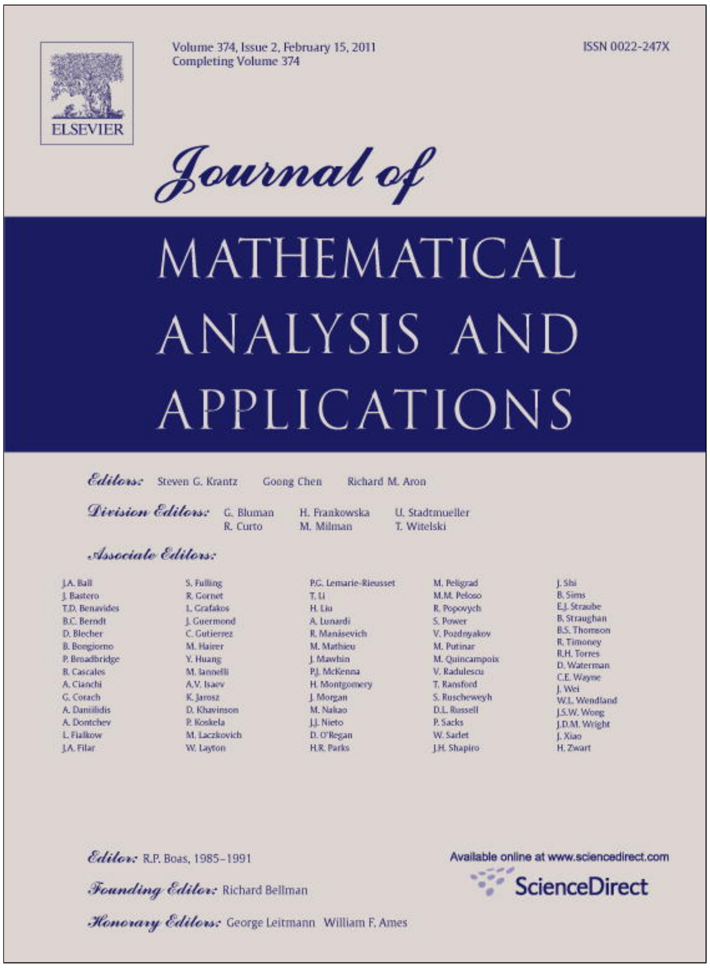
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Classification of homogeneous CR-manifolds in dimension 4

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ABSTRACT

Locally homogeneous CR-manifolds in dimension 3 were classified, up to local CR-equivalence, by E. Cartan. We classify, up to local CR-equivalence, all locally homogeneous CR-manifolds in dimension 4. The classification theorem enables us also to classify all symmetric CR-manifolds in dimension 4, up to local CR-equivalence.

1. Introduction

Let M be a real-analytic CR-manifold, generically embedded to the complex space $\mathbb{C}^{n+k}$, $n = \mathrm{CR}\dim M$, $k = \operatorname{codim} M$. An important class of CR-manifolds is the class of homogeneous CR-manifolds. A CR-manifold M is called *homogeneous*, if its CR-automorphism group acts transitively on it. The CR-manifold M is called *locally homogeneous*, if germs of M at any two points are CR-equivalent. In other words, locally homogeneous CR-manifolds are that ones which are "the same at all points" up to local biholomorphic transformations (in what follows we suppose all CR-manifolds and CR-mappings to be real-analytic, so, due to the classical result of Tomassini [24], any local CR-equivalence is reduced to a local biholomorphic equivalence and we don't make a difference in these notions in what follows; the consideration of just real-analytic generically embedded CR-submanifolds in $\mathbb{C}^N$ in this paper is motivated by the fact that a CR-manifold, admitting a local transitive action of a Lie group, is automatically real-analytic, and by the fact that any real-analytic CR-manifold can be locally generically embedded to $\mathbb{C}^N$, see [25]). A very useful equivalent definition of local homogeneity is as follows (see [25] for possible equivalent definitions of local homogeneity). A *holomorphic vector field on M at a point $p \in M$* is a vector field X of the form

$$f_1(z)\frac{\partial}{\partial z_1} + \cdots + f_{n+k}(z)\frac{\partial}{\partial z_{n+k}},$$

such that the real vector field $2\operatorname{Re} X = X + \overline{X}$ (the real form of X) is tangent to M, and the functions $f_j(z)$ are holomorphic in a neighborhood of p in the ambient space. In what follows we use the following convention: by *values* of a holomorphic vector field X we mean the values of its real form $X + \overline{X}$ and by $X|_p$ denote $(X + \overline{X})|_p$ as well.

The real forms of holomorphic vector fields at p are exactly the vector fields, which generate local flows of biholomorphic transformations at p, preserving M. Holomorphic vector fields on M at p constitute a Lie algebra with respect to the Lie bracket of vector fields, which is called *the infinitesimal automorphism algebra of M at p* and is denoted by $\mathfrak{aut}\, M_p$.

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The *stability subalgebra* of M at the point p is the subalgebra $\mathfrak{aut}_p M_p \subset \mathfrak{aut} M_p$, which consists of vector fields, vanishing at p. This algebra generates the identity component of the *stability group* (or the *isotropy group*) of M at p, which consists of biholomorphic automorphisms of the germ of M at p, preserving the fixed point p. An *evaluation mapping* is the natural mapping $\varepsilon_p : \mathfrak{aut} M_p \to T_p M$, given by the formula $\varepsilon_p(X) = X|_p$. M is called *locally homogeneous at* p, if the mapping ε_p is surjective (i.e. the values of vector fields from $\mathfrak{aut} M_p$ at p form all the tangent space $T_p M$). M now is called *locally homogeneous*, if it is locally homogeneous at all points.

Locally homogeneous CR-manifolds in dimension 3 were classified by E. Cartan [10]. The classification is based on the fundamental trichotomy for locally homogeneous real hypersurfaces in $\mathbb{C}^2$, demonstrated by H. Poincaré in [21]. Due to this trichotomy, a locally homogeneous real hypersurface in $\mathbb{C}^2$ is either Levi flat and in this case it has an infinite-dimensional isotropy subalgebra, or is locally equivalent to the 3-dimensional sphere (is *spherical*) and in this case it has isotropy subalgebra of dimension 5, or is Levi non-degenerate and non-spherical and in this case it has trivial isotropy subalgebra. Using the trichotomy and Bianchi's classification of 3-dimensional real Lie algebras, E. Cartan presented the desired list of locally homogeneous hypersurfaces in $\mathbb{C}^2$, which consists of the real hyperplane

$$\operatorname{Im} w = 0$$

(the case $\dim \mathfrak{aut}_p M_p = \infty$), the hypersphere

$$|z|^2 + |w|^2 = 1$$

(the case $\dim \mathfrak{aut}_p M_p = 5$), the Cartan's tube surfaces:

$$\text{(t1)} \qquad v = y^\lambda, \quad y > 0,\ |\lambda| \geqslant 1,\ \lambda \neq 1, 2,$$

$$\text{(t2)} \qquad v = y \ln y,$$

$$\text{(t3)} \qquad r = e^{a\varphi}, \quad a \geqslant 0$$

($\dim \mathfrak{aut}_p M_p = 0$), and the Cartan's projective surfaces:

$$\text{(p1)} \qquad 1 + |z|^2 + |w|^2 = a\left|1 + z^2 + w^2\right|, \quad a > 1,$$

$$\text{(p2)} \qquad 1 + |z|^2 - |w|^2 = a\left|1 + z^2 - w^2\right|, \quad a > 1,$$

$$\text{(p3)} \qquad -1 + |z|^2 + |w|^2 = a\left|-1 + z^2 + w^2\right|, \quad 0 < |a| < 1$$

($\dim \mathfrak{aut}_p M_p = 0$ as well). Here we denote by z, w the coordinates in $\mathbb{C}^2$, put $z = x + iy$, $w = u + iv$ and denote by r, φ the polar coordinates on the plane (y, v). Note that for a locally homogeneous CR-manifold we clearly have $\dim \mathfrak{aut} M_p = \dim \mathfrak{aut}_p M_p + \dim M$, so in the mentioned four cases we have $\dim \mathfrak{aut} M_p = \infty$, $\dim \mathfrak{aut} M_p = 8$, $\dim \mathfrak{aut} M_p = 3$, $\dim \mathfrak{aut} M_p = 3$ correspondingly.

Classification of locally homogeneous CR-manifolds in dimension 5 (which is essentially the classification of locally homogeneous hypersurfaces in $\mathbb{C}^3$) is in progress. Partial results in the Levi non-degenerate case have been obtained by A. Loboda, who presented the desired classification for CR-manifolds with the condition $\dim \mathfrak{aut}_p M_p > 1$ (see [16,17]) and partial classification in the case $\dim \mathfrak{aut}_p M_p = 1$ (this case is to be completed soon; see [18] for details). Unfortunately, in the case $\dim \mathfrak{aut} M_p = 0$ A. Loboda's approach is unapplicable and this case is to be studied by different methods (see [9] for some examples concerned with this case). In the Levi-degenerate case the complete classification has been obtained by G. Fels and W. Kaup in [13] (see also [14]).

In the present paper we classify all locally homogeneous CR-manifolds in dimension 4, i.e. we do the next step after E. Cartan's classification. Since for a generic CR-submanifold $\dim M = 2n + k$ holds, from $k, n > 0$ we get $n = 1$, $k = 2$ (the classification of locally homogeneous CR-manifolds in the case of a complex manifold ($k = 0$) and in the case of a totally real manifold ($n = 0$) is trivial: in the first case M is locally CR-equivalent to $\mathbb{C}^2$, in the second case M is locally CR-equivalent to a totally real 4-plane in $\mathbb{C}^4$). So in what follows M is supposed to be a real-analytic locally homogeneous CR-manifold of dimension 4 and codimension 2, generically embedded to the complex space $\mathbb{C}^3$.

A natural example of homogeneous CR-manifolds in the case under consideration is as follows. Let L be an affinely homogeneous curve in $\mathbb{R}^3$. Then the tube manifold $M = \{z \in \mathbb{C}^3\colon\ \operatorname{Im} z \in L\}$ is a homogeneous 4-dimensional CR-manifold. The homogeneity is provided by the abelian group of real translations $z \mapsto z + a$, $a \in \mathbb{R}^3$, and by the 1-dimensional affine group $z \mapsto A(t)z + b(t)$, where $y \mapsto A(t)y + b(t)$ is the 1-dimensional affine group, providing the homogeneity of L.

Another important example of a homogeneous 4-dimensional CR-manifold is concerned with the *main trichotomy* for CR-manifolds under consideration, demonstrated by V. Beloshapka, V. Ezhov and G. Schmalz in [5,6]. This trichotomy is some analogue of the trichotomy for real hypersurfaces in $\mathbb{C}^2$ by H. Poincaré. To formulate it, we firstly introduce the notion of total non-degeneracy. A 4-dimensional generic CR-manifold M in $\mathbb{C}^3$ is called *totally non-degenerate at a point* p, if it is Levi non-degenerate at p and, moreover,

$$T_p^{\mathbb{C}} M + \left[T^{\mathbb{C}} M, T^{\mathbb{C}} M\right]_p + \left[T^{\mathbb{C}} M, \left[T^{\mathbb{C}} M, T^{\mathbb{C}} M\right]\right]_p = T_p M,$$

where $T^{\mathbb{C}}M$ is the bundle of complex tangent planes to M; otherwise we call M just *degenerate* at p. 4-Dimensional totally non-degenerate CR-manifolds are 2-non-degenerate at each point [2]. They are also known as *Engel-type manifolds* (see [7]). As it was shown in [6], the total non-degeneracy condition is also equivalent to the existence of local holomorphic coordinates $(z, w_2, w_3) \in \mathbb{C}^3$, in which p is the origin and M is given as

$$\operatorname{Im} w_2 = |z|^2 + O(3), \qquad \operatorname{Im} w_3 = 2|z|^2 \operatorname{Re} z + O(4),$$

where z, w_2, w_3 are assigned the weights

$$[z] = 1, \qquad [w_2] = 2, \qquad [w_3] = 3$$

and $O(3)$, $O(4)$ are terms of weights $\geqslant 3$ and $\geqslant 4$ correspondingly. The manifold, for which the remainders vanish, is called *the* 4*-dimensional CR-cubic* (we will call it just *the cubic* and denote by C). The cubic is the main example of a homogeneous totally non-degenerate CR-manifold. The cubic is a particular case of a *model manifold* (see, for example, [4]) and has many remarkable properties (see [5–7,9] for details), the main one is given by the following trichotomy for a 4-dimensional locally homogeneous CR-submanifold M in $\mathbb{C}^3$:

(1) $\dim \mathfrak{aut}_p M_p = \infty$, which occurs if and only if M is locally biholomorphically equivalent to a direct product $M^3 \times \mathbb{R}^1$, where $M^3 \subset \mathbb{C}^2_{z_1,z_2}$ is a locally homogeneous hypersurface in $\mathbb{C}^2$, $\mathbb{R}^1 \subset \mathbb{C}^1_{z_3}$ is a real line (the degenerate case).

