, we always mean a truncated k-jet (i.e., a k-jet from which the components of (x, y, z) are excluded). This concerns both the functions of two variables and the functions of one variable. One can readily evaluate the dimension of a k-jet of this kind. It is equal to k for the functions of one variable and to $k(k + 3)/2$ for the functions of two variables.

One can represent the equation $(\log(z'_x/z'_y))''_{xy} = 0$ defining the first class in the form

$$\Delta_1(z) = z'_x z'_y (z'''_{xxy} z'_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0,$$

or, passing to another notation, in the form

$$\Delta_1(z) = z_{10} z_{01}(z_{21} z_{01} - z_{12} z_{10}) + z_{11}((z_{10})^2 z_{02} - (z_{01})^2 z_{20}) = 0.$$

To this relation one can assign three quantities, k, l, and m, where $k = 3$ is the differential order, $l = 4$ is the algebraic degree of the defining equation, and $m = 1$ is the codimension of Cl_1 in the k-jet. One could guess the values of some of these parameters without writing out the relation. Indeed, let $z = c(a(x)+b(y))$. Differentiating this relation, we obtain expressions for partial derivatives of the function z in terms of the derivatives of (a, b, c) of the corresponding orders. The dimension of the truncated k-jet (we take into account only derivatives whose order is at least one) for functions of two variables is equal to $k(k + 3)/2$, while for the family (a, b, c) it is equal to $3k$. For $k = 3$, both the quantities are equal to $d = 9$.

Here are the expressions for the derivatives of a function z of the first class using the derivatives of the functions (a, b, c).

$$\begin{gathered}
z_{10} = c_1 a_1, \quad z_{0,1} = c_1 b_1, \\
z_{20} = c_2 a_1^2 + c_1 a_2, \quad z_{11} = c_2 a_1 b_1, \quad z_{02} = c_2 b_1^2 + c_1 b_2, \\
z_{30} = c_3 a_1^3 + 3c_2 a_1 a_2 + c_1 a_3, \quad z_{21} = c_3 b_1 a_1^2 + c_2 b_1 a_2, \\
z_{03} = c_3 b_1^3 + 3c_2 b_1 b_2 + c_1 b_3, \quad z_{12} = c_3 a_1 b_1^2 + c_2 a_1 b_2
\end{gathered}$$

These expressions are homogeneous in the following sense. Under the change

$$(a_j \to t a_j, \quad b_j \to t^j b_j, \quad c_j \to t c_j),$$

the derivative $z_{x^n y^m}$ is transformed to $t^{m+n+1} z_{x^n y^m}$. This means that the weight of the first derivatives is equal to two, that of the second derivatives is three, and of the third ones is equal to four. These expressions define a polynomial mapping P_1 of weight $L = 4$ from $\mathbf{C}^9$ to $\mathbf{C}^9$, which is invariant with respect to the action of the multiplicative group $\mathbf{C}^*$. Namely, if $\lambda \in \mathbf{C}^*$, then

$$P(\lambda a_1, \lambda a_2, \lambda a_3, \lambda b_1, \lambda b_2, \lambda b_3, \lambda^{-1} c_1, \lambda^{-2} c_2, \lambda^{-3} c_3) = P(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3).$$

Therefore, the image of the one-dimensional orbit of any point in the preimage is a unique point in the image, and we conclude that the image is a proper algebraic subset of $\mathbf{C}^9$ of codimension m

(not less than one). The invariance with respect to the action of $\mathbf{C}^*$ follows from the invariance of Cl_1 with respect to linear change of variables of the form $\Lambda\colon z(x,y) \to z(\lambda x, \lambda y)$. The invariance of Cl_1 with respect to the change $z(x,y) \to z(\mu x, \nu y)$ has its own consequences. This change generates the mapping

$$(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \to (\mu a_1, \mu^2 a_2, \mu^3 a_3, \nu b_1, \nu^2 b_2, \nu^3 b_3, c_1, c_2, c_3).$$

