IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

ON THE DIMENSION OF THE GROUP OF AUTOMORPHISMS OF AN ANALYTIC HYPERSURFACE

This content has been downloaded from IOPscience. Please scroll down to see the full text.

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 130.56.64.29
This content was downloaded on 06/10/2015 at 19:31

Please note that terms and conditions apply.

Изв. Акад. Наук СССР
Сер. Мат. Том 43 (1979), Вып. 2

Math. USSR Izvestija
Vol. 14 (1980), No. 2

ON THE DIMENSION OF THE GROUP OF AUTOMORPHISMS OF AN ANALYTIC HYPERSURFACE

UDC 517.5

V. K. BELOŠAPKA

Abstract. Let M be a nondegenerate real analytic hypersurface in C^2, let $\xi \in M$, and let G_ξ consist of the automorphisms of M fixing the point ξ. Then, as follows from a theorem of Moser, the real dimension of G_ξ does not exceed 5. Here is is shown that 1) dimensions 2, 3, and 4 cannot be realized, but for 0, 1, and 5 examples are given; 2) if the point ξ is not umbilical, then G_ξ consists of not more than two mappings.

Bibliography: 4 titles.

Introduction

In this paper we study groups of automorphisms of real analytic hypersurfaces. Élie Cartan [1] observed that from results of Tresse [2] it follows that if the group of automorphisms of a real hypersurface in $\mathbf{C}^2$ contains a family of more than three parameters, then it depends on eight real parameters. Using the theorem of Moser on reducing a hypersurface to normal form [3], we obtain the following results in this direction.

THEOREM 1. *Let M be a nondegenerate real analytic hypersurface in $\mathbf{C}^n$, let $\xi \in M$, and let G_ξ be the stability group of the point ξ, i.e. G_ξ consists of the automorphisms of M leaving ξ fixed. Then the real dimension of G_ξ cannot equal n^2.*

From Moser's theorem it follows that this dimension is no larger than $n^2 + 1$.

THEOREM 2. *Let M be a nondegenerate real analytic hypersurface in $\mathbf{C}^2$, and let $\xi \in M$. If $\dim_{\mathbf{R}} G_\xi > 1$, then the hypersurface M is spherical, and consequently $\dim_{\mathbf{R}} G_\xi = 5$.*

Theorem 2 gives a complete description of the dimension of stability groups of real analytic hypersurfaces in $\mathbf{C}^2$. Moser's theorem admits the following possibilities for the dimension of such groups: 0, 1, 2, 3, 4, and 5. Dimension 5 is realized by the hyperquadric

$$S = \{(z, u + iv) \in \mathbf{C}^2 : v = |z|^2\}.$$

Dimensions 4, 3, and 2 are forbidden by Theorem 2. Dimension 1 is realized by the

1980 *Mathematics Subject Classification.* Primary 32C05, 53A55.

Copyright © 1980, American Mathematical Society

hypersurface

$$\{(z, u+iv)\in \mathbf{C}^2 : v = |z|^2 + |z|^8\}.$$

Dimension zero is the general case (see Theorem 3).

In the original version of this article Theorem 2 allowed dimension 2. A. V. Loboda observed that for dimension 2 to be realized it is necessary that the parameter λ be free, but this means that the polynomial H_e has a very special form, which does not allow any freedom to the parameter a (see §3, p. 235). The author thanks him for this important remark.

THEOREM 3. *Let M be a nondegenerate real analytic hypersurface in $\mathbf{C}^2$ and let $\{c_{mn}(n)\}$ be the coefficients of its normal form in a neighborhood of a point $\xi \in M$. If the point ξ is not umbilical, i.e. $c_{42}(0) \neq 0$, then G_ξ consists of not more than two mappings.*

If in addition $(c_{42} \cdot c_{25} + 3c_{24} \cdot c_{43})|_{u=0} \neq 0$, then G_ξ contains no mapping besides the identity.

Of analogous earlier results, that of Burns, Shnider, and Wells [4] should be mentioned. They showed that in the space of functions defining strongly pseudoconvex domains, the set of functions defining domains without automorphisms is a set of Baire second category.

The author thanks his scientific advisor, A. G. Vituškin.

§1. Study of the initial components of a mapping

We consider in the complex linear space $\mathbf{C}^{n+1}$ $(n \geqslant 1)$ with coordinate functions $z^1, \ldots, z^n$, $w = u + iv$ a real analytic hypersurface M. Let ρ be its defining function; that is, ρ is a real-valued real analytic function in a domain $V \subset \mathbf{C}^{n+1}$ such that $M = \{\xi \in V: \rho(\xi) = 0\}$, where grad $\rho \neq 0$ at points of M. Further let $0 \in V$ and $\rho(0) = 0$; that is, M contains the origin. In a neighborhood of 0 the equation $\rho(\xi) = 0$ may be solved for one of the real coordinates, and after a linear change the equation of M takes the form

$$v = F(z, \bar{z}; u), \tag{1}$$

where $z = (z^1, \ldots, z^n)$ and F is a real-valued real analytic function with $dF|_0 = 0$.

We make the following assumption on the hypersurface M. Its Levi form at 0

$$\langle z, z\rangle = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} z^\alpha \bar{z}^\beta, \quad \text{where} \quad h_{\alpha\beta} = \left.\frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta}\right|_0,$$

is nondegenerate, i.e. $\det(h_{\alpha\beta}) \neq 0$.

In the space of convergent power series in the variables z, $\bar{z}$, and u we introduce the following decomposition:

$$G(z, \bar{z}; u) = \sum_{k,l} G_{kl}(z, \bar{z}; u),$$

where G_{kl} satisfies

$$G_{kl}(t_1 z, t_2 \bar{z}; u) = t_1^k t_2^l G_{kl}(z, \bar{z}; u)$$

for all complex numbers t_1 and t_2. We will call this decomposition the *δ-decomposition,* and G_{kl} the (k, l) *δ-component* of G.

Let $\mathfrak{F}$ be the space of convergent real power series in z, $\bar{z}$, and u without constant or linear terms, $\mathfrak{R}$ the subspace consisting of series of the form

$$R = \sum_{\min(k,l)\leqslant 1} R_{kl} + r_{1\bar{1}}\langle z, z\rangle + (r_{10} + r_{01})\langle z, z\rangle^2 + r_{00}\langle z, z\rangle^3,$$

and $\mathfrak{N}$ the subspace defined by

$$\mathfrak{N} = \{ N \in \mathfrak{F}\colon N_{kl} = 0, \text{ if } \min(k, l) \leqslant 1, \ \operatorname{tr} N_{22} = \operatorname{tr}^2 N_{32} = \operatorname{tr}^3 N_{33} = 0\}.$$

The operator tr is a linear operator on the space $\mathfrak{F}$ taking series of type (k, l) to series of type $(k - 1, l - 1)$ (for the definition of this operator see [3], p. 232). The properties of the operator tr which we will need we state as the following lemma.

LEMMA 1 (see [3], p. 233). a) $\mathfrak{F} = \mathfrak{R} \oplus \mathfrak{N}$, *i.e. every series* $F \in \mathfrak{F}$ *can be uniquely represented in the form* $F = R + N$, *where* $R \in \mathfrak{R}$ *and* $N \in \mathfrak{N}$.

b) *This decomposition is invariant with respect to linear mappings preserving the form* $\langle z, z\rangle$. *In particular, if* $N(z, \bar{z}) \in \mathfrak{N}$ *and* $\mathscr{U} \in SU(n)$, *then* $N(\mathscr{U} z, \overline{\mathscr{U}} \bar{z}) \in \mathfrak{N}$.

c) *The subspace* $\mathfrak{N}$ *is an ideal of* $\mathfrak{F}$.

We also remark that for $n = 1$ the condition $\operatorname{tr} N_{22} = \operatorname{tr}^2 N_{32} = \operatorname{tr}^3 N_{33} = 0$ means that $N_{22} = N_{32} = N_{33} = 0$.

By making in a neighborhood of 0 a holomorphic change of coordinates preserving the form of equation (1), we may change the function F. We state a fundamental result of Moser.

LEMMA 2 (see [3], p. 234, Theorem 2.2). *By a holomorphic change of coordinates the equation of the hypersurface* M *may be brought to the form*

$$v = \langle z, z\rangle + N(z, \bar{z}; u), \quad \textit{where} \quad N \in \mathfrak{N}. \tag{2}$$

This change is called *the reduction of* M *to normal form.*

Now let the hypersurface M be reduced to normal form and let (2) be its equation. Further, let h be an automorphism of this hypersurface leaving the origin fixed, i.e. $h\colon V \longrightarrow \mathbf{C}^{n+1}$ is a holomorphic mapping such that

$$\det h'|_0 \neq 0, \tag{3}$$

$$h(0) = 0, \tag{4}$$

$$h(M) \cap V \subset M. \tag{5}$$

The first n coordinates of the function h are denoted by f, and the last is denoted by g, i.e. $h = \binom{f}{g}$. Then from (5) follows the identity

$$\begin{aligned} &\operatorname{Im} g(z, u + i(\langle z, z\rangle + N)) \\ &= \langle f(z, u + i(\langle z, z\rangle + N)), f(z, u + i(\langle z, z\rangle + N))\rangle \\ &+ N(f(z, u + i(\langle z, z\rangle + N)), \bar{f}(z, u + i(\langle z, z\rangle + N)), \operatorname{Re} g(z, u + i(\langle z, z\rangle + N))). \end{aligned} \tag{6}$$

This identity is the analytic expression of the fact that when the argument of h satisfies (2), the value of h also satisfies (2). In order to make the calculations connected with this identity manageable, we introduce the following decomposition of the space $\mathfrak{F}$:

$$F(z, \bar{z}; u) = \sum_k F_k(z, \bar{z}; u),$$

where

$$F_k(tz, t\bar{z}; t^2u) = t^k F_k(z, \bar{z}; u)$$

for all complex numbers t. We call this decomposition the *ϵ-decomposition.* We introduce an analogous decomposition of the space of convergent power series in z and w:

$$g(z, w) = \sum_k g_k(z, w),$$

where the polynomials g_k satisfy

$$g_k(tz, t^2w) = t^k g_k(z, w)$$

for all complex numbers t.