(2) $\dim \mathfrak{aut}_p M_p = 1$, which occurs if and only if M is locally biholomorphically equivalent to the cubic C.

(3) $\dim \mathfrak{aut}_p M_p = 0$ for all other totally non-degenerate manifolds (the *rigidity phenomenon*).

As it was demonstrated in [7], the cubic can be also singled out among totally non-degenerate surfaces as the (essentially) unique *flat* surface with respect to some special CR-curvature (some analogue of Chern and Moser curvature for real hypersurfaces in $\mathbb{C}^N$ [11]).

It is seen from the above trichotomy that *the homogeneity of a* 4*-dimensional totally non-degenerate CR-manifold, which is locally non-equivalent to the cubic, is provided by a* 4*-dimensional algebra of holomorphic vector fields, namely by its infinitesimal automorphism algebra.* This enables us to consider totally non-degenerate 4-dimensional CR-manifolds as orbits of local actions of 4-dimensional real Lie groups in $\mathbb{C}^3$ by holomorphic transformations with 0-dimensional isotropy. So we start with the G. Mubarakzyanov's classification of 4-dimensional real Lie algebras [19] and find (up to local biholomorphic transformations) all possible realizations of these algebras as holomorphic vector field algebras, providing the homogeneity of some totally non-degenerate 4-dimensional CR-manifold in $\mathbb{C}^3$. This enables us to present a certain list of homogeneous totally non-degenerate 4-dimensional surfaces in $\mathbb{C}^3$ such that any other locally homogeneous totally non-degenerate 4-dimensional real-analytic CR-manifold in $\mathbb{C}^3$ is locally biholomorphically equivalent to one of the surfaces in the list. This approach is realized in Section 2.

It also follows from the trichotomy that *a local CR-diffeomorphism between two* 4*-dimensional totally non-degenerate CR-manifolds, locally non-equivalent to the cubic, induces a local biholomorphic mappings between the vector field algebras, providing the homogeneity of the manifolds* (see Proposition 3.2 for details). This observation enables us to classify, up to local biholomorphic equivalence, the homogeneous surfaces in the list in the following way: we firstly single out the *spherical* surfaces (i.e. the surfaces, locally CR-equivalent to the cubic C), using the sphericity criterion from [7], and for the remaining surfaces we find all possible mapping between holomorphic vector filed algebras, providing their homogeneity. This approach is realized in Section 3 and finally allows us to obtain the desired classification, which is given by the following theorem (we use notations, associated to the cubic and denote the coordinates in $\mathbb{C}^3$ by z, w_2, w_3 and also set $z = x + iy$, $w_j = u_j + iv_j$):

Main Theorem. *Any real-analytic* 4*-dimensional locally homogeneous CR-manifold M is either flat and in this case it is locally CR-equivalent to one of the following flat CR-manifolds*:

(0.1) $\mathbb{C}^2$,
(0.2) $\mathbb{C} \times \mathbb{R}^2 \subset \mathbb{C}^3$,
(0.3) $\mathbb{R}^4 \subset \mathbb{C}^4$,

or is non-flat and in this case it is locally CR-equivalent to one of the following pairwise locally CR-inequivalent homogeneous surfaces in $\mathbb{C}^3$:

Case 1. $\dim \mathfrak{aut}_p M_p = \infty$ (*the degenerate case*):

(1.1) $M^3 \times \mathbb{R}^1$, *where* $M^3 \subset \mathbb{C}^2_{z,w_2}$ *is either a sphere, or one of the tube surfaces* (t1)–(t3), *or one of the projective surfaces* (p1)–(p3), *and* $\mathbb{R}^1 \subset \mathbb{C}^1_{w_3}$ *is a real line.*

Case 2. $\dim \mathfrak{aut}_p M_p = 1$ (*the spherical case*):

(2.1) $v_2 = y^2$, $v_3 = y^3 \sim v_2 = |z|^2$, $v_3 = 2|z|^2 \operatorname{Re} z$ (*the cubic*).

Case 3. $\dim \mathfrak{aut}_p M_p = 0$:

(3.1) $v_2 = xe^y + \gamma ye^y$, $v_3 = e^y$, $\gamma \in \mathbb{R}$.
(3.2) $v_2 = \frac{x}{y} + \gamma \ln y$, $v_3 = \frac{1}{y}$, $\gamma \in \mathbb{R}$.
(3.3) $v_2 = xy^\alpha + \gamma y^{\alpha+1}$, $v_3 = y^\alpha$, $y > 0$, $|\alpha| > 1$, $\alpha \neq 2$, $\gamma \in \mathbb{R}$.
(3.4) $v_2 = xy \ln y + \gamma y^2$, $v_3 = y \ln y$, $\gamma \in \mathbb{R}$.
(3.5) $v_2 = x\sqrt{1-y^2} + \gamma \arcsin y$, $v_3 = \sqrt{1-y^2}$, $|y| < 1$, $\gamma \in \mathbb{R}$.
(3.6) $v_2 = xv_3 + \gamma(v_3^2 + y^2)$, $\exp(q \operatorname{arctg} \frac{v_3}{y}) = v_3^2 + y^2$, $y > 0$, $v_3 > 0$, $q > 0$, $\gamma \in \mathbb{R}$.
(3.7) $v_2 = e^y$, $v_3 = e^{x+\delta y}$, $\delta \in \mathbb{R}$.
(3.8) $v_2 = e^{x+\alpha y} \cos \beta y$, $v_3 = e^{x+\alpha y} \sin \beta y$, $\beta > 0$, $(\alpha, \beta) \neq (0, 1)$.
(3.9) $v_2 = e^x y \cos y$, $v_3 = e^x y \sin y$.
(3.10) $v_2 = y^\alpha$, $v_3 = y^\beta$, $y > 0$, $1 \leqslant |\alpha| \leqslant |\beta|$, $\alpha, \beta \neq 1$, $\alpha \neq \beta$, $(\alpha, \beta) \neq (2, 3)$.
(3.11) $v_2 = e^{ay}$, $v_3 = e^y$, $0 < |a| < 1$.
(3.12) $v_2 = \operatorname{ch} y$, $v_3 = \operatorname{sh} y$.
(3.13) $v_2 = y \ln y$, $v_3 = y^\alpha$, $\alpha \neq \{0; 1\}$.
(3.14) $v_2 = ye^y$, $v_3 = e^y$.
(3.15) $v_2 = y^2$, $v_3 = e^y$.
(3.16) $v_2 = y \ln^2 y$, $v_3 = y \ln y$.
(3.17) $v_2 = e^y \cos \beta y$, $v_3 = e^y \sin \beta y$, $\beta > 0$.
(3.18) $v_2 = y^\alpha \cos(\beta \ln y)$, $v_3 = y^\alpha \sin(\beta \ln y)$, $\beta > 0$.
(3.19) $v_2 = \cos y$, $v_3 = \sin y$.

As a corollary we get the following statement.

Theorem 1.2. *Any locally homogeneous real-analytic totally non-degenerate* 4*-dimensional CR-manifold is locally CR-equivalent to an affinely homogeneous one, i.e. has a local affine realization.*

Note that the last property holds for G. Fels and W. Kaup's list of locally homogeneous 2-nondegenerate hypersurfaces in $\mathbb{C}^3$, but does not hold for E. Cartan's and A. Loboda's lists. Also note, that in contrast with G. Fels and W. Kaup's list, our list contains *essentially* non-tube manifolds (namely, the surfaces (3.1)–(3.9)), i.e. manifolds, locally CR-inequivalent to *any* tube 4-dimensional CR-manifold.

The Main Theorem also allows us to formulate the following classification theorem, specifying the main trichotomy and giving complete classification of locally homogeneous totally non-degenerate 4-dimensional CR-manifolds with non-trivial isotropy group.

Theorem 1.3. *Any locally homogeneous real-analytic totally non-degenerate* 4*-dimensional CR-manifold with non-trivial stability group is locally CR-equivalent to one of the following pairwise locally CR-inequivalent homogeneous surfaces*: (2.1) (*the cubic*), (3.2), (3.5), (3.12), (3.19).

In the first case the stability group at the origin looks as

$$z \mapsto \lambda z, \qquad w_2 \mapsto \lambda^2 w_2, \qquad w_3 \mapsto \lambda^3 w_3, \quad \lambda \in \mathbb{R}^*$$

and thus is isomorphic to $\mathbb{R}^*$; *in all other cases the stability group is of* $\mathbb{Z}_2$*-type and is generated by the automorphism*

$$z \mapsto -z, \qquad w_2 \mapsto -w_2, \qquad w_3 \mapsto w_3$$

at the point $(0, 0, i)$ *for the surfaces* (3.5), *by the automorphism*

$$z \mapsto -z, \qquad w_2 \mapsto w_2, \qquad w_3 \mapsto -w_3$$

at the point $(0, i, 0)$ *for the surfaces* (3.12), (3.19) *and by the automorphism*

$$z \mapsto w_3, \qquad w_2 \mapsto -w_2 + zw_3 + 1, \qquad w_3 \mapsto z$$

at the point $(i, 0, i)$ *for the surfaces* (3.2).

Comparing the obtained classification of locally homogeneous CR-manifolds in dimension 4 with Cartan's list of homogeneous surfaces we see that increase of the dimension of homogeneous CR-manifolds dramatically complicates the classification and one can assume, for example, that any list of (essentially) all locally homogeneous CR-manifolds in dimension 5 will be too long to present a sensible description of it, which means that the local homogeneity condition becomes too weak. This motivates to consider some classes of "super-symmetric" CR-manifolds, wider than the class of locally homogeneous CR-manifolds. One of the possible approaches in this direction was suggested by W. Kaup and D. Zaitsev in [15], who introduced the class of *symmetric* CR-manifolds. This class can be described as follows. Let M be a Riemannian CR-manifold. M is called *Hermitian CR-manifold*, if the Riemannian metric is compatible with the almost complex structure J on M in the sense that $\|Jv\|_p = \|v\|_p$ holds for each $v \in T_p^{\mathbb{C}}M$, $p \in M$. A Hermitian CR-manifold M is called *CR-symmetric*, if for each point $p \in M$ there exists a CR-isometry s_p of M, preserving the point p and such that the differential ds_p, restricted on the subspace $T_p^{\mathbb{C}}M \oplus T_p^{TR}M \subset T_pM$, is minus identical. Here $T_p^{TR}M$ is the *totally real part of* T_pM, i.e. the orthogonal complement to the subspace, spanned by $T_p^{\mathbb{C}}M$ and by the values at p of arbitrary order Lie brackets of vector fields $X \in TM$ with the condition $X_a \in T_a^{\mathbb{C}}M$ for each $a \in M$ (for finite type CR-manifolds, in particular for totally non-degenerate manifolds, the subspace $T_p^{TR}M$ is trivial; for infinite type CR-manifolds it is non-trivial and the condition for ds_p to be minus identical on $T_p^{TR}M$ guarantees the uniqueness of the involution s_p). In the paper [15] some beautiful connections between symmetric CR-manifolds and Hermitian symmetric spaces are demonstrated (see also [1]). In particular, it is proved that any symmetric CR-manifold is also CR-homogeneous. The cubic C is given in [15] as an example of a symmetric CR-manifold in dimension 4. Using Theorem 3.1 and the Cartan's classification theorem, we obtain the classification of *all* symmetric CR-manifolds in dimension 4.