To this mapping, there corresponds an action of $\mathbf{C}^* \times \mathbf{C}^*$ on the coordinates of the mapping P, namely, $S_{\mu\nu} : z_{ij} \to \mu^i \nu^j z_{ij}$. The invariance of the class means that the differential polynomial Δ_1 defining this class must be homogeneous with respect to this action. It can readily be seen that this polynomial has the differential bidegree $(l,l) = (3,3)$, i.e.,

$$\Delta_1(S_{\mu\nu}(z)) = \mu^3 \nu^3 \Delta_1(z).$$

The fact that the homogeneity degree with respect to μ is equal to the degree with respect to ν follows from the invariance of the class with respect to the $\mathbf{Z}_2$ action $z(x,y) \to z(y,x)$.

One can readily evaluate the rank of the mapping P_1^2, which is the restriction of P_1 to the 2-jet, and the rank of P_1^3, which is the restriction of P_1 to the 3-jet. Namely, P_1^2 is a mapping of $\mathbf{C}^6$ to $\mathbf{C}^5$, and it has full rank equal to 5. This means that there are no second-order differential relations satisfied by the second class. The mapping P_1^3 acts from $\mathbf{C}^9$ to $\mathbf{C}^9$, and its rank is equal to 8. This means that the codimension of the first class in the three-jet is equal to one indeed.

How these parameters are transformed when passing to the second class? A typical function of the second class is of the form

$$z = s(c(a(x) + b(y)) + r(p(x) + q(y))).$$

Its formation involves seven functions of one variable. The index of the critical jet is $k_2 = 11$, and the dimension of this jet is $d_2 = 77$. According to the rule of differentiating composite functions, the derivatives of the function z are polynomials in derivatives of the seven defining functions (a, b, c, p, q, r, s). Here are several first coordinates of this mapping:

$$z_{10} = s_1(c_1 a_1 + r_1 p_1), \quad z_{01} = s_1(c_1 b_1 + r_1 q_1),$$

$$\begin{aligned} z_{20} &= s_2(c_1 a_1 + r_1 p_1)^2 + s_1(c_2 a_1^2 + c_1 a_2 + r_2 p_1^2 + r_1 p_2), \\ z_{11} &= s_2(c_1 a_1 + r_1 p_1)(c_1 b_1 + r_1 q_1) + s_1(c_2 a_1 b_1 + r_2 p_1 q_1), \\ z_{02} &= s_2(c_1 b_1 + r_1 q_1)^2 + s_1(c_2 b_1^2 + c_1 b_2 + r_2 q_1^2 + r_1 q_2), \end{aligned}$$

$$\begin{aligned} z_{30} = s_3(c_1 a_1 + r_1 p_1)^3 &+ 2s_2(c_1 a_1 + r_1 p_1)(c_2 a_1^2 + c_1 a_2 + r_2 p_1^2 + r_1 p_2) \\ &+ s_1(c_3 a_1^3 + 3c_2 a_1 a_2 + c_1 a_3 + r_3 p_1^3 + 3r_2 p_1 p_2 + r_1 p_3), \end{aligned}$$

$$\begin{aligned} z_{21} &= s_3(c_1 a_1 + r_1 p_1)^2(c_1 b_1 + r_1 q_1) + 2s_2(c_1 a_1 + r_1 p_1)(c_2 a_1 b_1 + r_2 p_1 q_1) \\ &\quad + s_2(c_1 b_1 + r_1 q_1)(c_2 a_1^2 + c_1 a_2 + r_2 p_1^2 + r_1 p_2) + s_1(c_3 a_1^2 b_1 + c_2 b_1 a_2 + r_3 q_1 p_1^2 + r_2 q_1 p_2), \end{aligned}$$