Equation (2) is rewritten as

$$v = \langle z, z\rangle + \sum_k H_k(z, \bar{z}; u),$$

where H_k is the kth ϵ-component of the series N. From the condition $N \in \mathfrak{N}$ it follows that the first component of N which can be nonvanishing is $H_4 = N_{22}(z, \bar{z})$, and H_6 when $n = 1$. We rewrite (6) in this notation:

$$\begin{aligned}
&\operatorname{Re} ig(z, u + i(\langle z, z\rangle + H_4 + \ldots))\\
&+ \langle f(z, u + i(\langle z, z\rangle + H_4 + \ldots)), f(z, u + i(\langle z, z\rangle + H_4 + \ldots))\rangle\\
&+ H_4(f(z, u + i(\langle z, z\rangle + H_4 + \ldots)), \bar{f}(z, u + i(\langle z, z\rangle + H_4 + \ldots)) + \ldots = 0. \quad (7)
\end{aligned}$$

We consider the hyperquadric

$$S = \{(z, w) \in \mathbf{C}^{n+1} : v = \langle z, z\rangle\}.$$

This surface is transformed via the linear-fractional mapping

$$Z^\alpha = \frac{2z^\alpha}{w + i}, \quad W = \frac{w - i}{w + i}$$

into the surface $\langle Z, Z\rangle + W\bar{W} = 1$, which, if $\langle z, z\rangle = z^1\bar{z}^1 + \cdots + z^n\bar{z}^n$, is the unit sphere in $\mathbf{C}^{n+1}$.

The group of automorphisms of S leaving 0 fixed consists of the linear-fractional mappings

$$h^s = \begin{pmatrix} f^s \\ g^s \end{pmatrix}$$

of the form

$$f^s = \lambda' \mathcal{U}' (z + wa)/\Delta, \quad g^s = \pi' |\lambda'|^2 w/\Delta,$$
$$\Delta = 1 - [2i\langle z, a'\rangle + (r' + i\langle a', a'\rangle) w], \tag{8}$$

where $\lambda' \in \mathbf{C}^1 - \{0\}$, $\pi' = \pm 1$ and $\mathcal{U}' \in SU(n, \pi)$, i.e. $\mathcal{U}'$ is a linear transformation such that $\langle \mathcal{U}'z, \mathcal{U}'z\rangle = \pi\langle z, z\rangle$ and det $\mathcal{U}' = 1$, $a' \in \mathbf{C}^n$, $r' \in \mathbf{R}^1$ (see [3], p. 225).

We remark that if h^S is an automorphism of S, then we have the identity, analogous to (7),

$$\operatorname{Re} ig^s(z, u + i\langle z, z\rangle) + \langle f^s(z, u + i\langle z, z\rangle), f^s(z, u + i\langle z, z\rangle)\rangle = 0. \tag{9}$$

We will view (7) as an equation for the unknown mapping h. Separating the ϵ-components of this identity and successively setting them equal to zero, we obtain equations for the successive ϵ-components of h.

The equations obtained from the initial components of (7) turn out to be insufficient for a unique determination of the corresponding components of h. It becomes necessary to introduce parameters. The aim of this section is to show that if these parameters are introduced in a manner compatible with the system of parameters giving the automorphisms of S, then the initial components of h turn out to be equal to the corresponding components of h^S.

Turning now to the calculations, we denote the kth ϵ-component of (7) by Φ_k. We have $\Phi_k = 0$, $k = 0, 1, 2, \ldots$.

First step: $\Phi_1 = \operatorname{Re} ig_1$, $g_1 = \alpha z$. We obtain $\alpha = 0$, i.e. $g_1 = 0$.

Second step: $\Phi_2 = \operatorname{Re}(ig_2 + \langle f_1, f_1\rangle)|_{v=\langle z,z\rangle}$, $f_1 = Az$ and $g_2 = \alpha w + \beta(z)$, where A is a linear operator on $\mathbf{C}^n$, $\alpha \in \mathbf{C}^1$, and β is a quadratic form in z. Substituting, we obtain

$$\Phi_2 = -\operatorname{Im} \alpha \cdot u - \operatorname{Re} \alpha\langle z, z\rangle + \operatorname{Re} i\beta(z) + \langle Az, Az\rangle.$$

Separating the ϵ-components of Φ_2, we find

$$\begin{array}{ll} (2, 0) & \beta(z) = 0, \\ (1, 1) & \langle Az, Az\rangle - \operatorname{Re} \alpha\langle z, z\rangle = 0, \\ (0, 0) & \operatorname{Im} \alpha = 0. \end{array}$$

Therefore we may write $f_1 = \lambda \mathcal{U} z$ and $g_2 = \pi|\lambda|^2 w$, where $\lambda \in \mathbf{C}^1$, $\pi = \pm 1$ and $\mathcal{U} \in SU(n, \pi)$ (see (8)).

We note that if $\lambda = 0$, then det $h'|_0 = 0$, so from (3) it follows that $\lambda \neq 0$.

For the sequel we need the following fact.

LEMMA 3. *Let $\varphi(z)$ and $\psi(z)$ be two holomorphic vector-valued functions of $z^1, \ldots, z^n$. Then from the identity $\langle \varphi(z), z\rangle = \langle \psi(z), z\rangle$ is follows that $\varphi(z) = \psi(z)$.*

PROOF. We have $\langle (\varphi - \psi)(z), z\rangle = 0$, i.e.

$$\sum_{\alpha,\beta} h_{\alpha\beta} (\varphi - \psi)^\alpha (z) \bar{z}^\beta = 0.$$

Differentiating with respect to $\bar{z}^{\beta_0}$, we obtain

$$\sum_{\alpha} h_{\alpha\beta_0} (\varphi - \psi)^\alpha (z) = 0.$$

Writing out this identity for $\beta_0 = 1, \ldots, n$, we obtain a linear system in $(\varphi - \psi)(z)$. Its determinant $\det(h_{\alpha\beta}) \neq 0$, and so $(\varphi - \psi)(z) = 0$. The lemma is proved.

Third step: $\Phi_3 = \operatorname{Re}(ig_3 + 2\langle f_2, f_1\rangle)|_{v=\langle z,z\rangle}$, $f_2 = A(z) + Bw$ and $g_3 = wl(z) + \beta(z)$, where $A(z)$ is a vector-valued quadratic form in z, $B \in \mathbf{C}^n$, $l(z)$ is a linear form and $\beta(z)$ is a cubic form in z.

We represent $l(z)$ as $\langle z, \alpha\rangle$, where $\alpha \in \mathbf{C}^n$. Substituting, we obtain

$$\begin{aligned}\Phi_3 &= \operatorname{Re}(i(u+i\langle z, z\rangle)\langle z, \alpha\rangle + i\beta(z) + 2\langle A(z), \lambda\mathcal{U}z\rangle \\ &\quad + 2(u+i\langle z, z\rangle)\langle B, \lambda\mathcal{U}z\rangle) \\ &= \operatorname{Re}(iu\langle z, \alpha\rangle - \langle z, z\rangle\langle z, \alpha\rangle + i\beta(z) + 2\langle A(z), \lambda\mathcal{U}z\rangle \\ &\quad + 2u\langle B, \lambda\mathcal{U}z\rangle + 2i\langle z, z\rangle\langle B, \lambda\mathcal{U}z\rangle.\end{aligned}$$

We separate into ϵ-components:

$$\begin{array}{ll}(3,0) & \beta(z)=0, \\ (2,1) & -\langle z, z\rangle\langle z, \alpha\rangle + 2\langle A(z), \lambda\mathcal{U}z\rangle - 2i\langle z, z\rangle\langle\lambda\mathcal{U}z, B\rangle = 0, \\ (1,0) & i\langle z, \alpha\rangle + 2\langle\lambda\mathcal{U}z, B\rangle = 0\end{array}$$

We have $\langle z, \alpha\rangle = 2i\langle\lambda\mathcal{U}z, B\rangle$, or $\langle z, \alpha\rangle = \langle z, \mathcal{U}^{-1}(-2i\pi\bar{\lambda}B)\rangle$, whence in view of Lemma 3 we obtain $\alpha = \mathcal{U}^{-1}(-2i\pi\bar{\lambda}B)$ or $-2i\pi\bar{\lambda}B = \mathcal{U}\alpha$. Since $\lambda \neq 0$, we may set $\alpha = -2i\pi|\lambda|^2 a$, where $a \in \mathbf{C}^n$, and then we obtain $B = \lambda\mathcal{U}a$.

Moreover,

$$2\langle A(z), \lambda\mathcal{U}z\rangle = 2i\langle z, z\rangle\langle\lambda\mathcal{U}z, B\rangle + \langle z, z\rangle\langle z, \alpha\rangle,$$

or

$$\langle A(z), \lambda\mathcal{U}z\rangle = 2i|\lambda|^2\langle z, z\rangle\langle z, a\rangle.$$

Replacing z by $\mathcal{U}^{-1}z$ in this identity, we obtain

$$\langle A(\mathcal{U}^{-1}z), z\rangle = \langle 2i\lambda\langle\mathcal{U}^{-1}z, a\rangle z, z\rangle.$$

Consequently $A(\mathcal{U}^{-1}z) = 2i\lambda\langle\mathcal{U}^{-1}z, a\rangle z$, and finally $A(z) = 2i\lambda\langle z, a\rangle z$.