Theorem 1.4. *Any real-analytic symmetric CR-manifold of dimension* 4 *is either flat and in this case it is locally CR-equivalent to one of the following flat CR-manifolds*:

(0.1) $\mathbb{C}^2$,
(0.2) $\mathbb{C} \times \mathbb{R}^2 \subset \mathbb{C}^3$,
(0.3) $\mathbb{R}^4 \subset \mathbb{C}^4$,

or is non-flat and in this case it is locally CR-equivalent to one of the following pairwise locally CR-inequivalent symmetric surfaces in $\mathbb{C}^3$:

Case 1. *Degenerate manifolds*:

(1.a) $v_2 = y^2$, $v_3 = 0$.
(1.b) $y^2 + v_2^2 = 1$, $v_3 = 0$.
(1.c) $y^2 - v_2^2 = 1$, $v_3 = 0$.
(1.d) $1 + |z|^2 + |w_2|^2 = a|1 + z^2 + w_2^2|$, $v_3 = 0$, $a > 1$.
(1.e) $1 + |z|^2 - |w_2|^2 = a|1 + z^2 - w_2^2|$, $v_3 = 0$, $a > 1$.
(1.f) $-1 + |z|^2 + |w_2|^2 = a|-1 + z^2 + w_2^2|$, $v_3 = 0$, $0 < |a| < 1$.

Case 2. *Totally non-degenerate manifolds*:

(2.a) $v_2 = y^2$, $v_3 = y^3$.
(2.b) $v_2 = \cos y$, $v_3 = \sin y$.
(2.c) $v_2 = \operatorname{ch} y$, $v_3 = \operatorname{sh} y$.
(2.d) $v_2 = x\sqrt{1-y^2} + \gamma \arcsin y$, $y^2 + v_3^2 = 1$, $|y| < 1$, $\gamma \in \mathbb{R}$.
(2.e) $v_2 = \frac{x}{y} + \gamma \ln y$, $v_3 = \frac{1}{y}$, $\gamma \in \mathbb{R}$.

Remark 1.5. As it was explained above, the key point in the obtained classification theorems is the main trichotomy, enabling us to assume that the homogeneity of a non-degenerate 4-dimensional CR-manifold is provided by a 4-dimensional local transitively acting Lie group, which is a priori not clear at all (see, for example, [13]). The main trichotomy, as well as the sphericity criterion for non-degenerate 4-dimensional CR-manifolds in $\mathbb{C}^3$ (see Section 3), are based on the model surface method (see [4,3]). Thus the model surface method for dimension 4 CR-manifolds enables to classify all (sensible) classes of CR-manifolds with symmetries in the dimension under consideration.

Remark 1.6. It follows from Theorem 1.3 that the holomorphic automorphism groups of the homogeneous surfaces (2.1)–(3.19) coincide with $\exp(\mathfrak{g}(M))$ except the cases of symmetric surfaces, when the automorphism group is a semidirect product of $\exp(\mathfrak{g}(M))$ and the stability group of a fixed point, described in Theorem 1.3. Since in all cases $\mathfrak{g}(M)$ can be easily integrated, this gives a description of the holomorphic automorphism groups of the homogeneous surfaces (2.1)–(3.19).

Remark 1.7. Note the following similarity of our list of homogeneous CR-manifolds with E. Cartan's list and G. Fels and W. Kaup's list: (the non-degenerate part of) our list consists of one "model" object with positive-dimensional stability subalgebra and "rigid" objects with trivial stability subalgebra (the "rigidity phenomenon").

Remark 1.8. The Main Theorem can be considered as the classification of all totally non-degenerate CR-structures on 4-dimensional real Lie groups.

2. Homogeneous CR-manifolds and 4-dimensional Lie algebras of holomorphic vector fields in $\mathbb{C}^3$

In what follows M is supposed to be a real-analytic 4-dimensional totally non-degenerate locally homogeneous generic CR-submanifold in $\mathbb{C}^3$, $p=(p_1,p_2,p_3)$ is a fixed point on M. As it follows from the above discussion, the homogeneity of M is provided by some 4-dimensional real Lie algebra of holomorphic vector fields, which coincides with $\mathfrak{aut}\,M_p$ if M is locally non-equivalent to the cubic C, or is a subalgebra of the 5-dimensional Lie algebra $\mathfrak{aut}\,M_p$ in case when M is locally CR-equivalent to C (see [9] for precise description of $\mathfrak{aut}\,C$). We denote this algebra by $\mathfrak{g}(M)$. M is the orbit of the natural local action of $\mathfrak{g}(M)$ in $\mathbb{C}^3$ at the point p. Also we denote by X_1,X_2,X_3,X_4 a basis of the Lie algebra $\mathfrak{g}(M)$.

As the values of X_1,X_2,X_3,X_4 at p span the 4-dimensional linear space T_pM, these values are linearly independent over $\mathbb{R}$. The next proposition shows that the total non-degeneracy property gives stronger restriction on these values.

Proposition 2.1. *If the vector fields X_1,X_2,X_3 span a 3-dimensional Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}(M)$, then their values at p are linearly independent over $\mathbb{C}$.*

Proof. Suppose that $\mathrm{rk}_{\mathbb{C}}\{X_1|_p, X_2|_p, X_3|_p\}<3$. Let $\mathfrak{a}^{\mathbb{C}}$ be the complexification of $\mathfrak{a}$. Consider the real action of $\mathfrak{a}$ in $\mathbb{C}^3$ as well as the complex action of $\mathfrak{a}^{\mathbb{C}}$ in $\mathbb{C}^3$ and denote the orbits of these local actions at p by N and L correspondingly. Since the values of X_1,X_2,X_3 at p are linearly independent over $\mathbb{R}$, N is a real 3-manifold. The inequality $\mathrm{rk}_{\mathbb{C}}\{X_1|_p, X_2|_p, X_3|_p\}<2$ is impossible because X_1,X_2,X_3,X_4 span T_pM. So we have $\mathrm{rk}_{\mathbb{C}}\{X_1|_p, X_2|_p, X_3|_p\}=2$, L is a 2-dimensional complex manifold and $N\subset L$ is a real submanifold. Consider $T_p^{\mathbb{C}}N\subset T_p^{\mathbb{C}}L$. Since $N\subset M$ and $T_p^{\mathbb{C}}M$ is of dimension 1, we conclude that $T_p^{\mathbb{C}}N=T_p^{\mathbb{C}}M$, consequently $T_p^{\mathbb{C}}M\subset T_p^{\mathbb{C}}L$ and the same holds for all neighbor points of N, which is a contradiction with the total non-degeneracy condition. Hence $\mathrm{rk}_{\mathbb{C}}\{X_1|_p, X_2|_p, X_3|_p\}=3$, as required. □

In this section we obtain a partial classification of the class of CR-manifolds under consideration, considering them as orbits of the natural local action of 4-dimensional real Lie algebras in $\mathbb{C}^3$. We use the classification of 4-dimensional real Lie algebras, given in [19] (the results of [19] are also described, for example, in [20]). There are 22 types of such algebras: 10 solvable decomposable ones, 10 solvable indecomposable ones, and 2 non-solvable decomposable ones. Some types contain real parameters. For our purposes it will be more convenient to single out five types of solvable algebras, which do not contain a 3-dimensional abelian ideal (according to [19], these are types $A_{4.8}$, $A_{4.7}$, $A_{4.9}$, $A_{2.2}\oplus A_{2.2}$ and $A_{4.10}$ correspondingly). We also denote by Type VI all solvable algebras, which contain a 3-dimensional abelian ideal (according to [19], these are algebras of types $A_{3.1}\oplus A_1$, $A_{2.2}\oplus A_{2.1}$, $A_{3.j}\oplus A_1$, $j=3,\dots,9$ and $A_{4.j}$, $j=1,\dots,6$) and denote by Types VII and VIII correspondingly the two non-solvable algebras $\mathfrak{so}_{2,1}(\mathbb{R})\oplus\mathbb{R}^1$ and $\mathfrak{so}_3(\mathbb{R})\oplus\mathbb{R}^1$. Now the classification looks as follows:

4-Dimensional real Lie algebras

I.
$$[X_1,X_2]=0,\qquad [X_1,X_3]=0,\qquad [X_2,X_3]=X_1,$$
$$[X_1,X_4]=(q+1)X_1,\qquad [X_2,X_4]=X_2,\qquad [X_3,X_4]=qX_3,\quad |q|\leqslant 1.$$

II.
$$[X_1,X_2]=0,\qquad [X_1,X_3]=0,\qquad [X_2,X_3]=X_1,$$
$$[X_1,X_4]=2X_1,\qquad [X_2,X_4]=X_2,\qquad [X_3,X_4]=X_2+X_3.$$

III. $$[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_1,$$
$$[X_1, X_4] = 2qX_1, \qquad [X_2, X_4] = qX_2 - X_3, \qquad [X_3, X_4] = X_2 + qX_3, \quad q \geqslant 0.$$

IV. $$[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_2,$$
$$[X_1, X_4] = X_1, \qquad [X_2, X_4] = 0, \qquad [X_3, X_4] = 0.$$

V. $$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = X_2,$$
$$[X_1, X_4] = X_2, \qquad [X_2, X_4] = -X_1, \qquad [X_3, X_4] = 0.$$

VI. $$[X_i, X_j] = 0, \quad 1 \leqslant i, j \leqslant 3, \qquad [X_i, X_4] = \sum_{j=1}^{3} c_{ij} X_j, \quad 1 \leqslant i \leqslant 3, \ c_{ij} \in \mathbb{R}.$$

VII. $$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3,$$
$$[X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0.$$

VIII. $$[X_1, X_2] = X_3, \qquad [X_1, X_3] = -X_2, \qquad [X_2, X_3] = X_1,$$
$$[X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0.$$

Here any two Lie algebras, corresponding to different types, are non-equivalent and any two algebras of the same type, corresponding to different parameters (except Type VI algebras), are non-equivalent as well. Algebras of Types I–V are solvable and do not contain 3-dimensional abelian subalgebras, Type VI algebras are solvable and have 3-dimensional abelian ideals, and algebras of Types VII, VIII are not solvable (they have simple 3-dimensional ideals).

According to the classification, we consider 8 cases depending on the type of the Lie algebra $\mathfrak{g}(M)$.

Type I. Lie algebras of this type have the following commuting relations:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_1,$$
$$[X_1, X_4] = (q+1)X_1, \qquad [X_2, X_4] = X_2, \qquad [X_3, X_4] = qX_3, \quad |q| \leqslant 1.$$

Applying now for an algebra of Type I Proposition 2.1, we conclude, that the values of X_1, X_2, X_3 at p are linearly independent over $\mathbb{C}$ and we can rectify the commuting vector fields X_1, X_2 simultaneously in some neighborhood of p, so in this neighborhood we have: $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$ (the notations are taken from the introduction). From the commuting relations we then have $X_3 = a(z)\frac{\partial}{\partial z} + (b(z) + w_3)\frac{\partial}{\partial w_2} + c(z)\frac{\partial}{\partial w_3}$ for some analytic functions $a(z), b(z), c(z)$, $a(p_1) \neq 0$. Then we firstly rectify the non-zero vector field $a(z)\frac{\partial}{\partial z}$ and after that make a variable change of kind $w_3 \to w_3 + C(z)$. Then in the new coordinates

$$X_1 = \frac{\partial}{\partial w_2}, \qquad X_2 = \frac{\partial}{\partial w_3}, \qquad X_3 = \frac{\partial}{\partial z} + \big(\tilde{b}(z) + w_3\big)\frac{\partial}{\partial w_2} + \big(\tilde{c}(z) + C_z\big)\frac{\partial}{\partial w_3}.$$

Now taking $C(z)$ from the equation $C_z + \tilde{c}(z) = 0$ we get $X_3 = \frac{\partial}{\partial z} + (\tilde{b}(z) + w_3)\frac{\partial}{\partial w_2}$, and after a transformation $w_2 \to w_2 + B(z)$ for a function $B(z)$, satisfying $B_z + \tilde{b}(z) = 0$, we finally get $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$.