$$\begin{aligned} z_{12} &= s_3(c_1 b_1 + r_1 q_1)^2(c_1 a_1 + r_1 p_1) + 2s_2(c_1 b_1 + r_1 q_1)(c_2 a_1 b_1 + r_2 p_1 q_1) \\ &\quad + s_2(c_1 a_1 + r_1 p_1)(c_2 b_1^2 + c_1 b_2 + r_2 q_1^2 + r_1 q_2) + s_1(c_3 b_1^2 a_1 + c_2 a_1 b_2 + r_3 p_1 q_1^2 + r_2 p_1 q_2), \end{aligned}$$

$$\begin{aligned} z_{03} = s_3(c_1 b_1 + r_1 q_1)^3 &+ 2s_2(c_1 b_1 + r_1 q_1)(c_2 b_1^2 + c_1 b_2 + r_2 q_1^2 + r_1 q_2) \\ &+ s_1(c_3 b_1^3 + 3c_2 b_1 b_2 + c_1 b_3 + r_3 q_1^3 + 3r_2 q_1 q_2 + r_1 q_3), \end{aligned}$$

and so on, up to the eleventh jet, which define a polynomial mapping P_2 from $\mathbf{C}^{77}$ to $\mathbf{C}^{77}$. This mapping is invariant with respect to three actions of the multiplicative group $\mathbf{C}^*$, namely,

$$L : (a_1, \dots, a_{11}, b_1, \dots, b_{11}, c_1, \dots, c_{11}) \to (\lambda a_1, \dots, \lambda a_{11}, \lambda b_1, \dots, \lambda b_{11}, \lambda^{-1} c_1, \dots, \lambda^{-11} c_{11})$$
$$M : (p_1, \dots, p_{11}, q_1, \dots, q_{11}, r_1, \dots, r_{11}) \to (\mu p_1, \dots, \mu p_{11}, \mu q_1, \dots, \mu q_{11}, \mu^{-1} r_1, \dots, \mu^{-11} r_{11})$$
$$N : (c_1, \dots, c_{11}, r_1, \dots, r_{11}, s_1, \dots, s_{11}) \to (\nu c_1, \dots, \nu c_{11}, \nu r_1, \dots, \nu r_{11}, \nu^{-1} s_1, \dots, \nu^{-11} s_{11}).$$

This means that the image of the mapping P_2 is a proper algebraic subset of codimension $m_2 \geqslant 3$. With regard to the second class, the 11-jet is remarkable, namely, the dimension of the jet for it (the number of derivatives of the function of two variables) is equal to the number of derivatives of the functions of one variable used when expressing the derivatives of the function of two variables. In this case, we refer to the jet as a *critical* one. In particular, we say that the 11-jet is a critical jet of the second class. By the symmetry of the mapping, the image for the critical jet is an algebraic set of positive codimension. However, one can assume here that this situation is possible for some jets with a lesser index. In this connection, we introduce another notion. By the *precise critical jet* of a class we mean a jet with minimal index for which the image of this class has positive codimension. For the first class, both the precise critical jet and simply a critical jet id the three-jet. For the second class, the 11-jet is a critical jet.

Recently, Shvetsova, using computer calculations using the Maple system (for the files with calculations, see the Internet source [8]), showed that the restriction of P_2 to the 10-jet has full rank, which implies the following claim.

Claim 6. *The 11-let is the precise critical jet of the second class.*

Proof. Consider the differential of the mapping P_2^{10}, which is the restriction of P_2 to the 10-jet. This is a mapping from the 70-dimensional space of the variables

$$(a_1, \dots, a_{10}, b_1, \dots, b_{10}, c_1, \dots, c_{10}, p_1, \dots, p_{10}, q_1, \dots, q_{10}, r_1, \dots, r_3, s_1, \dots, s_{10})$$

to the 65-dimensional space of 10-jets of functions of two variables. The differential is a matrix with 65 rows of length 70. It turns out that the minor of order 65 corresponding to the differentiations with respect to the variables

