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Analytic Complexity of Functions of Two Variables

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Abstract. The definition of analytic complexity of an analytic function of two variables is given. It is proved that the class of functions of a chosen complexity is a differential-algebraic set. A differential polynomial defining the functions of first class is constructed. An algorithm for obtaining relations defining an arbitrary class is described. Examples of functions are given whose order of complexity is equal to zero, one, two, and infinity. It is shown that the formal order of complexity of the Cardano and Ferrari formulas is significantly higher than their analytic complexity. The complexity classes turn out to be invariant with respect to a certain infinite-dimensional transformation pseudogroup. In this connection, we describe the orbits of the action of this pseudogroup in the jets of orders one, two, and three. The notion of complexity order is extended to plane (or "planar") 3-webs. It is discovered that webs of complexity order one are the hexagonal webs. Some problems are posed.

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The main hero of this story is the differential polynomial of differential order three and algebraic degree four,

$$\Delta_1(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}), \tag{1}$$

which is treated in what follows from three points of view.

1. HIERARCHY OF CLASSES

Using an arbitrarily rich family of functions of a single variable only, one cannot obtain any function of two variables as their superposition. We need at least one function of two variables. Admittedly, the simplest function of this kind is the function $x + y$. The family of functions of two variables x and y which can be obtained in this way can be described as the hierarchy of classes defined by induction,

$$Cl_0 \subset Cl_1 \subset Cl_2 \subset Cl_n \subset \cdots \subset \mathbf{Cl},$$

where Cl_0 is formed by the functions of a single variable (x or y), Cl_1 is formed by the functions $c(a(x) + b(y))$, and Cl_{n+1} consists of the functions of the form $C(A_n(x, y) + B_n(x, y))$, where C is the family of functions of a single variable and A_n and B_n are functions in Cl_n. The symbol $\mathbf{Cl}$ stands for the union of all classes with finite indices. Let us now make two refining remarks. First: we deal with analytic functions only. Second: we understand the representability in the form of superposition as a local representability in a neighborhood of a generic point. For instance, the function $\tan(\sqrt{x}+\log y)$ should be regarded as a first-class function, despite all specifications related both to the domain of definition and to multivaluedness.

The zero class can be given by the following relation

$$Cl_0 = \{z : \Delta_0(z) = z'_x z'_y = 0\}.$$

One can obtain a similar criterion for the germ of a function to locally belong to the first class. Indeed, if $z = c(a(x)+b(y))$, then $z'_x = c'a'$, $z'_y = c'b'$, and hence $z'_y/z'_x = b'(y)/a'(x)$, which implies that the derivative of $\log(z'_y/z'_x)$ with respect to xy vanishes

$$\delta_1(z) = (\log(z'_y/z'_x))''_{xy} = 0. \tag{2}$$

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Computing the numerator of this rational expression, we obtain

$$\Delta_1(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0. \tag{3}$$

Conversely, any solution of (2) or (3) is a germ of a first-class function. Indeed, it follows from (2) that $\log(z'_y/z'_x) = \log(B(y)/A(x))$. Moreover, if we assume that $z(x, y)$ does not belong to Cl_0, then $A(x)$ and $B(y)$ are nonzero. Let us now rectify the vector field $A(x)\frac{\partial}{\partial x}$ by a change of the variable x and the vector field $B(y)\frac{\partial}{\partial y}$ by a change of the variable y. In this case, in the new variables (X, Y), we have $z'_X = z'_Y$, and therefore $z = c(X + Y)$ and, finally, $z = c(a(x) + b(y))$. Thus, taking into account the above discussion and the uniqueness theorem for analytic functions, we obtain

Proposition 1. *The following conditions are equivalent.*
(1) *Some germ of an analytic function $z(x, y)$ is a first-class germ.*
(2) *All germs of the analytic function $z(x, y)$ are first-class germs.*
(3) $\Delta_1(z) = 0$ *for some germ of the analytic function z.*
(4) $\Delta_1(z) = 0$ *for all germs of the analytic function z.*

The fact that there is a differential-algebraic criterion for a function to belong to an arbitrarily given class can be explained as follows. Let $U^{(k)}$ be the space of k-jets of functions of two variables, i.e., the space of families of the form $(x, y, z, z'_x, z'_y, \ldots$ derivatives of order not exceeding $k)$. As k increases, the number of derivatives increases quadratically, and the number of derivatives of the functions of a single variable entering the general expression for any function of the kth class increases linearly. The expression for the derivatives of the function z in terms of the functions of a single variable is polynomial. This means that, beginning with some k, the class is an algebraic subset of $U^{(k)}$. If $pr^{(k)}$ denotes the extension mapping for a function, i.e., the mapping that assigns to any function its image in $U^{(k)}$, then we can assert the following.