Now from the commuting relations for X_4 it is not difficult to verify that X_4 must have the form $X_4 = (qz + l)\frac{\partial}{\partial z} + ((q+1)w_2 + mz + n)\frac{\partial}{\partial w_2} + (w_3 + m)\frac{\partial}{\partial w_3}$, $l, m, n \in \mathbb{C}$. Also from the fact that the vector fields X_1, X_2, X_3 are tangent to M, we can conclude that M is given by equations

$$v_2 = x\psi(y) + \tau(y), \qquad v_3 = \psi(y) \tag{1}$$

for some real-analytic functions $\psi(y), \tau(y)$. To see that, we present M in the form $v_2 = F(x, y, u_2, u_3)$, $v_3 = G(x, y, u_2, u_3)$, which is possible since iX_1, iX_2 are transversal to M, and get from the tangency conditions $F_{u_2} = F_{u_3} = G_{u_2} = G_{u_3} = G_x = 0$, $F_x = G$. Since X_4 is tangent to M, we get the following conditions for ψ, τ:

$$(q+1)x\psi(y) + (q+1)\tau(y) + m_1 y + m_2 x + n_2 = qx\psi + qxy\psi_y + l_2 x\psi_y + l_1\psi + l_2\tau_y + qy\tau_y;$$
$$\psi + m_2 = l_2\psi_y + qy\psi_y.$$

Here $l = l_1 + il_2$, $m = m_1 + im_2$, $n = n_1 + in_2$. The second equation is a separable variables differential equation on ψ. Putting its solution to the first equation, we get a separable variables differential equation on τ. Note that any linear terms in ψ and τ can be eliminated in (1) by a polynomial transformation of $\mathbb{C}^3$. Also note that the total non-degeneracy condition requires $\psi(y) \neq ay + b$. Then, solving the equations on ψ, τ, making a polynomial transformation of $\mathbb{C}^3$ as well as a linear variable change with respect to z, we can put:

Case $q = 0$: $\psi = Ae^y$, $\tau = Bye^y$, $A \neq 0$. After a scaling in (1) we may suppose $A = 1$.

Case $q = -1$: $\psi = \frac{A}{y}$, $\tau = B \ln y$, $A \neq 0$. After a scaling we may suppose $A = 1$.

Case $q \neq \{0, \pm 1\}$: $\psi(y) = Ay^\alpha$, $\tau(y) = By^{\alpha+1}$, $A \neq 0$, where $\alpha = \frac{1}{q}$. After a scaling we may suppose $A = 1$.

It is interesting that the case $q = 1$ can't occur for a totally non-degenerate manifold, i.e. all orbits of the Lie algebra of Type I with $q = 1$ in $\mathbb{C}^3$ are degenerate.

Proposition 2.2. *Any totally non-degenerate CR-manifold with* $\mathfrak{g}(M)$ *of Type I is locally CR-equivalent to one of the following surfaces*:

$$\begin{aligned}
&\text{Type Ia:} \quad v_2 = xe^y + \gamma ye^y, \qquad v_3 = e^y, \quad \gamma \in \mathbb{R},\\
&\text{Type Ib:} \quad v_2 = \frac{x}{y} + \gamma \ln y, \qquad v_3 = \frac{1}{y}, \quad \gamma \in \mathbb{R},\\
&\text{Type Ic:} \quad v_2 = xy^\alpha + \gamma y^{\alpha+1}, \qquad v_3 = y^\alpha, \quad |\alpha| > 1,\ \gamma \in \mathbb{R}.
\end{aligned}$$

The restriction on α follows from the condition $|q| \leqslant 1$.

Type II. Lie algebras of this type have the following commuting relations:

$$\begin{aligned}
&[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_1,\\
&[X_1, X_4] = 2X_1, \qquad [X_2, X_4] = X_2, \qquad [X_3, X_4] = X_2 + X_3.
\end{aligned}$$

Since the commuting relations among X_1, X_2, X_3 are the same as in case I, we conclude that in appropriate coordinates these vector fields have the form $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3 \frac{\partial}{\partial w_2}$. From the commuting relations for X_4 it is also not difficult to verify that $X_4 = (z + l)\frac{\partial}{\partial z} + (2w_2 + \frac{1}{2}z^2 + mz + n)\frac{\partial}{\partial w_2} + (z + w_3 + m)\frac{\partial}{\partial w_3}$, $l, m, n \in \mathbb{C}$. After a translation along z (which do not change X_1, X_2, X_3) we may suppose $l = 0$. Since X_1, X_2, X_3 are the same as in case I, we can also conclude that M has the form (1). The tangency conditions for the vector field X_4 have the form:

$$\begin{aligned}
&2x\psi + 2\tau + xy + m_1 y + m_2 x = x\psi + xy\psi_y + y\tau_y;\\
&\psi + y + m_2 = y\psi_y.
\end{aligned}$$

The second equation is a linear differential equation on ψ. Putting its solution to the first equation, we get a linear differential equation on τ. Solving the equations and eliminating the linear terms for ψ and τ in (1) by a polynomial transformation of $\mathbb{C}^3$, we can put: $\psi(y) = Ay \ln|y|$, $\tau(y) = By^2$, $A \neq 0$. After a scaling in (1) we may suppose $A = 1$. Thus we have proved the following proposition.

Proposition 2.3. *Any totally non-degenerate CR-manifold with* $\mathfrak{g}(M)$ *of Type* II *is locally CR-equivalent to one of the following surfaces*:

$$\text{Type II:} \quad v_2 = xy \ln y + \gamma y^2, \qquad v_3 = y \ln y, \quad \gamma \in \mathbb{R}.$$

Type III. Lie algebras of this type have the following commuting relations:

$$\begin{aligned}
&[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_1,\\
&[X_1, X_4] = 2qX_1, \qquad [X_2, X_4] = qX_2 - X_3, \qquad [X_3, X_4] = X_2 + qX_3, \quad q \geqslant 0.
\end{aligned}$$

Since the commuting relations among X_1, X_2, X_3 are the same as in case I, we conclude that in appropriate coordinates these vector fields have the form $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3 \frac{\partial}{\partial w_2}$. From the commuting relations for X_4 it is also not difficult to verify that $X_4 = (qz - w_3 + l)\frac{\partial}{\partial z} + (2qw_2 + \frac{1}{2}z^2 - \frac{1}{2}w_3^2 + mz + n)\frac{\partial}{\partial w_2} + (z + qw_3 + m)\frac{\partial}{\partial w_3}$, $l, m, n \in \mathbb{C}$. After a translation along z (which does not change X_1, X_2, X_3) we may suppose $m = 0$. Now we consider two cases.

Case 1. $q = 0$. Since X_1, X_2, X_3 are the same as in case I, we can also conclude that M has the form (1). The tangency conditions for the vector field X_4 have the form:

$$xy - u_3\psi + n_2 = (-u_3 + l_1)\psi + x\psi_y(-\psi + l_2) + \tau_y(-\psi + l_2);$$
$$y = \psi_y(-\psi + l_2).$$

The second equation is a separable variables differential equation on ψ. Putting its solution to the first equation, we get a separable variables differential equation on τ. Solving the equations and eliminating the linear part of ψ and τ in (1) by a polynomial transformation of $\mathbb{C}^3$, we can put: $\psi(y) = \sqrt{R^2 - y^2}$, $\tau(y) = B \arcsin \frac{y}{R}$, $R > 0$. After a scaling in (1) we may suppose $R = 1$, and M is finally given as

$$v_2 = x\sqrt{1 - y^2} + \gamma \arcsin y, \qquad v_3 = \sqrt{1 - y^2}, \quad \gamma \in \mathbb{R}.$$

Case 2. $q > 0$. In that case after a translation along w_2 we may suppose $n = 0$. Also we make a variable change, which linearizes our vector field algebra:

$$z \mapsto z, \qquad w_2 \mapsto 2w_2 - zw_3, \qquad w_3 \mapsto w_3. \tag{2}$$

As a result we have $X_1 = 2\frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3} - z\frac{\partial}{\partial w_2}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = (qz - w_3 + l)\frac{\partial}{\partial z} + (2qw_2 - lw_3)\frac{\partial}{\partial w_2} + (qw_3 + z)\frac{\partial}{\partial w_3}$. Replacing X_4 by $X_4 - l_1 X_3$, we may suppose that $\operatorname{Re} l = 0$. Now after the translations $z \mapsto z + qib$, $w_3 \mapsto w_3 - ib$, where $b = \frac{l_1}{q^2+1}$, we get $X_1 = 2\frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3} - (z - qib)\frac{\partial}{\partial w_2}$, $X_3 = \frac{\partial}{\partial z} + (w_3 + ib)\frac{\partial}{\partial w_2}$, $X_4 = (qz - w_3)\frac{\partial}{\partial z} + (2qw_2 - lw_3)\frac{\partial}{\partial w_2} + (z + qw_3)\frac{\partial}{\partial w_3}$ (we replace X_4 by $X_4 - bl_2 X_1$). Presenting now M in the form $v_2 = F(x, y, u_2, u_3)$, $v_3 = G(x, y, u_2, u_3)$, from the tangency conditions for X_1, X_2, X_3 we get $F_{u_2} = G_x = G_{u_2} = G_{u_3} = 0$, $F_x = G + b$, $F_{u_3} = -y + bq$. Then M is given as

$$v_2 = x\psi(y) + \tau(y) - yu_3 + bqu_3 + bx, \qquad v_3 = \psi(y).$$

The tangency conditions for X_4 imply

$$2q(x\psi + \tau - yu_3 + bqu_3 + bx) - bu_3\big(q^2 + 1\big) = (qx - u_3)\psi + x\psi_y(qy - \psi) + \tau_y(qy - \psi) - (qy - \psi)u_3$$
$$- y(qu_3 + x) + bq(qu_3 + x) + b(qx - u_3);$$
$$q\psi + y = \psi_y(qy - \psi).$$

From the second equation we get

$$\psi_y = \frac{q\psi + y}{qy - \psi}.$$

This is a first order homogeneous differential equation. The general solution is $\exp(2q \operatorname{arctg} \frac{\psi}{y}) = c^2(\psi^2 + y^2)$, $c > 0$. Hence we get $\psi_c(y) = \frac{1}{c}\psi_1(cy)$ and after a scaling in the defining equations of M we may suppose for the original manifold M that $\psi(y) = \psi_1(y)$.

For τ from the first tangency condition we get

$$\frac{\tau_y}{\tau} = \frac{2q}{qy - \psi(y)}. \tag{3}$$

Hence the general solution is $\tau_c(y) = c\tau_1(y)$. It is straightforward to check that $\tau = c(\psi^2 + y^2)$ is actually the general solution of (3). Eliminating the pluriharmonic terms in the right-hand side of the defining equations of M by a polynomial transformation of $\mathbb{C}^3$ and replacing $-yu_3$ by xv_3 (since the difference is pluriharmonic), we can present M after a scaling in the following way:

$$v_2 = xv_3 + \gamma\big(v_3^2 + y^2\big), \qquad \exp\left(2q \operatorname{arctg} \frac{v_3}{y}\right) = v_3^2 + y^2, \quad q > 0,\ \gamma \in \mathbb{R}.$$

Proposition 2.4. *Any totally non-degenerate CR-manifold with $\mathfrak{g}(M)$ of Type* III *is locally CR-equivalent to one of the following surfaces*:

Type IIIa: $\quad v_2 = x\sqrt{1 - y^2} + \gamma \arcsin y, \qquad v_3 = \sqrt{1 - y^2}, \quad \gamma \in \mathbb{R},$

Type IIIb: $\quad v_2 = xv_3 + \gamma\big(v_3^2 + y^2\big), \qquad \exp\left(2q \operatorname{arctg} \frac{v_3}{y}\right) = v_3^2 + y^2, \quad q > 0,\ \gamma \in \mathbb{R}.$

Type IV. Lie algebras of this type have the following commuting relations:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = X_2,$$
$$[X_1, X_4] = X_1, \qquad [X_2, X_4] = 0, \qquad [X_3, X_4] = 0.$$

Using Proposition 2.1, we can rectify the commuting vector fields X_1, X_2, so $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$. From the commuting relations

$$X_3 = a(z)\frac{\partial}{\partial z} + b(z)\frac{\partial}{\partial w_2} + \bigl(w_3 + d(z)\bigr)\frac{\partial}{\partial w_3}, \quad a(z) \neq 0.$$

Now we firstly rectify $a(z)\frac{\partial}{\partial z}$, then make a variable change $w_2 \to w_2 + B(z)$, $w_3 \to w_3 + D(z)$ for functions $B(z)$, $D(z)$, satisfying $B_z + b(z) = 0$, $D_z - D(z) + d(z) = 0$ and finally get $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_3}$.