Thus

$$f_2 = \lambda\mathcal{U}(wa + 2i\langle z, a\rangle z),$$

$$g_3 = 2i\pi|\lambda|^2 w\langle z, a\rangle.$$

The computation of Φ_4 is rather complicated; therefore we give it in detail. For this we introduce the following notation. We will write $(x+1)^7 \longrightarrow 7x^6$ to signify that by selecting from the expression $(x+1)^7$ the terms with a certain property we obtain $7x^6$. In this case it is terms of degree 6; in our calculations it will be terms containing members of a given ϵ-weight.

Fourth step: The term entering into Φ_4 not containing H_4 is

$$\operatorname{Re}(ig_4 + 2\langle f_3, f_1\rangle + \langle f_2, f_2\rangle)|_{v=\langle z,z\rangle}.$$

We display the terms of (7) of ϵ-weight 4 containing H_4:

$$\operatorname{Re} ig \to \operatorname{Re} i\pi|\lambda|^2 w \to \operatorname{Re} i|\lambda|^2 iH_4(z, \bar{z}) = -\pi|\lambda|^2 H_4(z, \bar{z}),$$

$$H_4(f, \bar{f}) \to H_4(f_1, \bar{f}_1) = H_4(\lambda \mathcal{U} z, \lambda \overline{\mathcal{U} z}).$$

Now

$$\Phi_4 = \operatorname{Re}(ig_4 + 2\langle f_3, f_1\rangle + \langle f_2, f_2\rangle)|_{v=\langle z, z\rangle}$$

$$+ H_4(\lambda \mathcal{U} z, \lambda \overline{\mathcal{U} z}) - \pi|\lambda|^2 H_4(z, \bar{z}),$$

$$f_3 = A(z) + wB(z), \qquad g_4 = \alpha w^2 + \beta(z) w + \gamma(z),$$

where $A(z)$ is a vector-valued form of third order, $B(z)$ is a linear operator, $\alpha \in \mathbf{C}^1$, and $B(z)$ and $\gamma(z)$ are forms of second and fourth order respectively. Substituting and separating into ϵ-components, we obtain

$$\begin{array}{ll}
(4, 0) & \gamma(z) = 0, \\
(3, 1) & -\langle z, z\rangle \beta(z) + 2\langle A(z), \lambda \mathcal{U} z\rangle + 4\pi|\lambda|^2 \langle z, z\rangle \langle z, a\rangle^2 = 0, \\
(2, 2) & \operatorname{Re}(-i\alpha)\langle z, z\rangle^2 + \langle z, z\rangle 2\operatorname{Re}(i\langle B(z), \lambda \mathcal{U} z\rangle) \\
\multicolumn{2}{l}{+H_4(\lambda \mathcal{U} z, \lambda \overline{\mathcal{U} z}) - \pi|\lambda|^2 H_4(z, \bar{z}) = 0,} \\
(2, 0) & i\beta(z) + 4i\pi|\lambda|^2 \langle z, a\rangle = 0, \\
(1, 1) & -2\operatorname{Re} \alpha\langle z, z\rangle + 2\operatorname{Re}\langle B z, \lambda \mathcal{U} z\rangle = 0, \\
(0, 0) & \operatorname{Re}(i\alpha) + \pi|\lambda|^2 \langle a, a\rangle = 0.
\end{array}$$

The (0, 0) equation gives $\operatorname{Im} \alpha = \pi|\lambda|^2\langle a, a\rangle$. We set $\alpha = \pi|\lambda|^2(r + i\langle a, a\rangle)$, where $r \in \mathbf{R}^1$. From (2, 0) we obtain $B(z) = -4\pi|\lambda|^2\langle z, a\rangle^2$, in view of which (3, 1) takes the form

$$\langle A(z), \lambda \mathcal{U} z\rangle = -4\pi|\lambda|^2 \langle z, z\rangle \langle z, a\rangle^2,$$

whence

$$\langle A(\mathcal{U}^{-1} z), z\rangle = \langle -4\lambda\langle z, \mathcal{U} a\rangle^2 z, z\rangle.$$

From Lemma 3 we obtain

$$A(\mathcal{U}^{-1} z) = -4\lambda\langle z, \mathcal{U} a\rangle^2 z,$$

or

$$A(z) = -4\lambda\langle z, a\rangle^2 \mathcal{U} z.$$

LEMMA 4. *From the identity* (2, 2) *it follows that*

$$H_4(\lambda \mathcal{U} z, \lambda \overline{\mathcal{U} z}) = \pi|\lambda|^2 H_4(z, \bar{z}).$$

PROOF. By b) of Lemma 1, $H_4(\mathcal{U} z, \overline{\mathcal{U} z}) \in \mathfrak{N}$, and so

$$(H_4(\lambda \mathcal{U} z, \lambda \overline{\mathcal{U} z}) - \pi|\lambda|^2 H_4(z, \bar{z})) \in \mathfrak{N}.$$

We observe further that the remaining terms of (2, 2) are elements of the supspace $\mathfrak{N}$. Separating (2, 2) by $\mathfrak{N}$-components, we obtain the desired relation (see Lemma 1, a)).

Now from (2, 2) it follows that

$$\operatorname{Im}\langle B(z), \lambda \mathcal{U} z\rangle = \pi|\lambda|^2(\langle a, a\rangle\langle z, z\rangle + 2\langle z, a\rangle\langle a, z\rangle).$$

Taking (1, 1) into account, we obtain

$$\langle B(z), \lambda\mathcal{U}z\rangle = \pi|\lambda|^2((r+i\langle a, a\rangle)\langle z, z\rangle + 2i\langle z, a\rangle\langle a, z\rangle),$$

or

$$\langle B\mathcal{U}^{-1}z, z\rangle = \langle\lambda((r+i\langle a, a\rangle)z + 2i\langle z, \mathcal{U}a\rangle\mathcal{U}a), z\rangle,$$

whence we find that

$$B\mathcal{U}^{-1}z = \lambda((r+i\langle a, a\rangle) + 2i\langle z, \mathcal{U}a\rangle\mathcal{U}a),$$

or

$$Bz = \lambda\mathcal{U}((r+i\langle a, a\rangle)z + 2i\langle z, a\rangle a).$$

Thus

$$f_3 = \lambda\mathcal{U}((r+i\langle a, a\rangle)z - 4\langle z, a\rangle^2 z + 2i\langle z, a\rangle wa),$$
$$g_4 = \pi|\lambda|^2((r+i\langle a, a\rangle)w^2 - 4\langle z, a\rangle^2 w).$$

We have obtained an interesting result.

LEMMA 5. *If in* (8) *we set* $\lambda' = \lambda$, $\pi' = \pi$, $\mathcal{U}' = \mathcal{U}$, $a' = a$, *and* $r' = r$, *then for* $k = 1, 2, 3, 4$ *we have*

$$f^s_{k-1} = f_{k-1}, \quad g^s_k = g_k. \tag{10}$$

PROOF. The reader may check this directly by separating (8) into ϵ-components.

§2. The existence of forbidden dimensions for the stability group

For $k > 4$ the situation changes in the following sense.

LEMMA 6. *Let* $k > 4$. *For fixed values of the parameters* λ, $\mathcal{U}$, a, *and* r *the equation* $\Phi_k = 0$ *admits uniquely defined* f_{k-1} *and* g_k.

This lemma is a simple consequence of the work of Moser [3]. We present the proof in the notation of that paper. We have

$$\Phi_k = L(f_{k-1}, g_k) + \text{ terms depending on } f_{\mu-1} \text{ and } g_\mu,$$

where $\mu < k$. The equation $Lh = F \pmod{\mathfrak{N}}$ admits a unique solution $h \in D_0$ (see [3], p. 234). But for $k > 4$

$$h_k = \begin{pmatrix} f_{k-1} \\ g_k \end{pmatrix} \in \mathfrak{D}_0.$$

The lemma is proved.

Thus for $k > 4$ the equation $\Phi_k = 0$ allows no freedom for the introduction of new parameters. However, in calculating f_{k-1} and g_k we will not in general use all the information contained in this equation. The aim of this section is to obtain further identities which give the connection with the parameters previously introduced.

In the case where the hypersurface M is spherical, i.e. coincides with the hyperquadric S, these parameters are free, so we can hope to obtain such a relation only in the case where M is nonspherical.

Now let M be nonspherical and let e be the index of the first nonvanishing component in the ϵ-decomposition of $N(z, \bar{z}; u)$, i.e.

$$v = \langle z, z\rangle + H_e(z, \bar{z}; u) + \dots \tag{11}$$

is the equation of M.

We note that $e \geqslant 4$, and $e \geqslant 6$ for $n = 1$.

We denote by Φ_k^s the kth ϵ-component of (9) with $\lambda' = \lambda$, $\pi' = \pi$, $\mathcal{U}' = \mathcal{U}$, $a' = a$, and $r' = r$, and by D_k we mean the sum of those terms entering into Φ_k which depend on H_p for some p.

LEMMA 7. *For* $k = 1, 2, \dots, e-1$

$$f_{k-1} = f_{k-1}^s, \quad g_k = g_k^s. \tag{12}$$

PROOF. We proceed by induction on k. For $k = 1, 2, 3, 4$ the assertion is proved (see Lemma 5). Suppose for a given p, $4 < p < e - 1$, equation (12) holds for $k = 1, \dots, p-1$. This means, in view of (11), that $\Phi_p = 0$ and $\Phi_p^s = 0$ will agree as equations for f_{p-1} and g_p; but by Lemma 6 this equation has only a single solution. Accordingly (12) is fulfilled for $k = p$. The lemma is proved.