Proposition 2. *The set $pr^{(k)}(Cl_n)$ is an algebraic subset of $U^{(k)}$ which is proper, beginning with some $K(n)$.*

Thus, every class has a family of differential-polynomial relations defining it. Denote these relations by $\Delta_n(z) = 0$. With regard to the uniqueness theorem, we obtain the following corollary.

Corollary 3. *If at least one germ of an analytic functions belongs to Cl_n, then so do the other germs.*

Since every class contains all classes with lesser indices, it follows that the polynomials $\Delta_n(z)$ belong to the differential ideal generated by the polynomials $\Delta_m(z)$ for $m < n$ and by the relation $\Delta_n(z) = 0$ are differential-algebraic consequences of the relations $\Delta_m(z) = 0$.

Definition 4. An analytic function $z(x, y)$ is said to have *complexity order* n if z is contained in $Cl_n \setminus Cl_{n-1}$. If a function is not contained in $\mathbf{Cl}$, we say that the complexity order of f is equal to infinity.

In terms of relations defining the class, it is very simple to find the complexity order of a function.

Proposition 5. *The complexity order of a function $z(x, y)$ is equal to n if and only if $\Delta_n(z) = 0$ and $\Delta_{n-1}(z) \neq 0$.*

Let us now return to the very beginning and ask: What changes if one replaces the base function $x + y$ by another analytic function $\phi(x, y)$? Denote the hierarchy generated by a function $\phi(x, y)$ by

$$Cl_0(\phi) \subset Cl_1(\phi) \subset Cl_2(\phi) \subset Cl_n(\phi) \subset \cdots \subset \mathbf{Cl}(\phi).$$

One can formulate the following quite obvious assertion.

Proposition 6. *Let there be two hierarchies, $\mathbf{Cl}(\phi)$ and $\mathbf{Cl}(\psi)$. In this case, if $\phi \in Cl_1(\psi)$, then the inclusion $Cl_n(\phi) \subset Cl_n(\psi)$ holds for any n.*

Corollary 7. *If $\phi \in Cl_1(\psi)$ and $\psi \in Cl_1(\phi)$, then $Cl_n(\phi) = Cl_n(\psi)$ (the classes coincide) for any n.*

In particular, the hierarchies generated by distinct arithmetic operations coincide due to the relations $xy = \exp(\log(x) + \log(y))$ and $x + y = \log(\exp(x)\exp(y))$.

Consider several examples.

(1) The complexity order of the polynomial $x^2 + y^2$ is obviously equal to one and that of the polynomial $x^2 + xy$ is equal to two. Indeed, $\Delta_1(x^2 + xy) = 2$, i.e., the complexity exceeds one, and it is immediate that the complexity does not exceed two.

Question 8. For polynomials, along with analytic complexity, one can consider polynomial complexity. This means that one can construct a similar hierarchy by using polynomials in one variable. Is it true that the corresponding classes are the same? Let us pose a very specific question. Let a polynomial $z(x, y)$ have analytic complexity of order one. Does there exist a representation $z = c(a(x) + b(y))$, where (a, b, c) are polynomials?

(2) A. Ostrowski [1] showed that the generalized Riemann ζ-function

$$\zeta(x, y) = \sum_{n=1}^{\infty} \frac{x^n}{n^y}$$

cannot satisfy any polynomial-differential relation, and therefore its analytic complexity is infinite.

If we equip the space of analytic functions with some reasonable topology, then the standard argument shows that **Cl** is a rather meager subset (of the first Baire category, i.e., the countable union of nowhere dense subsets). However, the existential status of this great set of quite obscure functions is doubtful. There is a point of view claiming that these functions do not exist at all. On the other hand, polynomials, rational functions, and all elementary functions (one can add special functions of one variable to the generators) clearly belong to **Cl**.