Now from the commuting relations for X_4 it is not difficult to verify that X_4 must have the form $X_4 = l\frac{\partial}{\partial z} + (w_2 + m)\frac{\partial}{\partial w_2} + ne^z\frac{\partial}{\partial w_3}$, $l, m, n \in \mathbb{C}$. Also from the fact that the vector fields X_1, X_2, X_3 are tangent to M, we can conclude that M is given by equations

$$v_2 = \psi(y), \qquad v_3 = e^x\tau(y) \tag{4}$$

for some real-analytic functions $\psi(y)$, $\tau(y)$. To see that, we present M in the form $v_2 = F(x, y, u_2, u_3)$, $v_3 = G(x, y, u_2, u_3)$ and get from the tangency conditions $F_{u_2} = F_{u_3} = G_{u_2} = G_{u_3} = F_x = 0$, $G_x = G$. Since X_4 is tangent to M, we get the following conditions on ψ, τ:

$$\psi + m_2 = l_2\psi_y;$$
$$n_1e^x \sin y + n_2e^x \cos y = l_1e^x\tau + l_2e^x\tau_y.$$

The first equation is a separable variables differential equation on ψ. Putting its solution to the second equation, we get a separable variables differential equation on τ. Note that linear terms in ψ and terms of kind $a\sin y + b\cos y$ in τ in (4) can be eliminated by transformations of kind $w_2 \to w_2 + A(z)$, $w_3 \to w_3 + B(z)$. Also from the total non-degeneracy $\psi, \tau \neq 0$. Then after scalings in (4) we get $\psi = e^y$, $\tau = e^{\delta y}$. Thus we have proved the following proposition.

Proposition 2.5. *Any totally non-degenerate CR-manifold with $\mathfrak{g}(M)$ of Type* IV *is locally CR-equivalent to one of the following surfaces*:

$$\text{Type IV:} \quad v_2 = e^y, \qquad v_3 = e^{x+\delta y}, \quad \delta \in \mathbb{R}.$$

Type V. Lie algebras of this type have the following commuting relations:

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = X_2,$$
$$[X_1, X_4] = X_2, \qquad [X_2, X_4] = -X_1, \qquad [X_3, X_4] = 0.$$

Having $\mathfrak{g}(M)$ of Type V, we firstly rectify the commuting vector fields X_1, X_2 (using Proposition 2.1), so $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$. Then from the commuting relations $X_3 = a(z)\frac{\partial}{\partial z} + (w_2 + b(z))\frac{\partial}{\partial w_2} + (w_3 + d(z))\frac{\partial}{\partial w_3}$, $a(z) \neq 0$. After a rectification of $a(z)\frac{\partial}{\partial z}$ and a variable change $w_2 \to w_2 + B(z)$, $w_3 \to w_3 + D(z)$ for functions $B(z)$, $D(z)$, satisfying $B_z + b(z) = 0$; $D_z + d(z) = 0$, we finally get $X_3 = \frac{\partial}{\partial z} + w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3}$. From the commuting relations for the vector field X_4 we then get

$$X_4 = l\frac{\partial}{\partial z} + \bigl(me^z - w_3\bigr)\frac{\partial}{\partial w_2} + \bigl(ne^z + w_2\bigr)\frac{\partial}{\partial w_3}, \quad l, m, n \in \mathbb{C}.$$

Presenting M in the form $v_2 = F(x, y, u_2, u_3)$, $v_3 = G(x, y, u_2, u_3)$, we get from the tangency conditions for X_1, X_2, X_3: $F_{u_2} = F_{u_3} = G_{u_2} = G_{u_3} = 0$, $F_x = F$, $G_x = G$, which means that M is given as

$$v_2 = e^x\psi(y), \qquad v_3 = e^x\tau(y). \tag{5}$$

The tangency conditions for X_4 now give

$$m_1e^x \sin y + m_2e^x \cos y - e^x\tau = l_1e^x\psi + l_2e^x\psi_y;$$
$$n_1e^x \sin y + n_2e^x \cos y + e^x\psi = l_1e^x\tau + l_2e^x\tau_y.$$

These equations can be reduced to two non-homogeneous linear second order differential equations on ψ and τ correspondingly with constant coefficients and trigonometric polynomial in the right-hand side. Solving these equations and eliminating trigonometric terms $a\sin y + b\cos y$ by transformations of kind $w_j \to w_j + C_j e^z$ in (5), we get:

Case $l = l_1 + il_2 \neq \pm i$: $\psi = c_1 e^{\alpha y}\cos\beta y + c_2 e^{\alpha y}\sin\beta y$, $\tau = \tilde{c}_1 e^{\alpha y}\cos\beta y + \tilde{c}_2 e^{\alpha y}\sin\beta y$, where $\alpha = -\frac{l_1}{l_2}$, $\beta = \frac{1}{l_2}$ (non-resonant case);

Case $l = l_1 + il_2 = \pm i$: $\psi = c_1 y\cos y + c_2 y\sin y$, $\tau = \tilde{c}_1 y\cos y + \tilde{c}_2 y\sin y$ (resonant case). After a linear transformation of $\mathbb{C}^3$ we can put in (5) in both cases $c_1 = \tilde{c}_2 = 1$, $c_2 = \tilde{c}_1 = 0$. Thus we have proved the following proposition.

Proposition 2.6. *Any totally non-degenerate CR-manifold with* $\mathfrak{g}(M)$ *of Type* V *is locally CR-equivalent to one of the following surfaces*:

$$\text{Type Va:}\quad v_2 = e^{x+\alpha y}\cos\beta y,\qquad v_3 = e^{x+\alpha y}\sin\beta y,\quad \beta > 0,\ \alpha + \beta i \neq i,$$
$$\text{Type Vb:}\quad v_2 = e^x y\cos y,\qquad v_3 = e^x y\sin y.$$

Type VI. Lie algebras of this type are characterized by the property that they have an abelian 3-dimensional ideal. Hence we have the following commuting relations:

$$[X_i, X_j] = 0,\quad 1 \leqslant i, j \leqslant 3,\qquad [X_i, X_4] = \sum_{j=1}^{3} c_{ij} X_j,\quad 1 \leqslant i \leqslant 3,\ c_{ij} \in \mathbb{R}.$$

Applying Proposition 2.1, we can rectify the commuting vector fields X_1, X_2, X_3 simultaneously, then we have $X_1 = \frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial w_2}, X_3 = \frac{\partial}{\partial w_3}$. From the commuting relations we can conclude now that X_4 is an affine vector field with a real linear part and hence (since M is invariant under X_1, X_2, X_3) *M is a tube over a locally affinely homogeneous curve in* $\mathbb{R}^3$. All possible actions of affine 1-dimensional transformation groups in $\mathbb{R}^3$ were classified in [23]. It is not difficult to obtain from that classification the affine classification of locally affinely homogeneous curves in $\mathbb{R}^3$ and hence the real-affine classification of the corresponding tubes in $\mathbb{C}^3$. Rejecting the totally degenerate surfaces, we get the following proposition.

Proposition 2.7. *Any totally non-degenerate CR-manifold with* $\mathfrak{g}(M)$ *of Type* VI *is locally CR-equivalent to one of the following pairwise locally affinely non-equivalent affinely homogeneous tube surfaces*:

$$\text{Type VIa:}\quad v_2 = y^\alpha,\qquad v_3 = y^\beta,\quad 1 \leqslant |\alpha| \leqslant |\beta|,\ \alpha, \beta \neq 1,\ \alpha \neq \beta,$$
$$\text{Type VIb:}\quad v_2 = e^{ay},\qquad v_3 = e^y,\quad -1 \leqslant a < 1,$$
$$\text{Type VIc:}\quad v_2 = y\ln y,\qquad v_3 = y^\alpha,\quad \alpha \neq 0; 1,$$
$$\text{Type VId:}\quad v_2 = ye^y,\qquad v_3 = e^y,$$
$$\text{Type VIe:}\quad v_2 = y^2,\qquad v_3 = e^y,$$
$$\text{Type VIf:}\quad v_2 = y\ln^2 y,\qquad v_3 = y\ln y,$$
$$\text{Type VIg:}\quad v_2 = e^y\cos\beta y,\qquad v_3 = e^y\sin\beta y,\quad \beta > 0,$$
$$\text{Type VIh:}\quad v_2 = y^\alpha\cos(\beta\ln y),\qquad v_3 = y^\alpha\sin(\beta\ln y),\quad \beta > 0,$$
$$\text{Type VIi:}\quad v_2 = \cos y,\qquad v_3 = \sin y.$$

The surface VIb for $a = -1$ can be singled out among other surfaces VIb by its symmetry property (see Section 3) and thus is singled out in the Main Theorem to type (3.12).

Types VII–VIII. Lie algebras of these types have the following commuting relations:

$$[X_1, X_2] = X_1,\qquad [X_1, X_3] = 2X_2,\qquad [X_2, X_3] = X_3,$$
$$[X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0\quad \text{(Type VII)},$$
$$[X_1, X_2] = X_3,\qquad [X_1, X_3] = -X_2,\qquad [X_2, X_3] = X_1,$$
$$[X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0\quad \text{(Type VIII)}.$$

Consider firstly an algebra of Type VIII. It contains the subalgebra $\mathfrak{a} = \operatorname{span}\{X_1, X_2, X_3\}$, isomorphic to $\mathfrak{so}_3(\mathbb{R})$. By Proposition 2.1, the values of the vector fields X_1, X_2, X_3 at p are linearly independent over $\mathbb{C}$, which implies that the orbit of the natural local action of the complexified algebra $\mathfrak{a}^{\mathbb{C}}$ at p is an open set in $\mathbb{C}^3$. Hence there exists a local biholomorphic mapping from the Lie group $SO_3(\mathbb{C})$ to a neighborhood of p such that $\mathfrak{a}^{\mathbb{C}}$ is the image of the tangent algebra of the Lie group $SO_3(\mathbb{C})$ under this mapping and the point p corresponds to the element Id in $SO_3(\mathbb{C})$. Hence after a local holomorphic coordinate change we may suppose that $\mathfrak{a}^{\mathbb{C}}$ is the algebra of left-invariant vector fields on $SO_3(\mathbb{C})$ and that X_4 is a vector field on $SO_3(\mathbb{C})$, commuting with this algebra. Considering the flows of vector fields from $\mathfrak{a}^{\mathbb{C}}$ and of the vector field X_4, we conclude that these flows commute, which implies that any transformation from the flow of X_4 commute with the standard left multiplications in $SO_3(\mathbb{C})$. If φ is one of these transformations and $x, g \in SO_3(\mathbb{C})$, then we get $\varphi(g \cdot x) = g \cdot \varphi(x)$. Putting $x = e$, we get $\varphi(g) = g\varphi(e)$, which means that X_4 generates a one-parametric subgroup of right multiplications and hence is a right-invariant vector field.

Now the orbit of the given algebra at the point Id (corresponding to the original point p) is described as follows. Consider $\mathfrak{a}$ as a real subalgebra in the matrix Lie algebra $\mathfrak{so}_3(\mathbb{C})$. Since the values of the vector fields X_1, X_2, X_3 at p are linearly independent over $\mathbb{C}$, this subalgebra is a totally-real subspace. Let A_1, A_2, A_3 be a basis of this subspace. Then orbit of the action of the real Lie subgroup, corresponding to $\mathfrak{a}$, is given as $e^{x_1 A_1} \cdot e^{x_2 A_2} \cdot e^{x_3 A_3} \cdot Z$, $Z \in SO_3(\mathbb{C})$, $x_j \in \mathbb{R}$. The orbit of the action of X_4 is given as $Z \cdot e^{-Bt}$, $Z \in SO_3(\mathbb{C})$, $t \in \mathbb{R}$ for some matrix B from $\mathfrak{so}_3(\mathbb{C})$. Since $\mathfrak{a}$ is totally real, the matrix B can be presented as $B_1 + iB_2$, $B_1, B_2 \in \mathfrak{a}$. If $\tilde{X}_4$ is the right-invariant vector field, corresponding to the matrix iB_2, then (since $B_1 \in \mathfrak{a}$) at each point the vector fields X_1, X_2, X_3, X_4 and $X_1, X_2, X_3, \tilde{X}_4$ span the same 4-dimensional real linear space. This observation allows us to put $B_1 = 0$. Now we choose a basis in $\mathfrak{a}$ in such a way that $A_3 = -B_2$. Then we finally obtain that the desired orbit looks as follows:

$$e^{x_1 A_1} \cdot e^{x_2 A_2} \cdot e^{x_3 A_3} \cdot e^{itA_3}, \quad x_j, t \in \mathbb{R}.$$

Consider now the mapping $F : \mathbb{C}^3 \to SO_3(\mathbb{C})$, given as $F(z_1, z_2, z_3) = e^{z_1 A_1} \cdot e^{z_2 A_2} \cdot e^{z_3 A_3}$. It is biholomorphic at the origin, since $F_{z_j}(0) = A_j$ and A_j are linearly independent over $\mathbb{C}$, and the orbit turns out to be the image of the 4-plane $\{\operatorname{Im} z_1 = \operatorname{Im} z_2 = 0\}$ under F. Hence the orbit is degenerate, which is a contradiction. In the same way we obtain that all M with $\mathfrak{g}(M)$ of Type VII are degenerate (in that case the subalgebra, spanned by X_1, X_2, X_3 is isomorphic to $\mathfrak{so}_{2,1}(\mathbb{R})$). Thus we have proved the following proposition.

