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A Generic CR-Manifold as an $\{e\}$-Structure

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Abstract. In the paper it is proved that a "generic" CR-structure is reduced to an $\{e\}$-structure. Using the invariant frames obtained as a result of this reduction (an analog of the Darboux frame of classical differential geometry), we give a simple proof of an analog of the Vitushkin theorem of the extension of a germ for a rather broad class of real submanifolds of complex spaces.

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Greats teach us that the key to understanding an object is in the knowledge of its singularities, "singular points." If we study a class of objects rather than an individual object, then singularities of this kind are some special and rare objects that have unique properties in the given class. The knowledge of these singular objects gives a correct point of view at a generic object of the class. For instance, an "arbitrary" germ of a three-dimensional real hypersurface in $\mathbb{C}^2$ has no holomorphic symmetries at all, the È. Cartan homogeneous surfaces have a three-dimensional symmetry group (they form a one-parameter family), and only the sphere has a larger group, namely, an eight-dimensional one. Let us choose the sphere as the model and assume that every surface "wants to look like" the sphere. Here it is convenient to write out the sphere in the noncompact representation $\operatorname{Im} w = |z|^2$ (a point had sent to a point at infinity by a projective transformation); in this case, an "arbitrary" surface is a perturbation of the sphere,

$$\operatorname{Im} w = |z|^2 + \varepsilon(z, \bar{z}, \operatorname{Re} w).$$

An intension to choose local holomorphic coordinates in such a way that the equation of the surface becomes maximally similar (in some reasonable sense) to the equation of the sphere leads us to normal coordinates and to an equation in the normal form,

$$\operatorname{Im} w = |z|^2 + n(z, \bar{z}, \operatorname{Re} w).$$

The choice of normal coordinates is unique up to an origin-preserving automorphism of the surface (the stabilizer of any point on the sphere is five=dimensional). This ambiguity is unavoidable if the class contains an object with such a high symmetry. If we delete the symmetric objects from the class, then the choice of a normalization can become unique. This can be done as follows. Let us use the normalization suggested in [2]; in this case, the lower terms of a normal equation acquire the form

$$\operatorname{Im} w = |z|^2 + 2\operatorname{Re}(c_{42}(\operatorname{Re} w) z^4 \bar{z}^2 + c_{52}(\operatorname{Re} w) z^5 \bar{z}^3 + c_{43}(\operatorname{Re} w) z^4 \bar{z}^3) + \dots$$

If we impose the condition $c_{42}(0) \neq 0$ (in this case, the surface is said to be *nonumbilic* at the origin), then the stabilizer of a point contains at most two transformations, and if we additionally impose the condition

$$c_{42}(0)\bar{c}_{52}(0) + 3c_{43}(0)\bar{c}_{42}(0) \neq 0$$

(i.e., the surface is strongly nonumbilic), then the stabilizer becomes trivial [7], and the normal form can be refined in such a way that this form becomes unique (the so-called special normal form). These two conditions define two dense open subsets in a real-analytic hypersurface that are complements to proper analytic subsets provided that our surface is not the sphere. In terms of

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G-structure [3], our procedure can be described as follows. Originally, a nondegenerate hypersurface in $\mathbb{C}^2$ was an $\mathbf{H}$-structure, where $\mathbf{H}$ is the five-dimensional stabilizer of a point on the sphere [2]. On the set of nonumbilic points (our first condition) this $\mathbf{H}$-structure is reduced to a $\mathbf{Z}_2$-structure, and on the set of strongly nonumbilic points to an $\{\mathbf{e}\}$-structure, i.e., this is an invariant frame in the tangent bundle (absolute parallelism).One can readily construct such a frame on a strongly nonumbilic surface. Indeed, write

$$\theta_1(\xi) = \frac{\partial}{\partial x}, \quad \theta_2(\xi) = \frac{\partial}{\partial y}, \quad \theta_3(\xi) = \frac{\partial}{\partial u},$$

where $(z = x + iy,\ w = u + iv)$ are the coordinates at the point ξ that coincide with unique special formal normal coordinates at this point up to the 7th weight order. The dependence on the point is determined by the class of the hypersurface, namely, it is analytic if the hypersurface is and smooth if the hypersurface is. If there are two hypersurfaces of this kind, say, Γ and $\tilde{\Gamma}$, and if there is a biholomorphic mapping of the first hypersurface onto the other one, then the differential of this mapping takes the frame of the first hypersurface $(\theta_1, \theta_2, \theta_3)$ to the frame of the other hypersurface $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$.

This frame can be completed with the field

$$\nu(\xi) = \frac{\partial}{\partial v},$$

generating the normal bundle to our hypersurface. The quadruple of fields thus obtained is completely similar to the Darboux frame in the geometry of surfaces in the space $\mathbf{R}^3$.

Thus, let us sum up: every CR-structure on a strongly nonumbilic three-dimensional submanifold of two-dimensional complex manifold is reduced to an $\mathbf{e}$-structure. The singular object, which is the sphere in the case under consideration, is characterized by rich symmetries, whereas a generic object (an object in general position) is characterized by a rich system of invariants.