(3) Let the function z be given by the relation $z^m + xz + y = 0$. If $m = 1$, then its order of complexity is equal to one. Let us show that its complexity order is equal to two for all degrees $m \geqslant 2$. Indeed, differentiating the defining relation, we see that $z = z'_x/z'_y$, and thus the condition $\delta_1(z) = 0$ becomes $(\log(z))''_{xy} = 0$. Therefore, $(\log(z))''_{xy} = (z'_x/z)'_y = z''_{yy}$ and $z = P(x) + Q(x)y$, i.e., for any x, the function z is linear in y. This is possible for $m = 1$ only. Thus, if $m \geqslant 2$, then the complexity order is not less than two. On the other hand, after the change $Z = z/y^{\frac{1}{m}}$, $t = xy^{\frac{1}{m-1}}$, the equation becomes $Z^m + tZ + 1 = 0$, and its solution is an algebraic function of one variable, $Z(t)$. As a result, we obtain a representation of the original function in the form $z = y^{\frac{1}{m}} Z(xy^{\frac{1}{m-1}})$. Since both the functions $y^{\frac{1}{m}}$ and $Z(xy^{\frac{1}{m-1}})$ belong to the first class, their product belongs to the second one.

The result thus obtained somewhat contradicts the formal complexity of the known formulas for the roots of equations. Everything is good for the quadratic equation. Namely, the formula

$$z = \frac{1}{2}\left(-x + \sqrt{x^2 - 4y}\right)$$

has formal complexity order equal to two. However, for $m = 3$, we have

$$z = \frac{1}{6}\left(-108x + 12\sqrt{12x^3 + 81y^2}\right)^{1/3} - 2x\left(-108x + 12\sqrt{12x^3 + 81y^2}\right)^{-1/3},$$

and, as one can readily see, this formula has formal complexity order two. For $m = 4$ (I omit the formula), the formal complexity order is equal to eight (!). Certainly, there is no logical contradiction. The point is that Cardano's and Ferrari's formulas solve different problems and, from our point of view concerning complexity, the formulas are quite uneconomical.

How can one evaluate differential polynomials Δ_n defining the higher classes Cl_n? This problem has a well-known analog. Let a curve $z_1 = f(t)$, $z_2 = g(t)$ be given parametrically. How can one pass from the parametric equations to the equation on z_1 and z_2? One must eliminate the parameter t. If, moreover, f and g are polynomials, then one can use the modern version of the elimination theory. To this end, one must consider the ideal (in the polynomial ring in (z_1, z_2, t)) generated by the relations

$$z_1 - f(t) = 0, \qquad z_2 - g(t) = 0,$$

introduce an order on the variables in such a way that t turns out to be the leading variable, extend this order to the monomials, and evaluate (by the Buchberger algorithm) the standard basis (the Gröbner basis) of the ideal. In this case, in accordance with the general theory [3], the list of elements of the basis begins with a polynomial which does not contain t. The curve is the zero set of the polynomial. Our situation is completely similar to that described above. The relation

$$z = C(A_{n-1}(x, y) + B_{n-1}(x, y))$$

is a parametric representation of an arbitrary function in Cl_n, and it is determined by some family of functions of one variable, $t = (a_1, b_1, c_1, \dots)$. Let us regard this relation as the generator of the differential ideal in the differential ring in which the formal variables are the function z together with all its derivatives and the functions in the family t together with all of their derivatives. One must introduce an order on the variables of the ring in such a way that the function z and its derivatives are lower than t and the corresponding derivatives and then extend this order to the differential monomials. The Kolchin–Ritt algorithm constructs a differential Gröbner basis [3]. According to the theory, the list of elements of the basis begins with a polynomial containing no parameters. These are the desired expressions Δ_n.

Unfortunately, the complexity of this algorithm is so high that it is unclear whether or not this algorithm can be realized practically, at least to evaluate Δ_2.