LEMMA 8. *The following equations hold*:

$$H_e(\lambda\mathcal{U}z, \lambda\overline{\mathcal{U}z};\ |\lambda|^2 u) = \pi|\lambda|^2 H_e(z, \bar{z};\ u), \tag{13}$$

$$f_{e-1} = f_{e-1}^s, \quad g_e = g_e^s. \tag{14}$$

PROOF. We have

$$\Phi_e = \operatorname{Re}(ig_e + 2\langle f_{e-1}, f_1\rangle + 2\langle f_{e-2}, f_2\rangle + \dots)|_{v=\langle z,z\rangle} + D_e,$$

$$\Phi_e^s = \operatorname{Re}(ig_e^s + 2\langle f_{e-1}^s, f_1\rangle + 2\langle f_{e-2}^s, f_2\rangle + \dots)|_{v=\langle z,z\rangle}.$$

We set $\tilde{f} = f - f^s$ and $\tilde{g} = g - g^s$. Then from (12), (7), and (9) it follows that

$$\operatorname{Re}(i\tilde{g}_e + 2\langle\tilde{f}_{e-1}, \lambda\mathcal{U}z\rangle)|_{v=\langle z,z\rangle} + D_e = 0.$$

We compute D_e:

$$\operatorname{Re} ig \to \operatorname{Re} i\pi|\lambda|^2 w \to \pi|\lambda|^2 H_e(z, \bar{z};\ u),$$

$$H_e(f, \bar{f};\ \operatorname{Re} g) \to H_e(f, \bar{f};\ \operatorname{Re} g_2) = H_e(\lambda\mathcal{U}z, \overline{\lambda\mathcal{U}z};\ |\lambda|^2 u).$$

Thus

$$D_e = H_e(\lambda\mathcal{U}z, \overline{\lambda\mathcal{U}z}, |\lambda|^2 u) - \pi|\lambda|^2 H_e(z, \bar{z};\ u)$$

and consequently $D_e \in \mathfrak{N}$ (see Lemma 1 b)), i.e.

$$\operatorname{Re}[i\widetilde{g}_e + 2\langle \widetilde{f}_{e-1}, \lambda \mathcal{U} z\rangle]|_{v=\langle z, z\rangle} \equiv 0 \,(\operatorname{mod} \mathfrak{N}).$$

But from this it follows (see [3], p. 234) that $\widetilde{f}_{e-1} = 0$ and $\widetilde{g}_e = 0$; so $D_e = 0$. The lemma is proved.

COROLLARY 1. *If the form $\langle z, z\rangle$ is definite (i.e. all of its eigenvalues are of one sign), then $|\lambda| = 1$.*

PROOF. We observe at once that $\pi = 1$. Further let the (m, l) δ-component of the polynomial $H_e(z, \bar{z}; u)$ vanish. Separating (13) into (m, l)-components, we obtain

$$h_{ml}(\lambda \mathcal{U} z, \overline{\lambda \mathcal{U} z})(|\lambda|^2 u)^k = |\lambda|^2 h_{ml}(z, \bar{z}) u^k$$

or

$$h_{ml}(\mathcal{U} z, \overline{\mathcal{U} z}) = \mu h_{ml}(z, \bar{z}),$$

where $\mu = \lambda^{1-m-k}\bar{\lambda}^{1-l-k}$. We remark that $|\mu| = |\lambda|^{2-e}$.

Consider in $\mathbf{C}^n$ the surface K defined by the equation $\langle z, z\rangle = \sigma$, where

$$\sigma = \begin{cases} 1, & \text{if } \langle z, z\rangle \text{ is positive definite,} \\ -1, & \text{if } \langle z, z\rangle \text{ is negative definite.} \end{cases}$$

Since the form $\langle z, z\rangle$ is definite, the surface K is compact. Let the maximum of the function $|h_{ml}(z, \bar{z})|$ on this surface be attained at the point $z_0 \in K$. We obtain

$$|\mu| = \frac{|h_{ml}(\mathcal{U} z_0, \overline{\mathcal{U} z_0})|}{|h_{ml}(z_0, \bar{z}_0)|} \leqslant 1,$$

whence it follows that $|\lambda| \geqslant 1$.

In the same way, from the equation

$$h_{ml}(\mathcal{U}^{-1} z, \overline{\mathcal{U}^{-1} z}) = \mu^{-1} h_{ml}(z, \bar{z})$$

we obtain that $|\mu^{-1}| \leqslant 1$ and $|\lambda| \leqslant 1$. The corollary is proved.

LEMMA 9.

$$\begin{aligned} D_{e+1} = \pi|\lambda|^2 2\operatorname{Re}\Big(&-2i\langle z, a\rangle H_e + (u + i\langle z, z\rangle)\partial H_e(a) \\ &+ 2i\langle z, a\rangle \partial H_e(z) + i\langle z, a\rangle(u + i\langle z, z\rangle)\frac{\partial}{\partial u} H_e\Big) \\ &+ (H_{e+1}(\lambda \mathcal{U} z, \overline{\lambda \mathcal{U} z}; |\lambda|^2 u) - \pi|\lambda|^2 H_{e+1}(z, \bar{z}; u)). \end{aligned} \tag{15}$$

where $\partial H(v)$ is the value of the linear form ∂H on the vector v.

PROOF. We have

$$\operatorname{Re} ig \to (a) \to \operatorname{Re} i\pi|\lambda|^2 w \to -\pi|\lambda|^2 H_{e+1}(z, \bar{z}; u),$$

$$(b) \to \operatorname{Re} i(2i\pi|\lambda|^2\langle z, a\rangle w) \to -2\pi|\lambda|^2 \operatorname{Re}(i\langle z, a\rangle) H_e(z, \bar{z}; u).$$

We set

$$\varphi=\lambda^{-1}\mathcal{U}^{-1}f, \quad \psi=|\lambda|^{-2}g,$$

$$\langle f, f\rangle = \pi|\lambda|^2\langle\varphi,\varphi\rangle \to \pi|\lambda|^2\cdot 2\operatorname{Re}(\langle z, a\rangle\bar{w})$$

$$\to -2\pi|\lambda|^2\operatorname{Re}(i\langle z, a\rangle)\cdot H_e(z,\bar{z};u),$$

$$H_e(f,\bar{f};\operatorname{Re} g) = H_e(\lambda\mathcal{U}\varphi, \overline{\lambda\mathcal{U}\varphi}, |\lambda|^2\psi) = \pi|\lambda|^2 H_e(\varphi,\bar{\varphi};\operatorname{Re}\psi)$$

(see (13)). Then, decomposing $H_e(\varphi + \Delta\varphi, \bar{\varphi} + \overline{\Delta\varphi}; \operatorname{Re}\psi + \operatorname{Re}\Delta\psi)$ by increasing degrees we see that the terms of weight $e + 1$ of the expression $\pi|\lambda|^2H_e(\varphi_1 + \varphi_2 + \dots + \bar{\varphi}_1 + \bar{\varphi}_2 + \dots; \operatorname{Re}\psi_2 + \operatorname{Re}\psi_3 + \dots)$ are

$$\pi|\lambda|^2\Big(\partial H_e(\varphi_1,\bar{\varphi}_1,\operatorname{Re}\psi_2)(\varphi_2) + \bar{\partial} H_e(\varphi_1,\bar{\varphi}_1,\operatorname{Re}\psi_2)(\bar{\varphi}_2)$$
$$+\frac{\partial}{\partial u}H_e(\varphi_1,\bar{\varphi}_1;\operatorname{Re}\psi_2)(\operatorname{Re}\psi_3)\Big)\Big|_{v=\langle z,z\rangle}$$

Substituting the calculated values of $\varphi_1, \varphi_2, \psi_2$, and ψ_3, we obtain

$$\pi|\lambda|^2\Big(\partial H_e(z,\bar{z};u)(wa + 2i\langle z,a\rangle z) + \partial H_e(z,\bar{z};u)(\overline{wa + 2i\langle z,a\rangle z})$$
$$+\frac{\partial}{\partial u}H_e(z,\bar{z};u)\,2\operatorname{Re}(iw\langle z,a\rangle)\Big)\Big|_{v=\langle z,z\rangle}$$
$$=\pi|\lambda|^2\,2\operatorname{Re}\Big(\partial H_e(z,\bar{z};u)(wa + 2i\langle z,a\rangle z) + i\langle z,a\rangle w\frac{\partial}{\partial u}H_e(z,\bar{z};u)\Big)_{v=\langle z,z\rangle}$$

$$H_{e+1}(f,\bar{f};\operatorname{Re} g) \to H_{e+1}(f_1,\bar{f}_1;\operatorname{Re} g_2) = H_{e+1}(\lambda\mathcal{U}z, \overline{\lambda\mathcal{U}z}; |\lambda|^2 u).$$

Summing, we finally obtain (15).

COROLLARY 2. *$\tilde{f}_e$ and $\tilde{g}_{e+1}$ do not depend on r.*

PROOF. As in the proof of Lemma 8, we obtain from (7), (9), (12), and (14)

$$\operatorname{Re}(i\tilde{g}_{e+1} + 2\langle\tilde{f}_e, \lambda\mathcal{U}z\rangle)|_{v=\langle z,z\rangle} + D_{e+1} = 0. \tag{16}$$

But this allows us to define $\tilde{f}_e$ and $\tilde{g}_{e+1}$ (see [3], p. 234). The corollary is proved.