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10},$$
$$b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10},$$
$$c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9,$$
$$p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9,$$
$$q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9,$$
$$r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8,$$
$$s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10},$$

at the point

$$a_1 = 1, b_1 = 2, c_1 = 1, p_1 = 1, q_1 = 1, r_1 = 1, s_1 = 1,$$
$$a_2 = 1, b_2 = 2, c_2 = 2, p_2 = 1, q_2 = 1, r_2 = 1, s_2 = 1,$$
$$a_3 = 1, b_3 = 2, c_3 = 3, p_3 = 1, q_3 = 1, r_3 = 1, s_3 = 1,$$
$$a_4 = 1, b_4 = 2, c_4 = 3, p_4 = 1, q_4 = 1, r_4 = 1, s_4 = 1,$$
$$a_5 = 1, b_5 = 2, c_5 = 3, p_5 = 1, q_5 = 1, r_5 = 1, s_5 = 1,$$
$$a_6 = 1, b_6 = 2, c_6 = 3, p_6 = 1, q_6 = 1, r_6 = 1, s_6 = 1,$$
$$a_7 = 1, b_7 = 2, c_7 = 3, p_7 = 1, q_7 = 1, r_7 = 1, s_7 = 1,$$
$$a_8 = 1, b_8 = 2, c_8 = 3, p_8 = 1, q_8 = 1, r_8 = 1, s_8 = 1,$$
$$a_9 = 0, b_9 = 0, c_9 = 1, p_9 = 0, q_9 = 0, r_9 = 2, s_9 = 0,$$
$$a_{10} = 0, b_{10} = 0, c_{10} = 1, p_{10} = 0, q_{10} = 0, r_{10} = 1, s_{10} = 0$$

is equal to

$$172179306136957497872559389442205 28640.$$

As is well known, this number is nonzero; i.e., this minor is a nonzero polynomial, and the rank of the mapping is maximal (and equal to 65) outside the zero set of this polynomial. Therefore, the image is open, and therefore P_2^{10} is a mapping onto the entire 10-jet, and this completes the proof of the claim.

Thus, for the first and second classes, there is no difference between a critical jet and the precise critical jet. The problem of whether or not this holds for the classes of complexity three or more is open.

What can be said about the other classes? The following general assertion holds.

Proposition 7.

(1) The critical jet of the nth class is the k_n-jet with $k_n = 2^{n+2} - 5$; here the dimensions of the image and preimage for the mapping P_n^k (P_n^k stands for the restriction of the mapping of the nth class to the k_n-jet) are equal to $d_n = (2^{n+1} - 1)(2^{n+2} - 5)$.

(2) The codimension m_n of the image of the nth class in the $k_n$jet is not less than $2^n - 1$.

Proof. The representation of a function of the nth class contains $2^{n+1} - 1$ functions of one variable, which gives $(2^{n+1} - 1)k$ parameters when passing to the k-jet. Equating this variable to the number of parameters for the k-jet of functions of two variables, we obtain an equation for the index of the jet, $(2^{n+1} - 1)k = k(k + 3)/2$. Hence, $k_n = 2^{n+2} - 5$. If z is a function in Cl_n and P_n is a polynomial mapping from the k_n-jet of the defining functions of one variable to the k_n-jet of functions of two variables, then the dimensions of the spaces in the source and the target coincide and are equal to $d_n = (2^{n+1} - 1)(2^{n+2} - 5)$. Arguing by induction, one can readily note that the mapping P_n is invariant with respect to the action of the direct product of $2^n - 1$ counterparts of the group $\mathbf{C}^*$, which enables us to give a bound for the codimension.

8. DIFFERENTIATION, INTEGRATION, AND CHANGE OF THE VARIABLE

What happens to the complexity under differentiation?

Claim 8. *Let $N(z) \leqslant n$ and D be the operator of differentiation with respect to x or y. Then $N(Dz) \leqslant 2n$.*

Proof. One can use the induction on n and apply the chain rule for differentiating a composite function.