Proposition 2.8. *Any CR-manifold M with $\mathfrak{g}(M)$ of Types* VII–VIII *is degenerate.*

We resume this chapter by formulating the following partial classification theorem.

Theorem 2.9. *Any totally non-degenerate locally homogeneous* 4*-dimensional CR-manifold in $\mathbb{C}^3$ is locally CR-equivalent to one of the homogeneous surfaces* Ia, Ib, Ic, II, IIIa, IIIb, IV, Va, Vb, VIa–VIi.

Remark 2.10. Realizations of low-dimensional real Lie algebras as algebras of vector fields in a real linear space were considered in many papers. For example, realizations of 4-dimensional real Lie algebras as algebras of vector fields in $\mathbb{R}^3$ were considered in [22] and some formulas, obtained in the present section, are presented in [22], but the direct application of the results of [22] is impossible in our case since the situation of a real algebra of holomorphic vector fields in a complex space gives some restrictions on the possible realizations (as, for example, Proposition 2.1 shows) as well as some new possibilities, as the above examples show.

3. The classification

In this section we specify the partial classification Theorem 2.9. This finally allows us to prove the Main Theorem. More precisely, we study the local CR-equivalence relations among the surfaces Ia–VIi. To do so, we firstly give the following definition.

Definition 3.1. A totally non-degenerate locally homogeneous 4-dimensional CR-manifold M is called *spherical*, if at some point (and hence at each point) it is locally CR-equivalent to the cubic C. Otherwise M is called *non-spherical*.

The term "spherical" is used in analogue with the case of a hypersurface in $\mathbb{C}^2$, where the 3-dimensional sphere is the model surface for the class of Levi non-degenerate hypersurfaces [21]. Using the trichotomy for 4-dimensional locally homogeneous CR-submanifolds in $\mathbb{C}^3$ (see introduction), we get the following proposition.

Proposition 3.2. *Two non-spherical* 4*-dimensional totally non-degenerate locally homogeneous CR-submanifold M, M' are locally CR-equivalent if and only if there exists a local biholomorphic mapping F, translating a point $p \in M$ to a point $p' \in M'$ and* (*locally*) *translating the algebra $\mathfrak{g}(M) = \mathfrak{aut}\, M_p$ into the algebra $\mathfrak{g}(M') = \mathfrak{aut}\, M'_{p'}$. In particular, if M, M' are CR-equivalent, then $\mathfrak{g}(M)$, $\mathfrak{g}(M')$ are isomorphic as Lie algebras.*

It follows from the above proposition that for non-spherical manifolds the local CR-equivalence problem can be reduced to the biholomorphic equivalence problem for vector field algebras, providing the homogeneity of the manifolds. Hence it is important now to find out what surfaces in the extended list Ia–VIi are spherical. We firstly note that the sphericity of VIa with $\alpha = 2$, $\beta = 3$ follows from [9]. The sphericity of Ic for $\alpha = 2$ can be verified from the previous fact by applying the binomial formula for $(x + iy)^3$. To do the sphericity check for the other surfaces, we refer to the sphericity criterion, formulated in [7]. According to this criterion, a 4-dimensional totally non-degenerate locally homogeneous CR-manifold M is spherical if and only if in some local coordinates (z, w_2, w_3) it can be presented as

$$v_2 = |z|^2 + O(6), \qquad v_3 = 2|z|^2 \operatorname{Re} z + O(7),$$

where the variables are assigned the weights $[z] = 1$, $[w_2] = 2$, $[w_3] = 3$ and $O(6)$, $O(7)$ are terms of weights grater than 6 and 7 correspondingly (this is some analogue for the sphericity criterion for a hypersurface in $\mathbb{C}^N$, see [10,11]). To apply this criterion to the above list of homogeneous surfaces, we consider all possible holomorphic transformations, preserving the origin, and present them as

$$z \mapsto f_1 + \dots + f_n + O(n+1), \qquad w_2 \mapsto g_1 + \dots + g_{n+1} + O(n+2),$$
$$w_3 \mapsto h_1 + \dots + h_{n+2} + O(n+3),$$

where f_j, g_j, h_j are polynomials of weight j. We call the collection of all f_j, g_k, h_l for $j \leqslant n$, $k \leqslant n+1$, $l \leqslant n+2$ *the* $(n, n+1, n+2)$*-jet* of the transformation. We also present the manifold as

$$v_2 = |z|^2 + \sum_{j=3}^{\infty} F_j, \qquad v_3 = 2|z|^2 \operatorname{Re} z + \sum_{j=4}^{\infty} G_j,$$

where F_j, G_j are polynomials of weight j and call the collection of all F_j, G_{j+1} for $3 \leqslant j \leqslant m$ *the* $(m, m+1)$ *jet* of M. Then we note the following. Given a mapping of a manifold

$$v_2 = |z|^2 + F_3 + \dots + F_m + O(m+1),$$
$$v_3 = 2|z|^2 \operatorname{Re} z + G_4 + \dots + G_{m+1} + O(m+2)$$

to a manifold of the same kind

$$v_2 = |z|^2 + \hat{F}_3 + \dots + \hat{F}_m + O(m+1),$$
$$v_3 = 2|z|^2 \operatorname{Re} z + \hat{G}_4 + \dots + \hat{G}_{m+1} + O(m+2),$$

preserving the origin, for a fixed $(m, m+1)$-jet of the first manifold the $(m, m+1)$-jet of the second manifold depends only on the $(m-1, m, m+1)$-jet of the mapping. Then it is clear that the vanishing condition for the $(5, 6)$-jet of the manifold (which is equivalent to the sphericity) is a condition on the $(4, 5, 6)$-jet of the mapping. This condition is a system of equations on the coefficients of the $(4, 5, 6)$-jet of the mapping.

Now we describe the process of the sphericity inspection for a totally non-degenerate surface.

Step 0. We present the equations of the surface as

$$v_2 = |z|^2 + F_3 + F_4 + F_5 + O(6), \qquad v_3 = 2|z|^2 \operatorname{Re} z + G_4 + G_5 + G_6 + O(7).$$

Step 1. We write the condition on the coefficients of a mapping

$$z \mapsto z + f_2 + O(3), \qquad w_2 \mapsto w_2 + g_3 + O(4), \qquad w_3 \mapsto w_3 + h_4 + O(5),$$

which maps the original surface onto a surface

$$v_2 = |z|^2 + O(4), \qquad v_3 = 2|z|^2 \operatorname{Re} z + O(5).$$

This condition is a system of 17 real equations on 18 real variables. This system always has a solution (f_2, g_3, h_4) (see [6]).

Step 2. We write the condition on the coefficients of a mapping

$$z \mapsto z + f_2 + f_3 + O(4), \qquad w_2 \mapsto w_2 + g_3 + g_4 + O(5), \qquad w_3 \mapsto w_3 + h_4 + h_5 + O(6),$$

which maps the surface, obtained in Step 1, onto a surface

$$v_2 = |z|^2 + O(5), \qquad v_3 = 2|z|^2 \operatorname{Re} z + O(6).$$

This condition is a system of 26 real equations on 24 real variables, which might not have any solution. If this system has no solution, then we conclude that the surface is not spherical. Otherwise we get a solution (f_3, g_4, h_5) and go to Step 3.

Step 3. We write the condition on the coefficients of a mapping

$$z \mapsto z + f_2 + f_3 + f_4 + O(5), \qquad w_2 \mapsto w_2 + g_3 + g_4 + g_5 + O(6),$$
$$w_3 \mapsto w_3 + h_4 + h_5 + h_6 + O(7),$$

which maps the surface, obtained in Step 2, onto a surface

$$v_2 = |z|^2 + O(6), \qquad v_3 = 2|z|^2 \operatorname{Re} z + O(7).$$

This condition is a system of 39 real equations on 32 real variables, which may have no solution. If this system has no solution, then we conclude that the surface is not spherical, otherwise it is spherical.

To solve the systems of equations for the homogeneous surfaces Ia–Vb we used Maple (see [8] for the details of the computations). For the tube case VI it is possible to apply simpler arguments. We resume the results of our computations in the following proposition.

Proposition 3.3. *The following homogeneous surfaces from the list* Ia–VIi *are spherical*: *Ic for* $\alpha = 2$ *and* VIa *for* $\alpha = 2$, $\beta = 3$. *All other surfaces from the list* Ia–VIi *are non-spherical.*

Proof. After the above sphericity inspection it is remaining to prove the proposition for the tube surfaces VIa–VIi. We note the following: the subalgebra, spanned by the constant vector fields $\frac{\partial}{\partial z}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_3}$ is the unique abelian 3-dimensional subalgebra in the 5-dimensional infinitesimal automorphism algebra of the cubic C (it can be easily verified from the commuting relations in the algebra, see [9] for the details and the notations). As it was shown in [9], the cubic C is polynomially equivalent to the tube surface VIa for $\alpha = 2$, $\beta = 3$ (we denote this surface by $\tilde{C}$) and we conclude that $\operatorname{span}\{\frac{\partial}{\partial z}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_3}\}$ is the unique 3-dimensional abelian subalgebra in $\mathfrak{aut}\,\tilde{C}$. Hence if a tube surface M from the list VIa–VIi is locally biholomorphically equivalent to the cubic C, we get a biholomorphic mapping F, which maps $\mathfrak{g}(M)$ to a 4-dimensional subalgebra of the 5-dimensional algebra $\mathfrak{aut}\,\tilde{C}$. In particular, the abelian 3-algebra, spanned by X_1, X_2, X_3, is mapped to an abelian 3-dimensional subalgebra of $\mathfrak{aut}\,\tilde{C}$. Since such subalgebra is unique, we conclude that F maps the three coordinate vector field $\frac{\partial}{\partial z}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_3}$ to their linear combinations, which implies that F is in fact a linear mapping. Hence all the tube surfaces VIa–VIi except $\tilde{C}$ are locally biholomorphically non-equivalent to C, as required. □

It remains now to prove that the non-spherical homogeneous surfaces Ia–VIi are pairwise locally CR-inequivalent. Since homogeneous surfaces of different Types Ia–VIi correspond to non-isomorphic Lie algebras (except the cases Va, Vb, corresponding to the same algebra V), we just need to prove, using Proposition 3.2, that two vector field algebras, providing the homogeneity of two non-spherical surfaces of the same Types Ia–VIi, cannot be mapped onto each other by a local biholomorphic mapping and also to prove that two vector field algebras, providing the homogeneity of a Type Va surface and a Type Vb surface correspondingly, cannot be mapped onto each other by a local biholomorphic mapping. In what follows

$$\Phi : \mathbb{C}^3_{z,w_2,w_3} \mapsto \mathbb{C}^3_{\xi,\eta_2,\eta_3}$$

denotes a local biholomorphic mapping, which maps a germ of a homogeneous surface M at p onto a germ of a homogeneous surface M' at p'. X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3, Y_4 denote the basis of the vector field algebras $\mathfrak{g}(M)$ and $\mathfrak{g}(M')$ correspondingly. Now we consider different cases.