2. SYMMETRIES AND INVARIANTS

The hierarchy of classes constructed above admits an obvious symmetry. Namely, if $P(x)$, $Q(y)$, and $R(z)$ are locally invertible analytic changes of the corresponding single variable, if $z(x, y) \in Cl_n$, and if the expression $R(z(P(x), Q(y)))$ defines a function, then this function belongs to the same class Cl_n. These changes of variables form a pseudogroup [4], which we denote by G. Since G acts by smooth changes, the action has a natural extension to the space of jets. One can regard the coordinates of the space of ℓ-jets as coordinates in a finite-dimensional space. The dimension of this space is

$$D(\ell) = 2 + (1 + 2 + \dots + (\ell + 1)) = 2 + \frac{(\ell + 1)(\ell + 2)}{2},$$

the expressions for the derivatives whose order does not exceed ℓ are polynomials that depend on the derivatives of the change whose orders also do not exceed ℓ, and the number of the derivatives is $d(\ell) = 3(\ell + 1)$. From this point of view, our changes form a local Lie group (of dimension $d(\ell)$) acting on the linear space of dimension $D(\ell)$. Here is the beginning of the list of dimensions:

$$\begin{array}{cccccc} D(0) = 3, & D(1) = 5, & D(2) = 8, & D(3) = 12, & D(4) = 17, & \dots \\ d(0) = 3, & d(1) = 6, & d(2) = 9, & d(3) = 12, & d(4) = 15, & \dots \end{array} \tag{4}$$

If a vector field

$$v = p(x)\frac{\partial}{\partial x} + q(y)\frac{\partial}{\partial y} + r(z)\frac{\partial}{\partial z}$$

generates a local one-parameter group of analytic transformations in the space of 0-jets, then the extension of this action to jets of higher order also define one-parameter groups with the corresponding generators $pr^{(\ell))}v$. These generators have the following form (see [1]):

$$pr^{(1)}v = v + \Big(r'(z) - p'(x)\Big)z'_x\frac{\partial}{\partial z'_x} + \Big(r'(z) - q'(y)\Big)z'_y\frac{\partial}{\partial z'_y},$$

$$pr^{(2)}v = pr^{(1)}v + \Big(r''(z'_x)^2 + r'z_{xx} - p''z_x - 2p'z''_{xx}\Big)\frac{\partial}{\partial z''_{xx}}$$

$$+ \Big(r''z'_xz'_y + r'z''_{xy} - p'z''_{xy} - q'z''_{xy}\Big)\frac{\partial}{\partial z''_{xy}} + \Big(r''(z'_y)^2 + r'z_{yy} - q''z_y - 2q'z''_{yy}\Big)\frac{\partial}{\partial z''_{yy}},$$

$$pr^{(3)}v = pr^{(2)}v + \Big((p''' z'_x + 3p'' z''_{xx} + 3p' z'''_{xxx}) + (r'''(z'_x)^3 + 3r'' z'_x z''_{xx} + r' z'''_{xxx})\Big)\frac{\partial}{\partial z'''_{xxx}}$$
$$+ \Big((p'' z''_{xy} + 2p' z'''_{xxy}) + (q' z'''_{xxy}) + (r'''(z'_x)^2 z_y + r''(2z'_x z''_{xy} + z_{xx} z_y) + r' z'''_{xxy})\Big)\frac{\partial}{\partial z'''_{xxy}}$$
$$+ \Big((q'' z''_{xy} + 2q' z'''_{xyy}) + (p' z'''_{xyy}) + (r'''(z'_y)^2 z_x + r''(2z'_y z''_{xy} + z_{yy} z_x) + r' z'''_{xyy})\Big)\frac{\partial}{\partial z'''_{xyy}}$$
$$+ \Big((q''' z'_y + 3q'' z''_{yy} + 3q' z'''_{yyy}) + (r'''(z'_y)^3 + 3r'' z'_y z''_{yy} + r' z'''_{yyy})\Big)\frac{\partial}{\partial z'''_{yyy}}, \tag{5}$$

and so on. Let us choose an order ℓ of the jet. The quantities

$$(p, p', \dots, p^{(\ell)}, q, q', \dots, q^{(\ell)}, r, r', \dots, r^{(\ell)})$$

can be regarded as independent parameters. To each of these parameters, there corresponds an infinitesimal generator of the local Lie group. The dimension of the orbit is the rank of the corresponding matrix. For $\ell = 1$, this matrix looks as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z'_x & 0 \\ 0 & 0 & 0 & 0 & z'_y \\ 0 & 0 & 0 & z_x & z_y \end{pmatrix}. \tag{6}$$