It is unclear what happens to the complexity under integration (the passage to an antiderivative with respect to one of the variables). However, if the support of a series is finite or is contained in the union of several straight lines (see Proposition 4), then, since the support of both the derivative and the antiderivative is obtained from the support of the original function by a parallel translation, it follows that, in the case in question, both the operations do nor worse the upper bound for the complexity.

What happens to the complexity under a change of a variable? The following claim holds.

Claim 9. *Let the complexity of a function $z(X, Y)$ as a function of X and Y does not exceed n, and let $X = A(x, y)$, $Y = B(x, y)$ be a change of variables, where the complexities of A and B does not exceed m. Then the complexity of $w = z(A(x, y), B(x, y))$ does not exceed $n + m$.*

9. EQUIPOLLENCE OF VARIABLES AND WEBS

Let $(\{x(u, v) = c_1\},\ \{y(u, v) = c_2\},\ \{z(u, v) = c_3\})$ be a 3-web on a plane [2], i.e., three pairwise transversal smooth one-dimensional foliations of the plane. These objects, when considered up to change of coordinates on the plane, are studied by tools of differential geometry. In particular, connection forms and curvature forms of a web are constructed. On the other hand, to a web one can assign a web function. To this end, one can, for example, consider the pair (x, y) as the independent coordinates instead of the pair (u, v); in this case, the web acquires the following form.

This is a family of horizontal lines $y = c_2$ and vertical lines $x = c_1$ together with the level curves of the web function, $z(x, y) = c_3$. If a web is analytic, then, as was noted in [1], the condition that the web function has the analytic complexity one is equivalent to the condition that the curvature of the web is identically zero or, equivalently, the web can be straighten into three families of parallel lines (a hexagonal web). From the point of view of the theory of webs, the variables (x, y, z) are absolutely equipollent. This implies the following claim.

Claim 10. *Let $x = X(y, z)$ and $y = Y(x, z)$ be relations obtained by resolving the relation $z = Z(x, y)$ with respect to x and y. If $N(Z) = 1$, then $N(x) = N(Y) = 1$.*

Proof. The proof follows from the interpretation of Z as a web function (and can be obtained immediately). Indeed, if $z = c(a(x)+b(y))$, then $x = a^{-1}(c^{-1}(z)-b(y))$ and $y = b^{-1}(c^{-1}(z)-a(x))$.

In this connection, the following question arises. *Does this equipollency of variables hold for functions whose complexity exceeds one?* This is a question of whether or not it is possible to treat the complexity as a geometric characteristic of the corresponding 3-web, or the complexity is sensitive to the separation of variables into independent and dependent ones.

Examples.

(1) Let variables (x, y, z) satisfy the relation $z^m + xz + y = 0$, where $m \geqslant 2$. This relation can be solved with respect to one of the variables in three ways. It can readily be seen that $N(x) = N(y) = 2$. As was proved in [1], the complexity of z is also equal to 2.

(2) Suppose now that variables (x, y, z) satisfy the relation $z^3 + xz^2 + yz + 1 = 0$. Then it can readily be seen that $N(x) = N(y) = 2$. As far as z is concerned, note that, after the change of variable $Z = z - x/3$, the equation becomes reduced, i.e., of the form $Z^3 + XZ + Y = 0$, where

$$X = y - x^2/3, \qquad Y = \frac{2}{27}x^3 - xy/3 + 1.$$

In this case, it follows from the bound for the reduced equation that $N(z) \leqslant 4$. This is better than the bound which immediately follows from Cardano's formula. It is clear that $N(z) > 1$, and therefore there are three possibilities (2, 3, and 4), the precise answer is not known.

Note that both the examples are related to rectilinear webs, i.e., webs in which each family is a family of straight lines in some coordinates.