Ia $\mapsto$ Ia. In this case we may suppose $p = p' = (0, 0, i)$. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3 \frac{\partial}{\partial w_2}$, $X_4 = (i - \gamma)\frac{\partial}{\partial z} + w_2 \frac{\partial}{\partial w_2} + w_3 \frac{\partial}{\partial w_3}$ and the same for Y_j (with a parameter γ'). The vector field algebras are of Type I with $q = 0$. $\operatorname{Span}\{X_1, X_2\}$ is the commutant and hence is invariant under Φ. X_4 is the unique element, modulo the commutant, for which the corresponding adjoint operator is identical on the commutant. X_3 is the unique element, up to a scalar and modulo the commutant, for which the corresponding adjoint operator has zero eigenvalues on the commutant. $\operatorname{Span}\{X_1\}$ is the kernel of ad_{X_3}. From all the above invariant descriptions we conclude that Φ maps X_1 to aY_1, X_2 to $bY_1 + eY_2$, X_3 to $mY_1 + nY_2 + kY_3$, X_4 to $pY_1 + sY_2 + Y_4$, where $a, b, e, m, n, k, p, s \in \mathbb{R}$, $a, e, k \neq 0$. The first two conditions imply $\xi = F(z)$, $\eta_2 = G(z) + aw_2 + bw_3$, $\eta_3 = H(z) + ew_3$. The third one implies $F_z = k$, the fourth one implies $(i - \gamma)F_z = i - \gamma'$. Hence we get $k = 1$, $\gamma = \gamma'$ and conclude that $M = M'$.

Ib $\mapsto$ Ib. In this case we may suppose $p = p' = (i, 0, i)$. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3 \frac{\partial}{\partial w_2}$, $X_4 = -z\frac{\partial}{\partial z} - i\gamma \frac{\partial}{\partial w_2} + w_3 \frac{\partial}{\partial w_3}$ and the same for Y_j (with a parameter γ'). The vector field algebras are of Type I with $q = -1$. $\operatorname{Span}\{X_1, X_2, X_3\}$ is the commutant and hence is invariant under Φ. X_4 is the unique element, up to a sign and modulo the commutant, for which the corresponding adjoint operator has eigenvalues $\{0; 1; -1\}$ on the commutant. $\operatorname{Span}\{X_1\}$ is the second commutant. From the above invariant descriptions we conclude that Φ maps X_1 to aY_1, X_2 to $bY_1 + eY_2 + kY_3$, X_3 to $lY_1 + mY_2 + nY_3$, X_4 to $pY_1 + sY_2 + rY_3 \pm Y_4$. In case we have plus, the commuting relations imply $k = m = 0$. Then from the first two conditions we get $\xi = F(z)$, $\eta_2 = G(z) + aw_2 + bw_3$, $\eta_3 = H(z) + ew_3$. Comparing the $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta_2}$ and $\frac{\partial}{\partial\eta_3}$-coefficients for the third condition, we get $F_z = n$, $G_z + aw_3 = l + nH + new_3$, $H_z = m$ and hence $F = nz - ni + i$, $a = ne$, $G_z = nH + l$, $H_z = 0$ (because $F(i) = i$). Comparing the $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta_2}$-coefficients for the fourth condition, we get $-nz = -nz + ni - i + r$, $-zG_z - i\gamma a + bw_3 = p + rH + rew_3 - i\gamma'$, which implies $n = 1$, $r = 0$, $G_z = 0$, $\gamma' = \gamma a$. This finally gives us $H = -l \in \mathbb{R}$ and hence $e = 1$ (because $\Phi(p) = p'$). Now we conclude that $a = ne = 1$, $\gamma' = \gamma a = \gamma$ and $M = M'$, as required. In case we have minus in the fourth condition, we note that M has the following polynomial automorphism σ, preserving the point p:

$$z \mapsto w_3, \qquad w_2 \mapsto -w_2 + zw_3 + 1, \qquad w_3 \mapsto z.$$

Under this transformation

$$X_1 \mapsto -X_1, \qquad X_2 \mapsto X_3, \qquad X_3 \mapsto X_2, \qquad X_4 \mapsto -X_4. \tag{6}$$

Composing Φ with σ, we get a mapping $\tilde{\Phi}$ from M onto M' with a plus in the fourth condition and hence conclude that $\gamma' = \gamma$ and $M' = M$.

Ic $\mapsto$ **Ic**. In this case we may suppose that the alpha's are the same for M and M' (since different alpha's correspond to different q and hence to non-isomorphic Lie algebras). Also we suppose that $p = (i, i\gamma, i)$, $p' = (i, i\gamma', i)$. The vector field algebras are of Type I with $q \in (-1, 1)$, $q \neq 0$. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = qz\frac{\partial}{\partial z} + (q+1)w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3}$ and the same for Y_j. Span$\{X_1, X_2, X_3\}$ is the commutant. Span$\{X_1\}$ is the second commutant. X_4 is the unique element, modulo the commutant, for which the corresponding adjoint operator has the eigenvalue $q + 1$ on the second commutant. From the above invariant descriptions we conclude that Φ maps X_1 to aY_1, X_2 to $bY_1 + eY_2 + kY_3$, X_3 to $lY_1 + mY_2 + nY_3$, X_4 to $pY_1 + sY_2 + rY_3 + Y_4$. The commuting relations imply $k = m = 0$. Then from the first two conditions we get $\xi = F(z)$, $\eta_2 = G(z) + aw_2 + bw_3$, $\eta_3 = H(z) + ew_3$. Comparing the $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta_2}$ and $\frac{\partial}{\partial\eta_3}$-coefficients for the third condition, we get $F_z = n$, $G_z + aw_3 = l + nH + new_3$, $H_z = m$ and hence $a = ne$, $G_z = nH + l$, $H_z = 0$. In particular, $F = nz - ni + i$ (from $F(i) = i$) and G is linear. Comparing the $\frac{\partial}{\partial\xi}$-coefficients for the fourth condition, we get $qnz = q(nz - ni + i) + r$, which implies $n = 1$, $r = 0$. Comparing the $\frac{\partial}{\partial\eta_2}$-coefficients for the fourth condition, we get $zG_z + a(q+1)w_2 + bw_3 = p + rH + rew_3 + a(q+1)w_2 + (q+1)w_3 + (q+1)G + p$, which implies $b = b(q+1)$ and hence $b = 0$, and also $zG_z = \frac{q+1}{q}G$ and $G = Az^{\frac{q+1}{q}}$. Since G is linear and $q \neq -1$, we get $G = 0$. Then $H = -l \in \mathbb{R}$ and hence $e = 1$ (because $\Phi(p) = p'$). Now we conclude that $a = ne = 1$, and hence $\eta_2 = w_2$, which implies $i\gamma = i\gamma'$ and $M = M'$, as required.

II $\mapsto$ **II**. In this case $p = (i, i\gamma, 0)$, $p' = (i, i\gamma', 0)$. The vector field algebras are of Type II. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = z\frac{\partial}{\partial z} + (2w_2 + \frac{1}{2}z^2)\frac{\partial}{\partial w_2} + (z + w_3)\frac{\partial}{\partial w_3}$ and the same for Y_j. Span$\{X_1, X_2, X_3\}$ is the commutant. Span$\{X_1\}$ is the second commutant. X_4 is unique element, modulo the commutant, for which the corresponding adjoint operator has the eigenvalue 2 on the second commutant. From the above invariant descriptions we conclude that Φ maps X_1 to aY_1, X_2 to $bY_1 + eY_2 + kY_3$, X_3 to $lY_1 + mY_2 + nY_3$, X_4 to $pY_1 + sY_2 + rY_3 + Y_4$. The commuting relations imply $k = 0$, $n = e$, $a = e^2$. Then in the same way as in the previous case from the first two conditions we get $\xi = F(z)$, $\eta_2 = G(z) + e^2w_2 + bw_3$, $\eta_3 = H(z) + ew_3$, and from the third one $F_z = nz - ni + i$, $G_z = eH + l$, $H_z = m$. Comparing the $\frac{\partial}{\partial\xi}$-coefficients for the fourth condition, we get $nz = nz - ni + i + r$, which implies $n = 1, r = 0$. Comparing the $\frac{\partial}{\partial\eta_2}$-coefficients for the fourth condition, we get $zG_z + (2w_2 + \frac{1}{2}z^2)e^2 + b(z + w_3) = p + 2e^2w_2 + 2bw_3 + 2G$, which implies $e^2 = 1$, $b = 0$, $p = 0$, $zG_z = 2G$ and hence $G = Az^2$. Finally, comparing the $\frac{\partial}{\partial\eta_3}$-coefficients, we get $zH_z + (z + w_3)e = z + H + ew_3 + q$, which implies $zH_z = H + q$, $H = mz - q$. Now from $\Phi(p) = p'$ we get $mi - q = 0$ and hence $m = q = 0$, $H = 0$, $G = 0$ (because $G = Az^2$ and $G_z = l + eH$). Applying $\Phi(p) = p'$ again, we get $i\gamma = i\gamma'$ and $M = M'$, as required.

IIIa $\mapsto$ **IIIa**. In this case we may suppose $p = p' = (0, 0, i)$. The vector field algebras are of Type III with $q = 0$. Span$\{X_1, X_2, X_3\}$ is the commutant, span$\{X_1\}$ is the second commutant, X_4 is the unique element, up to a sign and modulo the commutant, for which the corresponding adjoint operator has eigenvalues $\{0, i, -i\}$ on the commutant. From the above invariant descriptions we conclude that Φ maps X_1 to aY_1, X_2 to $bY_1 + cY_2 + dY_3$, X_3 to $eY_1 + kY_2 + lY_3$, X_4 to $mY_1 + nY_2 + pY_3 \pm Y_4$. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = -w_3\frac{\partial}{\partial z} + (\frac{1}{2}z^2 - \frac{1}{2}w_3^2 - i\gamma)\frac{\partial}{\partial w_2} + z\frac{\partial}{\partial w_3}$ and the same for Y_j (with a parameter γ'). It is convenient now to make the variable change (2) and thus to get the affine vector fields $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3} - w_2\frac{\partial}{\partial z}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = -w_3\frac{\partial}{\partial z} - 2i\gamma\frac{\partial}{\partial w_2} + z\frac{\partial}{\partial w_3}$, and the same for Y_j (with a parameter γ'). The commuting relations now imply $l = c$, $k = -d$, $c^2 + d^2 = a$. Then from the first condition we get $\xi_{w_2} = (\eta_3)_{w_2} = 0$, $(\eta_2)_{w_2} = a$. The third and the second conditions imply $\xi_{w_3} = d$, $\xi_z = c$, $(\eta_3)_z = -d$, $(\eta_3)_{w_3} = c$ and hence

$$\xi = cz + dw_3 + s, \qquad \eta_3 = -dz + cw_3 + t.$$

Also we get $(\eta_2)_z = e + sd + ct$, $(\eta_2)_{w_3} = b - cs + dt$. From $\Phi(p) = p'$ we get $s = -di$, $t = i - ci$. In case we have plus in the fourth condition, we compare the $\frac{\partial}{\partial\xi}$ and $\frac{\partial}{\partial\eta_3}$-coefficients for the fourth condition and get $p = n = s = t = 0$. Hence $c = 1$, $d = 0$, $a = 1$. Thus we have $\eta_2 = w_2 + ez + bw_3 + h$. Comparing now the $\frac{\partial}{\partial\eta_2}$-coefficients for the fourth condition, we get

$b = e = 0$, $\gamma = \gamma'$, so $M = M'$. In case we have minus in the fourth condition, we firstly apply the following automorphism ε, preserving M and the fixed point p:

$$z \mapsto -z, \qquad w_2 \mapsto -w_2, \qquad w_3 \mapsto w_3.$$

As a result we get $X_1 \mapsto -X_1$, $X_2 \mapsto X_2$, $X_3 \mapsto -X_3$, $X_4 \mapsto -X_4$. Composing Φ with ε, we get a biholomorphic mapping Φ' of a germ of M at p onto a germ of M' at p' with a plus in the fourth condition and hence get $M = M'$, as required.