The matrix is of size 6×5, which corresponds to the fact that the local group of dimension 6 (with the parameters (p, p', q, q', r, r')) acts on the space of dimension 5 (with the parameters (x, y, z, z_x, z_y)). This matrix has a block structure; one can single out the identity 3×3 matrix as a direct summand. This corresponds to the fact that an explicit dependence on (p, q, r) occurs for the 0-jet only. Thus, a similar direct summand must also occur in the matrices constructed for arbitrary ℓ. In particular, this implies the following assertion.

Proposition 9. *Any invariant of the extended action does not depend explicitly on* (x, y, z).

Thus, the truncated local group (with the parameters $(p', \dots, p^{(\ell)}, q', \dots, q^{(\ell)}, r', \dots, r^{(\ell)})$) acting on the space of truncated jets (i.e., without (x, y, z)) is of special interest. To this group, there corresponds a truncated matrix which contains all the information on the rank of the system of generators. For $\ell = 1$, this is a 3×2 matrix whose rank is equal to 2 outside $\{z_x z_y = 0\} = Cl_0$ and does not exceed one on Cl_0. The orbit of a generic point is open, and therefore there are no invariants. Cl_0 is a singular orbit.

Let now $\ell = 2$. The truncated matrix is of the form

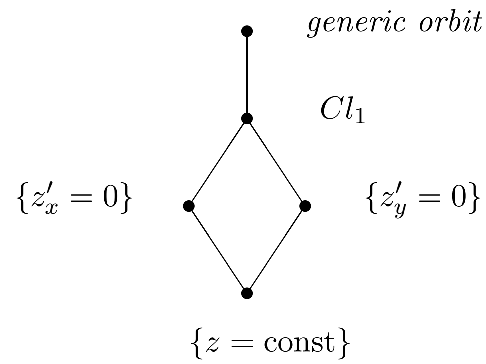
$$\begin{pmatrix} z'_x & 0 & 2z''_{xx} & z''_{xy} & 0 \\ 0 & z'_y & 0 & z''_{xy} & 2z''_{yy} \\ z'_x & z'_y & z''_{xx} & z''_{xy} & z''_{yy} \\ 0 & 0 & z'_x & 0 & 0 \\ 0 & 0 & 0 & 0 & z'_y \\ 0 & 0 & (z'_x)^2 & z'_x z'_y & (z'_y)^2 \end{pmatrix}. \tag{7}$$

The rank of this matrix is maximal and equal to 5 outside the same zero class $\{z_x z_y = 0\}$ and does not exceed two on Cl_0. The orbit of the generic point is open, and there are no invariants.

The next step is $\ell = 3$, the size of the truncated matrix is 9×9,

$$\begin{pmatrix} z'_x & 0 & 2z''_{xx} & z''_{xy} & 0 & 3z'_x & 2z'''_{xxy} & z'''_{xyy} & 0 \\ 0 & z'_y & 0 & z''_{xy} & 2z''_{yy} & 0 & z'''_{xxy} & 2z'''_{xyy} & 3z'''_{yyy} \\ z'_x & z'_y & z''_{xx} & z''_{xy} & z''_{yy} & z'''_{xxx} & z'''_{xxy} & z'''_{xyy} & z'''_{yyy} \\ 0 & 0 & z'_x & 0 & 0 & 3z''_{xx} & z''_{xy} & z'''_{xxy} & 0 \\ 0 & 0 & 0 & 0 & z'_y & 0 & 0 & z''_{xy} & 3z''_{yy} \\ 0 & 0 & {z'_x}^2 & z'_x z'_y & {z'_y}^2 & 3z'_x z''_{yy} & 2z'_x z''_{xy} + z'_y z''_{xx} & 2z'_y z''_{xy} + z'_x z''_{yy} & 3z'_y z''_{xx} \\ 0 & 0 & 0 & 0 & 0 & z'_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z'_y \\ 0 & 0 & 0 & 0 & 0 & {z'_x}^3 & {z'_x}^2 z'_y & z'_x {z'_y}^2 & {z'_y}^3 \end{pmatrix}, \tag{8}$$