LEMMA 10. *If the hypersurface M is nonspherical, then the parameter r is uniquely determined by λ, π, $\mathcal{U}$, and a.*

PROOF. As in the proof of Lemma 8, from (7), (9), (12), and (14) we obtain the following relation in weight $e + 2$:

$$\operatorname{Re}(i\tilde{g}_{e+2} + 2\langle\tilde{f}_{e+1}, \lambda\mathcal{U}z\rangle + 2\langle\tilde{f}_e, f_2\rangle)|_{v=\langle z,z\rangle} + D_{e+2}' = 0.$$

We compute the term D_{e+2} containing the parameter r:

$$\operatorname{Re} ig \to \operatorname{Re} i\pi|\lambda|^2 rw^2 \to \operatorname{Re} i\pi|\lambda|^2\cdot r\cdot 2i(u + i\langle z,z\rangle)H_e(z,\bar{z};u)$$
$$= 2\pi|\lambda|^2 uH_e(z,\bar{z};u)\,r,$$

$$H_e(f, \bar{f}; \operatorname{Re} g) = \pi|\lambda|^2 H_e(\varphi, \bar{\varphi}; R\psi)$$
$$\to \pi|\lambda|^2\left[\partial H_e(z, \bar{z}; u)(\varphi_3) + \bar{\partial} H_e(z, \bar{z}; u)(\bar{\varphi}_3) + \frac{\partial}{\partial u} H_e(z, \bar{z}; u)(\operatorname{Re}\psi_4)\right]$$
$$\to \pi|\lambda|^2 \operatorname{Re}\left[2(u + i\langle z, z\rangle)\partial H_e(z) + \frac{\partial}{\partial u} H_e(u^2 - \langle z, z\rangle^2)\right] r.$$

Consequently

$$D_{e+2} = r\pi|\lambda|^2 \operatorname{Re}\Big[-2H_e u + 2(u + i\langle z, z\rangle)\partial H_e(z)$$
$$+ \frac{\partial}{\partial u} H_e(z, \bar{z}; u)(u^2 - \langle z, z\rangle^2)\Big] + \text{terms not containing } r.$$

But by the corollary to Lemma 9, and because f_2 is independent of r, the product $\langle \tilde{f}_e, f_z\rangle$ also is independent of r, and so

$$0 = \operatorname{Re}(i\tilde{g}_{e+2} + 2\langle \tilde{f}_{e+1}, \lambda \mathcal{U} z\rangle)|_{v=\langle z,z\rangle} + r\pi|\lambda|^2 \operatorname{Re}[-2H_e u + 2(u + i\langle z, z\rangle)\partial H_e(z)$$
$$+ \frac{\partial}{\partial u} H_e(u^2 - \langle z, z\rangle^2)\Big] + \text{ terms not containing } r. \tag{17}$$

Suppose for fixed λ, π, $\mathcal{U}$, and a there are two values $r_1 \neq r_2$ satisfying (17). We set $\hat{r} = r_1 - r_2 \neq 0$, $\hat{f}_{e+1} = \tilde{f}_{e+1}(r_1) - \tilde{f}_{e+1}(r_2)$ and $\hat{g}_{e+2} = \tilde{g}_{e+2}(r_1) - \tilde{g}_{e+2}(r_2)$. Subtracting (17) for r_2 from (17) for r_1, we obtain

$$\operatorname{Re}(i\hat{g}_{e+2} + 2\langle \hat{f}_{e+1}, \lambda \mathcal{U} z\rangle)|_{v=\langle z,z\rangle}$$
$$= -\hat{r}\pi|\lambda|^2 \operatorname{Re}\left[-2H_e u + 2(u + i\langle z, z\rangle)\partial H_e(z) + \frac{\partial H_e}{\partial u}(u^2 - \langle z, z\rangle^2)\right]. \tag{18}$$

We observe furthermore that if h_{ml} is a polynomial of type (m, l), then

$$\partial h_{ml}(z) = m h_{ml}.$$

Therefore, in view of part c) of Lemma 1, the right-hand side of (18) lies in $\mathfrak{N}$. We have

$$\operatorname{Re}(i\hat{g}_{e+2} + 2\langle \hat{f}_{e+1}, \lambda \mathcal{U} z\rangle)|_{v=\langle z,z\rangle} \equiv 0 \pmod{\mathfrak{N}},$$

whence $\hat{f}_{e+1} = 0$ and $\hat{g}_{e+2} = 0$ (see [3], p. 234). Thus

$$\operatorname{Re}\left(-2H_e u + 2(u + i\langle z, z\rangle)\partial H_e(z) + \frac{\partial H_e}{\partial u}(u^2 - \langle z, z\rangle^2)\right) = 0. \tag{19}$$

Let $H_e = u^p h_p(z, \bar{z}) + \ldots + h_0(z, \bar{z})$, where $h_p(z, \bar{z}) \neq 0$. Then the highest term of (19) in u is

$$\operatorname{Re}(-2h_p + 2\partial h_p(z) + p h_p) u^{p+1} = 0,$$

or

$$(p - 2) h_p + \partial h_p(z) + \bar{\partial} h_p(\bar{z}) = 0.$$

But

$$\partial h_p(z) + \overline{\partial h}_p(\bar{z}) = (\deg h_p)_{(z,\bar{z})} \cdot h_p,$$

and so

$$[(p-2)+\deg h_{p(z,\bar z)}]\cdot h_p = 0. \tag{20}$$

Since $h_p \in \mathfrak{N}$, we have $\deg h_{p(z,\bar z)} \geqslant 4$, and from (20) we obtain that $h_p = 0$, a contradiction. This means $r_1 = r_2$. The lemma is proved.

Let G_ξ be the stability group of the point ξ, i.e. G_ξ consists of the automorphisms of the hypersurface M fixing ξ. If M is spherical, then G_ξ is isomorphic to the group of automorphisms of the hyperquadric S and depends on $(n+1)^2+1$ real parameters. In general, as will be shown in Theorem 3, for $n=1$ the typical case is that where the dimension of this group is zero. In connection with this the following theorem is of interest.

THEOREM 1. *Let $M \subset \mathbf{C}^{n+1}$ be a nondegenerate real analytic hypersurface, and let $\xi \in M$. Then the group G_ξ cannot depend on $(n+1)^2$ real parameters.*

REMARK. In fact a stronger assertion will be proved. Namely, if M is nonspherical, then the real dimension of G_ξ is less than $(n+1)^2$.

PROOF. We reduce M to normal form in a neighborhood of ξ. If M is spherical, then $\dim G_\xi = (n+1)^2+1$; if nonspherical, then from (13) and Lemma 10 it follows that $\dim_{\mathbf{R}} G_\xi < (n+1)^2$. The theorem is proved.

§3. Restrictions on the dimension in $\mathbf{C}^2$

This section is devoted to the proof of the following theorem.

THEOREM 2. *Let M be a nondegenerate real analytic hypersurface in $\mathbf{C}^2$, and let $\xi \in M$. If $\dim_{\mathbf{R}} G_\xi > 1$, then M is spherical, and consequently $\dim_{\mathbf{R}} G_\xi = 5$.*

For the sequel we need to develop some notions concerning CR-functions on the hyperquadric S.

We set $X = \partial(\ )(z) + i\langle z, z\rangle\partial/\partial u$, and correspondingly $\bar X = \bar\partial(\ )(\bar z) - i\langle z, z\rangle\partial/\partial u$. The properties of the operators X and $\bar X$ which we need are stated in the following lemma.

LEMMA 11. a) *If φ is a CR-function on S, then $\bar X\varphi = X\bar\varphi = 0$.*

b) *If ψ is a homogeneous polynomial in z of degree p whose coefficients are CR-functions on S, then $\bar X\psi = p\psi$ and $X\bar\psi = p\bar\psi$.*

In particular, a linear form is left invariant.

PROOF. Part a) follows from the definition of a CR-function. We get part b) by direct computation. We calculate $[\bar X, X]$:

$$\bar X X\varphi = \left(\bar\partial(\ )(\bar z) - i\langle z, z\rangle\frac{\partial}{\partial u}\right)(\partial\varphi(z) + i\langle z, z\rangle\varphi_u)$$

$$= \partial\bar\partial\varphi(z,\bar z) + i\langle z, z\rangle\varphi_u + i\langle z, z\rangle\bar\partial\varphi_u(\bar z) - i\langle z, z\rangle\partial\varphi_u(z) + \langle z, z\rangle^2\varphi_{uu};$$

$$X\bar X\varphi = \left(\partial(\ )(z) + i\langle z, z\rangle\frac{\partial}{\partial u}\right)(\bar\partial\varphi(\bar z) - i\langle z, z\rangle\varphi_u)$$

$$= \partial\bar\partial\varphi(z,\bar z) - i(z, z)\varphi_u - i\langle z, z\rangle\partial\varphi_u(z) + i\langle z, z\rangle\bar\partial\varphi_u(z) + \langle z, z\rangle^2\varphi_{uu}.$$

Thus

$$\Theta = [\bar{X}, X] = \bar{X}X - X\bar{X} = 2i\langle z, z\rangle \frac{\partial}{\partial u}.$$

We compute $[\bar{X}, \Theta]$:

$$\begin{aligned}\bar{X}\Theta\varphi &= \left(\bar{\partial}(\ )(\bar{z}) - i\langle z, z\rangle \frac{\partial}{\partial u}\right) 2i\langle z, z\rangle \varphi_u \\ &= 2i\langle z, z\rangle \varphi_u + 2i\langle z, z\rangle \bar{\partial}\varphi_u(\bar{z}) + 2\langle z, z\rangle^2 \varphi_{uu}.\end{aligned}$$

$$\Theta\bar{X}\varphi = 2i\langle z, z\rangle \frac{\partial}{\partial u}(\bar{\partial}\varphi(\bar{z}) - i\langle z, z\rangle \varphi_u) = 2i\langle z, z\rangle \frac{\partial}{\partial u}\bar{\partial}\varphi(\bar{z}) + 2\langle z, z\rangle^2 \varphi_{uu}.$$

Therefore

$$[\bar{X}, \Theta] = \bar{X}\Theta - \Theta\bar{X} = 2i\langle z, z\rangle \frac{\partial}{\partial u} = \Theta.$$

Conjugating this equation, we obtain $[X, \Theta] = \Theta$. We set $\varphi = \langle f, Cz\rangle$, where f is a vector-valued CR-function on S (i.e. an n-tuple of CR-functions on S), and C is a linear operator on $\mathbf{C}^n$. Letting $\psi = \varphi + \bar{\varphi}$, we have

LEMMA 12. *If* $\tau = (\bar{X} - 2)(\bar{X} - 1)(X - 1)$, *then* $\tau(\psi) = 0$.