IIIb $\mapsto$ **IIIb**. In this case we may suppose that the q-parameters are the same for M and M' (since different q correspond to non-isomorphic Lie algebras). Also we suppose that $p = (i, i\gamma, 0)$, $p' = (i, i\gamma', 0)$. The vector field algebras are of Type III with $q > 0$. $\mathrm{Span}\{X_1, X_2, X_3\}$ is the commutant, $\mathrm{span}\{X_1\}$ is the second commutant, X_4 is the unique element, modulo the commutant, for which the corresponding adjoint operator has the eigenvalue $2q$ on the second commutant. From the above invariant descriptions we conclude that Φ maps X_1 to aY_1, X_2 to $bY_1 + cY_2 + dY_3$, X_3 to $eY_1 + kY_2 + lY_3$, X_4 to $mY_1 + nY_2 + pY_3 + Y_4$. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = (qz - w_3)\frac{\partial}{\partial z} + (2qw_2 + \frac{1}{2}z^2 - \frac{1}{2}w_3^2)\frac{\partial}{\partial w_2} + (z + qw_3)\frac{\partial}{\partial w_3}$ and the same for Y_j. It is convenient now to make the variable change (2) and thus to get the affine vector fields $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3} - w_2\frac{\partial}{\partial z}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = (qz - w_3)\frac{\partial}{\partial z} + 2qw_2\frac{\partial}{\partial w_2} + (z + qw_3)\frac{\partial}{\partial w_3}$ (and the same for Y_j). The commuting relations now imply $l = c$, $k = -d$, $c^2 + d^2 = a$. Then from the first condition we get $\xi_{w_2} = (\eta_3)_{w_2} = 0$, $(\eta_2)_{w_2} = a$. The third and the second condition imply $\xi_{w_3} = d$, $\xi_z = c$, $(\eta_3)_z = -d$, $(\eta_3)_{w_3} = c$ and hence

$$\xi = cz + dw_3 + s, \qquad \eta_3 = -dz + cw_3 + t.$$

Also we get $(\eta_2)_z = e + sd + ct$, $(\eta_2)_{w_3} = b - cs + dt$. From $\Phi(p) = p'$ we get $s = i - ci$, $t = di$. Comparing the $\frac{\partial}{\partial\xi}$ and $\frac{\partial}{\partial\eta_3}$-coefficients for the fourth condition, it is not difficult to obtain $p = n = s = t = 0$ and hence $c = 1$, $d = 0$, $a = 1$. Thus we have $\eta_2 = w_2 + ez + bw_3 + h$. Comparing now the $\frac{\partial}{\partial\eta_2}$-coefficients for the fourth condition, we get $b = e = 0$, $m + 2qh = 0$, which implies $\mathrm{Im}\, h = 0$ and from $\Phi(p) = p'$ we now get $i\gamma = i\gamma'$ and $M = M'$, as required.

IV $\mapsto$ **IV**. In this case we may suppose $p = p' = (0, i, i)$. From Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_3}$, $X_4 = (i - \delta)\frac{\partial}{\partial z} + w_2\frac{\partial}{\partial w_2}$ and the same for Y_j (with a parameter δ'). The vector field algebras are of Type IV. $\mathrm{Span}\{X_1, X_2\}$ is the commutant. X_3 and X_4 are the unique elements, up to a permutation and modulo the commutant, for which the corresponding adjoint operators have a collection of eigenvalues $\{0; 1\}$ on the commutant. From the above invariant descriptions we conclude that Φ maps X_1 to $aY_1 + pY_2$, X_2 to $bY_1 + qY_2$, and also (first case) X_3 to $mY_1 + nY_2 + Y_3$, X_4 to $kY_1 + lY_2 + Y_4$, or (second case) X_3 to $mY_1 + nY_2 + Y_4$, X_4 to $kY_1 + lY_2 + Y_3$. The first two conditions in both cases imply $\xi = F(z)$, $\eta_2 = G(z) + aw_2 + bw_3$, $\eta_3 = H(z) + pw_2 + qw_3$. Comparing now the $\frac{\partial}{\partial\xi}$-coefficients for the third and the fourth conditions in the first case, we get $F_z = 1$ and $(i - \delta)F_z = i - \delta'$, which implies $\delta = \delta'$ and $M = M'$, as required. Comparing the $\frac{\partial}{\partial\xi}$-coefficients for the third and the fourth conditions in the second case, we get $F_z = i - \delta$ and $(i - \delta)F_z = 1$, which implies $(i - \delta)^2 = 1$, which is impossible since $\delta \in \mathbb{R}$, so the second case cannot occur and finally $M = M'$.

V $\mapsto$ **V**. We consider a mapping between germs of arbitrary surfaces with $\mathfrak{g}(M)$ of Type V. In this case from Section 2 we have $X_1 = \frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3}$, $X_3 = \frac{\partial}{\partial z} + w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3}$. For Type Va we have $X_4 = \frac{i-\alpha}{\beta}\frac{\partial}{\partial z} - w_3\frac{\partial}{\partial w_2} + w_2\frac{\partial}{\partial w_3}$, for Type Vb we have $X_4 = i\frac{\partial}{\partial z} + (ie^z - w_3)\frac{\partial}{\partial w_2} + (w_2 + e^z)\frac{\partial}{\partial w_3}$. $\mathrm{Span}\{X_1, X_2\}$ is the commutant. X_3 is the unique element, modulo the commutant, for which the corresponding adjoint operator is identical on the commutant. X_4 is the unique element, up to a sign and modulo the commutant, for which the corresponding adjoint operator has eigenvalues $\{i; -i\}$. From the above invariant descriptions we conclude that Φ maps X_1 to $aY_1 + pY_2$, X_2 to $bY_1 + qY_2$, X_3 to $mY_1 + nY_2 + Y_3$, and also (first case) X_4 to $kY_1 + lY_2 + Y_4$, or (second case) X_4 to $kY_1 + lY_2 - Y_4$. The first two conditions in both cases imply $\xi = F(z)$, $\eta_2 = G(z) + aw_2 + bw_3$, $\eta_3 = H(z) + pw_2 + qw_3$. Comparing now the $\frac{\partial}{\partial\xi}$-coefficients for the third and the fourth conditions in the first case, we get $F_z = 1$, and also: $\frac{i-\alpha}{\beta}F_z = \frac{i-\alpha'}{\beta'}$ (Va $\mapsto$ Va), or $\frac{i-\alpha}{\beta}F_z = i$ (Va $\mapsto$ Vb), which implies $\alpha = \alpha'$, $\beta = \beta'$ and $M = M'$ for Va $\mapsto$ Va, or $\alpha = 0$, $\beta = 1$ for Va $\mapsto$ Vb, which is a contradiction. Comparing the $\frac{\partial}{\partial\xi}$-coefficients for the third and the fourth conditions in the second case, we get $F_z = 1$, and also: $\frac{i-\alpha}{\beta}F_z = -\frac{i-\alpha'}{\beta'}$ (Va $\mapsto$ Va), or $\frac{i-\alpha}{\beta}F_z = -i$ (Va $\mapsto$ Vb), which implies $\beta = -\beta'$ for Va $\mapsto$ Va, which is a contradiction since $\beta, \beta' > 0$, or $\alpha = 0$, $\beta = -1$ for $Va \mapsto Vb$, which is also a contradiction since $\beta > 0$, as required.

VI $\mapsto$ **VI**. We claim that all non-spherical tube surfaces VIa–VIi are pairwise non-equivalent. To see that, we note that for all tube surfaces VIa–VIi the real matrix, defining the linear part of the affine vector field X_4, has rank at least 2, which implies that the centralizer of X_4 in $\mathfrak{g}(M)$ is not more than 1-dimensional. Then we can find no 3-dimensional abelian subalgebra in $\mathfrak{g}(M)$ other than the one spanned by X_1, X_2, X_3. Hence if F is a biholomorphic mapping between germs of two non-spherical tube surfaces M, M' from the list VIa–VIi, then by Proposition 3.2 F maps $\mathfrak{g}(M)$ to $\mathfrak{g}(M')$ and the unique 3-dimensional abelian subalgebras are also mapped onto each other. In the same way as in Proposition 3.3 we conclude now that F is a linear mapping. Hence different tube surfaces from the list VIa–VIi are pairwise locally CR non-equivalent, as required.

Thus, according to the claims of the main trichotomy, Theorem 2.9 and Proposition 3.3, the Main Theorem is completely proved.

Proof of Theorem 1.2. For all surfaces (2.1)–(3.19) except (3.4) and (3.9) the desired claim follows from the fact that in appropriate coordinates $\mathfrak{g}(M)$ consists actually of affine vector fields (see the above description of $\mathfrak{g}(M)$ for different types). For the exceptional surface (3.4) one should make the variable change (2). After that we have (see the case II $\mapsto$ II above): $X_1 = 2\frac{\partial}{\partial w_2}$, $X_2 = \frac{\partial}{\partial w_3} - z\frac{\partial}{\partial w_2}$, $X_3 = \frac{\partial}{\partial z} + w_3\frac{\partial}{\partial w_2}$, $X_4 = z\frac{\partial}{\partial z} + 2w_2\frac{\partial}{\partial w_2} + (z + w_3)\frac{\partial}{\partial w_3}$ and thus the homogeneity of (3.4) is provided by an algebra of affine vector fields. For the exceptional case (3.9) one should make the variable change $z^* = e^z$. Then $\mathfrak{g}(M)$ looks as follows (see the case V $\mapsto$ V above):

$$X_1 = \frac{\partial}{\partial w_2}, \qquad X_2 = \frac{\partial}{\partial w_3}, \qquad X_3 = z\frac{\partial}{\partial z} + w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3},$$
$$X_4 = iz\frac{\partial}{\partial z} + (iz - w_3)\frac{\partial}{\partial w_2} + (z + w_2)\frac{\partial}{\partial w_3}$$

and hence consists of affine vector fields, as required. □

Proof of Theorem 1.3. Following carefully the above arguments, one can see that in cases Ia–Vb any mapping of a germ of a homogeneous surface onto itself is actually identical except the mapping σ for the surfaces of Type Ib and ε for the surfaces of Type IIIa. For tube surfaces VIa–VIi each mapping of a germ onto itself must be linear (see the arguments above). It is not difficult to see that only the cubic, the surfaces VIb for $a = -1$ and the surface VIi have the desired linear automorphism, preserving a fixed point (the affinely homogeneous curves, corresponding to tube surfaces VIa, cannot be extended to the origin as affinely homogeneous curves except the case of the cubic). □

Proof of Theorem 1.4. As it was mentioned above, any symmetric CR-manifold is CR-homogeneous and thus is locally CR-equivalent to one of the surfaces (0.1)–(3.19). It follows from Theorem 1.3 that among the totally non-degenerate homogeneous surfaces (2.1)–(3.19) the following ones are symmetric: (2.1), (3.2), (3.5), (3.12) and (3.19), the desired involutions s_p are as they are presented in Theorem 1.3; for the cubic one should put $\lambda = -1$. The Hermitian metric is the restriction of the Euclidean metric for T_pM and is translated to the tangent planes in all other points by means of the Lie algebra $\mathfrak{g}(M)$. The isometry property for s_p is obvious in cases (2.1), (3.5), (3.12), (3.19). In the case of (3.2) the isometry property follows from (6). The symmetry property check in all cases is straightforward. For the degenerate surfaces (1.1) we note that a CR-manifold of kind $M^3 \times \mathbb{R}^1$, where $M^3 \subset \mathbb{C}^2_{z,w_2}$ is a locally homogeneous surface in $\mathbb{C}^2$, $\mathbb{R}^1 \subset \mathbb{C}^1_{w_3}$ is a real line, is CR-symmetric if and only if M^3 is symmetric. It follows from E. Cartan's classification theorem that among the hypersurfaces from his list only the surfaces, corresponding to (1.a)–(1.f), have non-trivial stability subgroups, which are of $\mathbb{Z}_2$-type for (1.b)–(1.f) and of $\mathbb{R}^*$-type for (1.a). The symmetry property check for these surfaces in $\mathbb{C}^2$ is straightforward. For the case of flat manifolds the symmetry property is obvious. This completely proves the theorem. □

Remark 3.4. It is an amazing consequence of Theorem 1.4 that any symmetric totally non-degenerate 4-manifold is associated to a second order plane curve in the sense that one of the defining equations of the manifold can be chosen as an equation of a second order plane curve.

Remark 3.5. Another possible approach to the description of homogeneous surfaces is to present a list of all possible normal forms (see, for example, [12,18]). For some cases this approach is actually realized in [8]. It would be interesting, taking Theorem 1.2 into account, to reformulate the Main Theorem in terms of some affine normal forms, as it was made, for example, in [12] for affinely homogeneous hypersurfaces in $\mathbb{R}^3$. It would be also interesting to find the specify of the normal form for the symmetric surfaces.

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