and the determinant is equal to $2(z'_x)^4(z'_y)^4\Delta_1(z)$. This is exactly the second way of looking at the polynomial (1). This polynomial defines a singular orbit in the 3-jet. Thus, the orbit of the generic point is open, and there are no invariants. The two singular orbits are Cl_0 and Cl_1. Let us use the notation of the book [5]: any point located above another stands for an orbit whose closure contains the orbit represented by the lower point. Then the action described above gives the following picture of abutting for the orbits:



To the fact that Cl_0 is partitioned into two orbits, there corresponds the fact that $\Delta_0(z)$ can be factorized into two irreducible factors, whereas $\Delta_1(z)$ is irreducible. Let $\ell(n)$ be the differential order of $\Delta_n(z)$. If the other classes are irreducible, then one can assume that the general picture of abutting for the orbits of the action on the $\ell(n)$-jet is a linear graph (excluding the lowest segment) whose vertices are all classes up to Cl_n plus the orbit of the generic point.

3. WEBS

Webs were introduced in mathematical usage by W. Blaschke [6]. A 3-web on the plane or on a domain in the plane is defined by three families of smooth planar curves

$$\{u_1(x,y) = \mathrm{const}, \quad u_2(x,y) = \mathrm{const}, \quad u_3(x,y) = \mathrm{const}\}$$

such that the Jacobian of any pair of functions vanishes nowhere. Two webs obtained from each other by a smooth change of coordinates on the plane are regarded as equivalent ones. Since we have no other webs here, by a web we mean a planar analytic 3-web in what follows. Analyticity here means the analyticity of the functions, curves, and changes of variables. Every web can be represented locally in the form

$$\{x = \mathrm{const}, \quad y = \mathrm{const}, \quad z(x,y) = \mathrm{const}, \quad \text{where } z \notin Cl_0\}. \tag{9}$$

In Blaschke's book cited above, the differential-geometric theory of webs was developed. To a web, one can assign the traditional family of objects: differential forms, connection, curvature, etc.

The curvature form of a web is a 2-form on the plane, $\Omega = k(x,y)dx \wedge dy$, which is determined by a single scalar coefficient, the so-called curvature, $k(x,y)$. The curvature form has geometric definition and, if two webs are equivalent by some change of variables, then the same change connects the curvature forms. For webs represented in the form (9), the equivalence is the action of the above pseudogroup G.

The following remarkable coincidence occurs. For any web represented in the form (9), the curvature is $k = \delta_1(z)$ ([6, Chap. 1, Sec. 9]). Therefore, the following assertion holds.

Proposition 10. *A function $z(x,y)$ has complexity order one if and only if the web of the form* (9) *corresponding to this function has zero curvature.*

A web whose curvature is identically zero has a local representation in the form of three families of parallel lines. A web of this kind is said to be *hexagonal*. Our definition of the complexity order for functions can be extended to webs.

Definition 11. A web is said to have *complexity order n* if n is the complexity order of the function z in the representation (9).

The definition is correct, i.e., equivalent webs have the same complexity order, because the action of the pseudogroup G does not modify the complexity order of the defining function. We can now state the following corollary.

Corollary 12. *A web has complexity order one if and only if it is a hexagonal web.*

Question 13. What geometric property characterizes webs of complexity order two?

This question is meaningful even for linear webs, i.e., webs formed by three families of straight lines. The dependence of the solution of an algebraic equation on two coefficients of the equation gives interesting examples of linear webs. For instance, let $z(x,y)$ be a solution of the equation $z^m + xz + y = 0$. In this case, the level curves are straight lines, and we obtain a linear web. As was shown above, the complexity order is equal to two for $m \geqslant 2$.

Question 14.

(A) What is the complexity order of the linear web given by the equation $z^m + xz^2 + yz + 1 = 0$?

(B) The rational functions and the webs defined by these functions have finite complexity order. Does there exist an algebraic function (an algebraic web) of infinite complexity order?

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