PROOF. We have

$$X\bar{\varphi} = X\langle Cz, f\rangle = \langle XCz, f\rangle + \langle Cz, \bar{X}f\rangle,$$

but since $XCz = Cz$ and $\bar{X}f = 0$, it follows that $X\bar{\varphi} = \langle Cz, f\rangle = \varphi$, i.e. $(X - 1)\bar{\varphi} = 0$. Analogously, $(\bar{X} - 1)\varphi = 0$. Then we have $(X - 1)\psi = (X - 1)\varphi$; but, since $[\bar{X} - 1, X - 1] = \Theta$,

$$(\bar{X} - 1)(X - 1)\psi = (\bar{X} - 1)(X - 1)\varphi = [(X - 1)(\bar{X} - 1) + \Theta]\varphi = \Theta\varphi.$$

Also

$$(\bar{X} - 1)^2(X - 1)\psi = (\bar{X} - 1)\Theta\varphi = \Theta(\bar{X} - 1)\varphi + \Theta\varphi = \Theta\varphi.$$

Subtracting, we obtain

$$[(\bar{X} - 1)^2(X - 1) - (\bar{X} - 1)(X - 1)]\psi = 0,$$

or, finally,

$$(\bar{X} - 2)(\bar{X} - 1)(X - 1)\psi = 0.$$

The lemma is proved.

LEMMA 13. *If* $\tilde{g}_{e+1}$ *satisfies* (16), *then*

$$\tilde{g}_{e+1} = \begin{cases} 0, & \text{if } e = 2k + 1, \\ \alpha_1(z)(u + i\langle z, z\rangle), & \text{if } e = 2k, \end{cases}$$

where $\alpha_1(z)$ *is a linear form.*

We give the proof for the case $e = 2k$. We set

$$\widetilde{f}_e = A_0 (u + i\langle z, z\rangle)^k + A_2 (z) (u + i\langle z, z\rangle)^{k-1} + \ldots + A_{2k}(z),$$
$$\widetilde{g}_{e+1} = \alpha_1 (z) (u + i\langle z, z\rangle)^k + \alpha_3 (z) (u + i\langle z, z\rangle)^{k-1} + \ldots + \alpha_{2k+1}(z),$$

where $\alpha_p(z)$ is a form of degree p, and $A_p(z)$ is a vector-valued form of degree p. From (15) it is clear that D_{e+1} does not contain terms of type $(p; 0)$, so the terms of this type must be equal in the expression

$$\mathrm{Re}\,(i\widetilde{g}_{e+1} + 2\langle \widetilde{f}_e, \lambda \mathcal{U} z\rangle)|_{v=\langle z,z\rangle}.$$

We obtain

$$(2k+1, 0) \to \alpha_{2k+1}(z) = 0,$$
$$(2k-1, 0) \to \alpha_{2k-1}(z) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(3, 0) \to \alpha_3(z) = 0,$$
$$(1, 0) \to \alpha_1(z) - \langle \lambda \mathcal{U} z, A_0\rangle = 0.$$

The lemma is proved.

We compute $\tau(\varphi)$, where $\varphi = (\beta(z)w^k + \overline{\beta(z)w^k})|_{v=\langle z,z\rangle}$ and $\beta(z)$ is a linear form. We have

$$X\varphi = \beta w^k + 2ik\langle z, z\rangle \beta w^{k-1},$$
$$(X-1)\varphi = 2ik\langle z, z\rangle \beta w^{k-1} - \bar{\beta}\bar{w}^k,$$
$$\overline{X}(X-1)\varphi = 2ik\langle z, z\rangle \beta w^{k-1} - \bar{\beta}\bar{w}^k + 2\bar{\beta} ik\langle z, z\rangle \bar{w}^{k-1},$$
$$(\overline{X}-2)(X-1)\varphi = -2ik\langle z, z\rangle \beta w^{k-1} + \bar{\beta}\bar{w}^k - \bar{\beta}\cdot 2ik\langle z, z\rangle \bar{w}^{k-1}$$
$$+ \beta\cdot 2ik\langle z, z\rangle \bar{w}^{k-1} + \beta\cdot 2ik\langle z, z\rangle \bar{w}^{k-1} + \beta\cdot 2ik\langle z, z\rangle\cdot 2i(k-1)\langle z, z\rangle \bar{w}^{k-2},$$
$$\tau(\varphi) = (\overline{X}-1)(\overline{X}-2)(X-1)\varphi = 4k(k-1)\langle z, z\rangle^2 \overline{\beta(z)}\,\bar{w}^{k-2}.$$

We note that $\tau(\varphi)$ is a quadratic form in z whose coefficients are CR-functions on S. Therefore (see Lemma 11 b))

$$(X-2)\tau(\varphi) = 0. \tag{21}$$

Suppose there exist two distinct vectors $a_1 \neq a_2$ satisfying (16). Introducing the notation $\hat{a} = a_1 - a_2$, $\hat{f}_e = \widetilde{f}_e(a_1) - \widetilde{f}_e(a_2)$ and $\hat{g}_{e+1} = \widetilde{g}_{e+1}(a_1) - \widetilde{g}_{e+1}(a_2)$, and subtracting (16) for a_2 from (16) for a_1, we obtain

$$\mathrm{Re}\,(i\hat{g}_{e+1} + 2\langle \hat{f}_e \lambda \mathcal{U} z\rangle)|_{v=\langle z,z\rangle} + \hat{D}_{e+1} = 0,$$

where

$$\hat{D}_{e+1} = 2\pi|\lambda|^2 \mathrm{Re}\Big(-2i\langle z, \hat{a}\rangle H_e + (u + i\langle z, z\rangle)\partial H_e(\hat{a})$$
$$+ 2i\langle z, \hat{a}\rangle \partial H_e(z) + i\langle z, \hat{a}\rangle (u + i\langle z, z\rangle)\frac{\partial H_e}{\partial u}\Big).$$

Letting $T = (X - 2)\tau$, we deduce from Lemma 13 and (21) that

$$T\hat{D}_{e+1} = 0. \tag{22}$$

If we view this as an equation for the vector a, then it represents a system of linear equations for the unknowns

$$\operatorname{Re}\hat{a}^1,\ \operatorname{Im}\hat{a}^1,\ \ldots,\ \operatorname{Re}\hat{a}^n,\ \operatorname{Im}\hat{a}^n.$$

Therefore the set of solutions is a linear subspace of $\mathbf{R}^{2n} = \mathbf{C}^n$. Let this subspace consist of the complex lines $\{\hat{a} = t\hat{a}_0$, where $t \in \mathbf{C}^1$ and $\hat{a}_0 \neq 0\}$. Then, introducing the variable t into (22) and separating the terms of degree (0, 1) in t, we obtain

$$T\Big(-2i\langle z, \hat{a}_0\rangle H_e + (u - i\langle z, z\rangle)\bar{\partial}H_e(\hat{a}_0)$$
$$+ 2i\langle z, \hat{a}_0\rangle \partial H_e(z) + i\langle z, \hat{a}_0\rangle(u + i\langle z, z\rangle)\frac{\partial H_e}{\partial u}\Big) = 0.$$

Let $x = \partial(\ )(z)$, $d = \langle z, z\rangle\partial/\partial u$, and $x_k = x - kE$, where E is the identity operator. Then

$$T = (x_2 + id)(\bar{x}_2 - id)(\bar{x}_1 - id)(x_1 + id).$$

LEMMA 14. *If $\alpha(z)$ is a linear form, then*

$$x_k(\alpha(z)\varphi) = \alpha(z)x_{k-1}\varphi.$$

PROOF.

$$x_k(\alpha(z)\varphi) = x(\alpha(z)\varphi) - k\alpha(z)\varphi$$
$$= \alpha(z)\varphi + \alpha(z)x\varphi - k\alpha(z)\varphi = \alpha(z)(x - (k-1))\varphi$$
$$= \alpha(z)x_{k-1}\varphi.$$

REMARK. We have correspondingly

$$\alpha(z)x_k\varphi = x_{k+1}(\alpha(z)\varphi).$$

We expand T in powers of the operator d:

$$T = (x_2 + id)(\bar{x}_2 - id)(\bar{x}_1 - id)(x_2 + id)$$
$$= (x_2\bar{x}_2 + id\,\bar{x}_2 - ix_2 d + d^2)(x_1\bar{x}_1 - id\,x_1 + id\,\bar{x}_1 + d^2)$$
$$= (x_2\bar{x}_2 + i(\bar{x}_3 - x_2)d + d^2)(x_1\bar{x}_1 + i(x_1 - x_2)d + d^2)$$
$$= x_2\bar{x}_2x_1\bar{x}_1 + i(x_2\bar{x}_2(\bar{x}_1 - x_2) + x_2\bar{x}_2(\bar{x}_3 - x_2))d$$
$$+ (x_2\bar{x}_2 + x_3\bar{x}_3 - (\bar{x}_3 - x_2)(\bar{x}_2 - x_3))d^2 + i((\bar{x}_3 - x_2) + (\bar{x}_3 - x_4))d^3 + d^4,$$

i. e. $T = l_0 + l_1 d + l_2 d^2 + l_3 d^3 + d^4$, where

$$l_0 = (x - 2)(\bar{x} - 2)(x - 1)(\bar{x} - 1),$$
$$l_1 = 2i(x - 2)(\bar{x} - 2)(\bar{x} - x),$$
$$l_2 = ((x - 2)(\bar{x} - 2) + (x - 3)(\bar{x} - 3) - (\bar{x} - x)^2 + 1),$$
$$l_3 = 2i(\bar{x} - x).$$

In the sequel we assume that $n = 1$. Let

$$P = u\frac{\partial}{\partial \bar{z}}, \quad Q = iz(2x - \bar{x} + \varkappa - 2), \quad R = -zd,$$

where $\kappa = u\partial/\partial u$. Then (22) may be written as

$$T(P+Q+R)(H_e) = 0.$$

$T(P + Q + R)$ may be represented in the form

$$T(P+Q+R) = ul_0\left(\frac{\partial}{\partial \bar{z}}\right) + z(A_0 + A_1 d + A_2 d^2 + A_3 d^3 + A_4 d^4 - d^5).$$

where the operators A_i transform monomials $cz^m\bar{z}^p u^k$ into monomials of the same type, i.e. each monomial is an eigenvector. Setting

$$G = \left(\frac{1}{z}\right) T(P+Q+R),$$

we have

$$G = \frac{u}{|z|^2} A_{-1} + A_0 + A_1 d + A_2 d^2 + A_3 d^3 + A_4 d^4 - d^5.$$