10. COMPLEXITY OF MAPPINGS AND THE JACOBIAN CONJECTURE

Problems concerning analytic complexity can be transferred from functions to mappings. The following approach to estimating the *complexity of mappings of the plane* is possible. If $\phi = (x \to z_1(x, y), y \to z_2(x, y))$, then by the complexity $N(\phi)$ we mean $\max(N(z_1), N(z_2))$, where the complexity of the components is the ordinary analytic complexity. Here are several obvious observations.

Claim 11. *(a) $N(\phi \circ \psi) \leqslant N(\phi) + N(\psi)$.*

(b) The complexity of every linear mapping does not exceed one.

(c) The complexity of a triangular mapping $(x \to x + a(y), y \to y)$, where a is a nonconstant analytic function, is equal to one.

(d) The complexity of the direct $(x \to x/y, y \to y)$ and inverse $(x \to xy, y \to y)$ σ-processes is equal to one.

The well-known "Jacobian conjecture" is as follows. Let $X = A(x, y)$, $Y = B(x, y)$ be a locally invertible polynomial mapping of the two-dimensional complex space onto itself. Then this mapping is globally invertible. The local invertibility means that the Jacobian of the mapping, $J = A_x B_y - A_y B_x$, is nowhere zero. This is possible only if this is a constant, which can be assumed to be equal to zero to be definite. However, by the Young theorem, the global invertibility is equivalent to the condition that the mapping in question is a composition of invertible linear transformations and the so-called "triangular" mappings, i.e., transformations of the form $X = x + b(y)$, $Y = y$, where $b(y)$ is a polynomial. The Jacobian problem is naturally contained, because of Hironaka's theorem on resolution of singularities, in problems concerning rational mappings of the plane, namely, the σ-processes [4].

One can suggest the proof of the following simplifies version of this conjecture.

Claim 12 (the Jacobian conjecture for the first class)**.** *Suppose that $X = C(A(x) + B(y))$, $Y = R(P(x)+Q(y))$ is a locally invertible mapping of a two-dimensional complex space onto itself, where all functions A, B, C, P, Q, and R are polynomials. Then the Jacobian conjecture holds for the mapping in question.*

Proof. We assume that the mapping preserves the origin. The polynomials C and R are linear, because otherwise their derivatives have zeros, which implies that the Jacobian also vanishes at these zeros. Therefore, we may assume that $X = A(x) + B(y)$, $Y = P(x) + Q(y)$. We have $a(x)q(y) - p(x)b(y) = 1$, where (a, b, p, q) are the derivatives of (A, B, P, Q), respectively. Let a be identically zero. Then we see that p and b are mutually inverse constants, and the conjecture is true. In a similar way, the conjecture is also true if one of the other three polynomials (b, p, or q) vanishes identically. We may now assume that none of the four polynomials is identically zero. Then

$$q(y) = \frac{p(x)}{a(x)}b(y) + \frac{1}{a(x)}.$$

If b is not constant, then, choosing a value of y at which b vanishes, we see that a is a constant, say, α, and, moreover, p is a constant, say, π. Then $P(x) = \pi x$, $A(x) = \alpha x$, and $Q(y) = \frac{\pi}{\alpha}B(y) + \frac{y}{\alpha}$. We see that the mapping becomes triangular after a linear change of variable. In our assumption that b is not constant, we can replace b by a, p, or q. Finally, if all derivatives are constant, then the mapping is linear.

This assertion can be interpreted as follows. *A counterexample to the Jacobian conjecture must be constructed for mapping functions of the second class.* However, the simplest expressions of the second class, like

$$X = A(x) + B(x)C(y), \qquad Y = D(y) + E(x)G(y),$$

cannot give a desired example either. Therefore, it is reasonable to construct an example for expressions of the form

$$X = A(x)B(y) + C(x)D(y), \qquad Y = E(x)F(y) + G(x)H(y).$$

The author of the present paper was informed by Sadykov that he successfully obtained bounds for the complexity of discriminants and of some classes of hypergeometric functions, and that Sadykov also suggested an interesting approach to estimating the complexity of nodes, which uses the polynomial complexity.

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