We compute A_{-1}, A_0, and A_1:

$$A_{-1} = \bar{z} l_0\left(\frac{\partial}{\partial \bar{z}}\right) = (x-2)(x-1)(\bar{x}-2)(\bar{x}-3)\bar{x},$$

$$A_0 = \left(\frac{1}{z}\right)[l_0(Q) + l_1 d(P)],$$

$$l_0(Q) = (x-2)(\bar{x}-2)(x-1)(\bar{x}-1)[iz(2x - \bar{x} + \varkappa - 2)]$$
$$= iz(x-1)x(\bar{x}-2)(\bar{x}-1)(2x - \bar{x} + \varkappa - 2),$$

$$d(P) = z\bar{z}\frac{\partial}{\partial u}\left(u\frac{\partial}{\partial \bar{z}}\right) = z\bar{z}\frac{\partial}{\partial \bar{z}}(1 + \varkappa) = z\bar{x}(\varkappa + 1),$$

$$l_1 d(P) = z \cdot 2i(x-1)(\bar{x}-2)(\bar{x} - x - 1)\bar{x}(\varkappa + 1).$$

As a result we obtain

$$A_0 = i(x-1)(\bar{x}-2)[x(\bar{x}-1)(2x - \bar{x} + \varkappa - 2) + 2(\bar{x} - x - 1)\bar{x}(\varkappa + 1)],$$

$$A_1 d = \left(\frac{1}{z}\right)[l_0(R) + l_1 d(Q) + l_2 d^2(P)],$$

$$l_0(R) = (x-2)(\bar{x}-2)(x-1)(\bar{x}-1)(-zd)$$
$$= -z(x-1)x(\bar{x}-2)(\bar{x}-1)d,$$
$$d(Q) = iz(2x - \bar{x} + \varkappa - 2)d,$$

$$l_1 d(Q) = -2z(x-1)(\bar{x}-2)(\bar{x} - x - 1)(2x - \bar{x} + \varkappa - 2)d,$$

$$d^2(P) = z^2 \cdot \bar{z}^2 \frac{\partial^2}{\partial u^2}\left(u\frac{\partial}{\partial \bar{z}}\right) = z^2 \cdot \bar{z}^2 \cdot \frac{\partial}{\partial z}(\varkappa + 2)\frac{\partial}{\partial u}$$

$$= z^2\bar{z}\bar{x}(\varkappa + 2)\frac{\partial}{\partial u} = z(\bar{x}-1)(\varkappa + 2)d,$$

$$l_2 d^2(P) = z[(x-1)(\bar{x}-2)+(x-2)+(x-2)(\bar{x}-3) \\ -(\bar{x}-x-1)^2+1](\bar{x}-1)(\varkappa+2)d,$$

so that

$$A_1 = (((x-1)(\bar{x}-2)+(x-2)(\bar{x}-3)-(\bar{x}-x-1)^2+1)(\bar{x}-1)(\varkappa+2) \\ -(x-1)x(\bar{x}-2)(\bar{x}-1)-2(x-1)(\bar{x}-2)(\bar{x}-x-1)(2x-\bar{x}+\varkappa-2)).$$

We have $G(H_e) = 0$.

Let

$$H_e = u^s(h_{42}z^4\bar{z}^2 + h_{24}z^2\bar{z}^4) + \ldots + (h_{m2}z^m \cdot \bar{z}^2 + \ldots + h_{2m}z^2 \cdot \bar{z}^m).$$

We will calculate the coefficients of $G(H_e)$. We denote by $A_i(m, p)$ the factor which appears in front of the monomial $z^m\bar{z}^p$ after application of the operator A_i, and by $G(m, p)$ the coefficient of $z^m\bar{z}^p$ in $G(H_e)$.

Each coefficient of $G(H_e)$ is a linear combination of the h_{mp}, so by setting these coefficients equal to zero we obtain linear equations connecting the h_{mp}. Our immediate goal is to prove that $h_{42} = 0$. This can be done using the fact that the operator G acts on the symmetric monomial H_e in an asymmetric manner. This enables us to obtain additional equations without introducing new unknowns.

We have

$$G(2, 4) = h_{35}A_{-1}(3, 5) + h_{24}A_0(2, 4),$$
$$A_{-1}(3, 5) = 1 \cdot 2 \cdot 3 \cdot 2 \cdot 5 = 60,$$

$$A_0(2, 4) = i \cdot 1 \cdot 2 \cdot [2 \cdot 3(4-4+s-2) + 2(4-2-1)4(s+1)] \\ = 2i[6(s-2)+8(s+1)] = 4i(7s-2).$$

Cancelling a factor of 4, we obtain

$$15h_{35} + i(7s-2)h_{24} = 0. \tag{23}$$

Furthermore,

$$G(5, 3) = h_{64}A_{-1}(6, 4) + h_{53}A_0(5, 3) + h_{42}A_1(5, 3),$$
$$A_{-1}(6, 4) = 4 \cdot 5 \cdot 2 \cdot 1 \cdot 4 = 160,$$
$$A_0(5, 3) = i \cdot 4 \cdot 1 \cdot [5 \cdot 2(10-3+s-1-2) + 2(3-5-1)3s] \\ = 4i[10(s+4-18s)] = -32i(s-5),$$
$$A_1(5, 3) = \{[4 \cdot 1 + 3 \cdot 0 - 9 + 1] \cdot 2(s+1) - 4 \cdot 5 \cdot 1 \cdot 2 \\ + 2 \cdot 4 \cdot 1 \cdot 3(10-3+s-1-2)\} = -8(s+1) - 40 + 24(s+4) = 16(s+3).$$

Cancelling a factor of 16, we obtain

$$10h_{64} - 2i(s-5)h_{53} + (s+3)h_{42} = 0. \tag{24}$$

Moreover,

$$G(3,\ 5) = h_{46}A_{-1}(4,\ 6) + h_{35}A_0(3,\ 5) + h_{24}A_1(3,\ 5),$$

$$\begin{aligned} A_{-1}(4,\ 6) &= 2 \cdot 3 \cdot 4 \cdot 3 \cdot 6 = 27 \cdot 16, \\ A_0(3,\ 5) &= i \cdot 2 \cdot 3[3 \cdot 4(6-5+s-1-2) + 2(5-3-1)5s] \\ &= 6i[12(s-2) + 10s] = 12i(11s-12), \\ A_1(3,\ 5) &= \{[2 \cdot 3 + 1 \cdot 2 - 1 + 1]4(s+1) - 2 \cdot 3 \cdot 3 \cdot 4 \\ &\quad -2 \cdot 2 \cdot 3 \cdot 1(6-5+s-1-2)\} = 32(s+2) - 72 - 12(s-2) = 4(5s-4). \end{aligned}$$

Cancelling a factor of 4, we obtain

$$27 \cdot 4h_{46} + 3i(11s-12)h_{35} + (5s-4)h_{24} = 0. \tag{25}$$

(23), (25), and the equation obtained by conjugating (24) form a system of three linear equations for the unknowns h_{46}, h_{35}, and h_{24}. We express it as a matrix

$$\begin{pmatrix} 0 & 15 & i(7s-2) \\ 10 & 2i(s-5) & (s+3) \\ 27 \cdot 4 & 3i(11s-12) & (5s-4) \end{pmatrix}.$$

Dividing the first column by 2 and taking the determinant, we obtain

$$\Delta(s) = 3(-19 \cdot 7 \cdot s^2 - 657 \cdot s + 2 \cdot 575).$$

We note that $\Delta(0) \neq 0$ and $\Delta(1) \neq 0$, but for $s \geqslant 2$ we have $657\,s > 2 \cdot 575$, so $\Delta(s) < 0$ for $s \geqslant 2$. Thus we have shown that $\Delta(s) \neq 0$ for $s = 0, 1, 2, \ldots$, and so $h_{42} = 0$.

By comparing the components of (16) of type (1, 0), (2, 1), and (3, 2) for $n = 1$ and $e = 2k$, we obtain in the notation of Lemma 13

$$\begin{aligned} \alpha + \bar{A}_0 &= 0, \\ A_2 - 2ik\bar{A}_0 &= 0, \\ i(k-1)A_2 &= 4ah_{42}. \end{aligned}$$

Hence in view of the condition $h_{42} = 0$ it follows that $\alpha = 0$. Thus $g_{e+1} = 0$, since it is independent of the parity of e. Now from Lemma 12 we obtain

$$\tau(P+Q+R)(H_e) = 0.$$

We expand the operator τ in powers of d:

$$\begin{aligned} \tau &= (\bar{x}_2 - id)(\bar{x}_1 - id)(x_1 + id) = (\bar{x}_2 - id)[\bar{x}_1x_1 + i(\bar{x}_1 - x_2)d + d^2] \\ &= \bar{x}_2\bar{x}_1x_1 + i[x - x - (\bar{x} - 2)(x - 2) + 1]d + (2\bar{x} - x - 1)d^2 - id^3. \end{aligned}$$

We introduce the following operator:

$$\Gamma = \left(\frac{1}{z}\right)\tau(P+Q+R).$$

Like G, it may be written in the form

$$\Gamma = \frac{u}{|z|^2} B_{-1} + B_0 + B_1 d + B_2 d^2 + B_3 d^3 + id^4,$$

where

$$B_{-1} = \bar{z}(\bar{x}-2)(\bar{x}-1)(x-1)\left(\frac{\partial}{\partial \bar{z}}\right) = (\bar{x}-3)(\bar{x}-2)\bar{x}(x-1).$$

Computing the coefficients of $z\bar{z}^{m-1}$ in $\Gamma(H_e)$, we obtain

$$\Gamma(1, m-1) = h_{2m} B_{-1}(2, m) = h_{2m}(m-3)(m-2)m.$$

If $m \geqslant 4$, it follows from the identity $\Gamma(H_e) = 0$ that $h_{m2} = 0$. We now prove by induction on p that $h_{2+p,m+p} = 0$.

Let $h_{2+q,m+q} = 0$ for $q < p$. Then

$$\begin{aligned}\Gamma(1+p, m+p-1) &= B_{-1}(2+p, m+p) h_{2+p,\, m+p} \\ &= (m+p-3)(m+p-2)(m+p)(p+1) h_{2+p,\, m+p}.\end{aligned}$$

From $\Gamma(H_e) = 0$ we obtain that $h_{2+p,m+p} = 0$. This also shows that $H_e = 0$. This is a contradiction; that is, the space of solutions of (22) for $\hat{a}$ is at most one-dimensional.

We summarize. If the hypersurface M is nonspherical, then the number of degrees of freedom of the parameter λ does not exceed unity (see Corollary 1), the parameter a, as shown in this section, also has at most one degree of freedom, and the parameter r has no freedom (see Lemma 10). Consequently, the dimension of G_ξ in this case does not exceed two.

Let $\dim_{\mathbf{R}} G_\xi = 2$. This means that (13) really has one degree of freedom in λ. This can be so only if

$$H_e = u^s L_{pp} z^p \bar{z}^p + u^{s-1} L_{p-1p-1} z^{p-1}\bar{z}^{p-1} + \ldots + L_{kk} z^k \bar{z}^k,$$

where $L_{pp} \neq 0$. But in this case the right-hand side of (16) lies in $\mathfrak{N}$, so $\tilde{f}_e = \tilde{g}_{e+1} = 0$. We obtain

$$2\operatorname{Re}\{\bar{a}[P+Q+R](H_e)\} = 0.$$

Selecting from this equation the leading component in u, we find that

$$u^{s+1} \cdot L_{pp} \cdot p2\operatorname{Re}\{\bar{a}[z^p\bar{z}^{p-1} + z^{p-1}\bar{z}^p]\} = 0.$$

whence it follows that $a = 0$, and there remains all told one degree of freedom in the whole group G_ξ. Theorem 2 is proved.

§4. The stability group of a nonumbilical point

The point $\xi \in M$ is called *umbilical* if the contact of a spherical surface at this point is of higher order than what the normal form guarantees.

Let $n = 1$, and let

$$v = |z|^2 + c_{42}(u)z^4\bar{z}^2 + c_{24}(u)z^2\bar{z}^4 + \sum_{\substack{m+l\geq 7\\ \min(m,l)\geq 2}} c_{ml}(u)z^m\bar{z}^l$$

be the equation of M in normal form. Then the point 0 is umbilical for this surface if $c_{42}(0) = 0$.

We denote $c_{42}(0)$ by h, $c_{52}(0)$ by p, and $c_{43}(0)$ by q.

THEOREM 3. *Let M be a nondegenerate real analytic hypersurface in $\mathbf{C}^2$, and let $\xi \in M$ be a nonumbilical point, i.e. $h \neq 0$. Then the group G_ξ consists of not more than two mappings.*

Moreover, if in addition $h\bar{p} + 3\bar{h}q \neq 0$, then $G_\xi = \{E\}$, i.e. there is no automorphism of the hypersurface leaving ξ fixed and different from the identity.

PROOF. Since ξ is not umbilical, we have $e = 6$ and

$$H_e = hz^4\bar{z}^2 + \bar{h}z^2\bar{z}^4. \tag{26}$$

As a consequence of (13) we obtain $\lambda^3 \cdot \bar{\lambda} = 1$. This equation has the two solutions 1 and -1. We will show that to each of these two values of λ corresponds at most one value of a.

Rewriting (16), we obtain

$$\begin{aligned}\operatorname{Re}(i\tilde{g}_7 + 2\bar{\lambda}\cdot\tilde{f}_6\bar{z})\big|_{v=|z|^2} = 2\operatorname{Re}\bar{a}\left[-2izH_6 - iz\bar{z}\frac{\partial H_6}{\partial\bar{z}} + 2iz^2\frac{\partial H_6}{\partial z} + u\frac{\partial H_6}{\partial\bar{z}}\right]\\ + (H_7(\lambda z, \overline{\lambda z}) - H_7(z, \bar{z})).\end{aligned} \tag{27}$$

Suppose there exist two values of a, say a_1 and a_2, satisfying (27). Set

$$\hat{a} = a_1 - a_2, \quad \hat{f}_6 = \tilde{f}_6(a_1) - \tilde{f}_6(a_2), \quad \hat{g}_7 = \tilde{g}_7(a_1) - \tilde{g}_7(a_2).$$

Subtracting (27) for a_2 from (27) for a_1, we obtain

$$\operatorname{Re}(i\hat{g}_7 + 2\bar{\lambda}\hat{f}_6\bar{z})\big|_{v=|z|^2} = 2\operatorname{Re}\bar{\hat{a}}\left[-2izH_6 - iz\bar{z}\frac{\partial H_6}{\partial\bar{z}} + 2iz^2\frac{\partial H_6}{\partial z} + u\frac{\partial H_6}{\partial z}\right].$$

Substituting (26) here, we find that

$$\begin{aligned}&\operatorname{Re}(i\hat{g}_7 + 2\bar{\lambda}\hat{f}_6\bar{z})\big|_{v=|z|^2}\\ &= 2\operatorname{Re}\bar{\hat{a}}[4ihz^5\bar{z}^2 - 2i\bar{h}z^3\cdot\bar{z}^4 + 2uhz^4\bar{z} + 4u\bar{h}z^2\cdot\bar{z}^3].\end{aligned} \tag{28}$$

We set

$$\bar{\lambda}\hat{f}_6\bar{z} = Az^6\bar{z} + Bz^4\bar{z}w + Cz^2\bar{z}w^2 + D\bar{z}w^3,$$

$$\frac{i}{2}\hat{g}_7 = \alpha zw^3 + \beta\cdot z^3w^2 + \gamma z^5w + \delta z^7.$$

The right-hand side of (28) does not contain terms of type (7, 0), (5, 0), or (3, 0); so, setting the corresponding components of the left-hand side equal to zero, we obtain

$\beta = \gamma = \delta = 0$. Now computing the coefficients of the components of the left-hand side of types (5, 2) and (4, 1), we find that $(5, 2) \longrightarrow iB$ and $(4, 1) \longrightarrow B$. Equating them to the corresponding components of the right-hand side, we obtain

$$4ih\bar{\hat{a}} = iB, \quad 2h\bar{\hat{a}} = B. \tag{29}$$

But $h \neq 0$, so (29) means that $\hat{a} = 0$.

To complete the proof of the first part of the theorem it remains to apply Lemmas 8 and 10.

Now taking into account that

$$H_7 = pz^5\bar{z}^2 + qz^4\bar{z}^3 + \bar{q}z^3\bar{z}^4 + \bar{p}z^2\bar{z}^5,$$

we write (27) for $\lambda = -1$. We obtain

$$\begin{aligned} &\operatorname{Re}(i\tilde{g}_7 + 2\bar{\lambda}\tilde{f}_6\bar{z})|_{v=|z|^2} \\ &= 2\operatorname{Re}[\bar{a}(4ihz^5\bar{z}^2 - 2i\bar{h}z^3\bar{z}^4 + 2uhz^4\bar{z} + 4u\bar{h}z^2\bar{z}^3) - 2pz^5\bar{z}^2 - 2qz^4\bar{z}^3]. \end{aligned}$$

We set

$$\begin{aligned} 2\bar{\lambda}\tilde{f}_6\bar{z} &= Az^6\bar{z} + Bz^4\bar{z}w + Cz^2\bar{z}w^2 + D\bar{z}w^3, \\ \frac{i}{2}\tilde{g}_7 &= \alpha zw^3 + \beta z^3w^2 + \gamma z^5w + \delta z^7. \end{aligned}$$

As before, from the comparison of the components of type (7, 0), (5, 0), and (3, 0) we have $\beta = \gamma = \delta = 0$. The remaining components give the equations

$$\begin{array}{ll} (6,1) & A=0, \\ (5,2) & iB=4ih\bar{a}-2p, \\ (4,3) & -C+iD-i\alpha=2iha-2q, \\ (4,1) & B=2h\bar{a}, \\ (3,2) & 2iC-3\bar{D}-3\alpha=4ha, \\ (2,1) & C-3i\bar{D}+3i\alpha=0, \\ (1,0) & \bar{D}+\alpha=0. \end{array}$$

From (5, 2) and (4, 1) we obtain

$$p = ih\bar{a}. \tag{30}$$

Equations (4, 3), (3, 2), (2, 1), and (1, 0) form a system of four linear equations for the three unknowns C, $\bar{D}$, and α. Eliminating unknowns, we obtain on the right-hand side the relation $h\alpha + 3iq = 0$, which together with (30) gives finally $\bar{p}h + 3qh = 0$. The theorem is proved.

Received 20/NOV/1978

BIBLIOGRAPHY

1. Élie Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes.* I, Ann. Math. Pura Appl. (4) **11** (1932), 17–90; reprinted in his *Oeuvres complètes,* Partie II, Vol. 2, Gauthier-Villars, Paris, 1953, pp. 1231–1304.

2. A. Tresse, *Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre* $y'' = \omega(x, y, y')$, Preisschriften Fürstlich Jablonowski'schen Ges. Leipzig, No. 32 (Math.-Nat. Sect., no. 13), Hirzel, Leipzig, 1896.

3. S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds,* Acta Math. **133** (1974), 219–271.

4. J. Burns, S. Shnider and R. Wells, *Deformation of strongly pseudoconvex domains,* Preprint, 1974.

Translated by H. P. BOAS