A three-dimensional real hypersurface in a two-dimensional complex manifold is a model and a source of analogies for more multidimensional situations. The generating d-dimensional real submanifold M of a complex space of dimension N is characterized by its type first of all. The type is a pair (n, K), where n is the CR-dimension, i.e., the complex dimension of the complex tangent, whereas K is the CR-codimension which is the real codimension in the enveloping space on one hand and the codimension of the complex tangent in the real tangent space, on the other hand. It is assumed that the type does not depend on the point and that the numbers n and K are positive. Moreover, $N = n + K$ and $d = 2n + K$. There is another parameter we need, this is ℓ, the length of the Levi–Tanaka algebra; this parameter is introduced as follows. This is the index of the distribution at which an increasing chain of distributions of real subspaces is stabilized, where the chain begins with the distribution D_1 of complex tangents and every subsequent distribution is obtained as the linear hull of the previous distribution and the brackets of the vector fields belonging to the previous distribution. Moreover, one can assume that the space in the last distribution D_ℓ is the entire tangent space (because the general case can be reduced to this special case).

Recently, in the paper [5], model manifolds that are "singular" for the class of a completely nondegenerate real submanifold of arbitrary type (n, K) in the same sense in which the sphere mentioned above is singular among the germs of the manifolds of type $(1, 1)$. The complete nondegeneracy condition has the following meaning: the brackets of the vector fields in the complex tangent not only give the entire tangent but also give it at minimally many operations for the given values of n and K. It is clear that, in this case, ℓ can be evaluated from the type and is not an independent parameter. The minimal value of ℓ is two and, for a chosen n, the parameter ℓ logarithmically slowly and unboundedly grows as K increases. In contrast to the type (1,1), a model surface need not be unique. In the general case we have a finite-dimensional family of pairwise inequivalent model surfaces which can be described as the quotient space of the space of polynomials of special kind by the action of some linear group [8]. Here the entire collection of germs of completely nondegenerate manifolds of the given type is partitioned into subtypes each of which is subjected to its own model surface. This model surface (the tangent model surface) $Q = Q(M_\xi)$ is related to the germ M_ξ like a tangent paraboloid of classical differential geometry. The group of holomorphic symmetries of the model surface is a Lie group, which is a subgroup of the group of birational transformations of the space. The Lie algebra of this Lie group is a Lie subalgebra of the polynomial vector fields with a natural grading structure of the form

$$g = g_{-\ell} + \dots + g_{-1} + g_0 + g_1 + \dots + g_d = g_- + g_0 + g_+.$$

Moreover, the subgroup G_- corresponding to the Lie subalgebra g_- acts on Q freely and fixed-point-freely and can naturally be identifies with the model surface Q by itself. In this case, the Lie algebra g_- (which then becomes the tangent space) can be identifies with the Levi–Tanaka algebra, where g_{-1} becomes a complex tangent and generates the entire subalgebra g_-. The group G_0 corresponding to the Lie subalgebra g_0 is a linear subgroup of the stabilizer (this subgroup admits an explicit description) and is isomorphic to some subgroup of $\mathrm{GL}(n,\mathbb{C})$. The following facts are known for the subgroup G_+ corresponding to g_+. If $\ell = 2$, then d can take the values 0, 1, and 2. For the overwhelming majority of types, to the germs in general position there correspond model surfaces for which $d = 0$; however, the number two is admissible in all well-studied situations. For $\ell \geqslant 3$, no example with $d > 0$ is known. Recently, I. Kossovskii [4] proved that, if $\ell = 3$, then d always vanishes, i.e., the subgroup G_+ is trivial. It seems probable that this holds for all ℓ with $\ell \geqslant 3$. If this is the case, then the Lie algebra of automorphisms of any model surface with $\ell > 2$ is of the form

$$g = g_{-\ell} + \dots + g_{-1} + g_0 = g_- + g_0.$$

Since the dimension of g_- is invariable and is equal to the dimension of the manifold, to $d = 2n+K$, the only variable quantity (in the sense of dimension) can be the component g_0 only. For some rare types, the group G_0 can coincide with the full linear group $\mathrm{GL}(n,\mathbb{C})$; however, for the overwhelming majority of types and for the model surfaces corresponding to germs in general position, this group is reduced to $\mathbb{R}^*$. These model surfaces are said to be *rigid*, and a germ is referred to as a *rigid* germ if its tangent model surface is. Our nearest objective is to show that the CR-structure of a rigid germ (i.e., of almost every germ) can almost always be reduced to an $\{\mathbf{e}\}$-structure (the only exclusion is the model surface). To this end, we recall some constructions and results of the paper [5].

A model surface of an arbitrary type (n,K) can be constructed as a result of some recurrence process. This is a process of evaluation of some sequence of data. Here is the list of the corresponding data with the index m: a direct decomposition of the space of polynomials with weight m into the sum of the space of harmonic forms $\mathcal{H}_m$ and a complementary space (the space of normalized polynomials of weight m, $\mathcal{CH}_m$), the dimension k_m of the space $\mathcal{CH}_m$, and the space $\mathbb{C}^{k_m}$ with the coordinates $w_m = u_m + iv_m$. Note that the weights are attributed to the variables thus arising as follows:

$$[z] = [\bar z] = 1, \qquad [u_m] = [w_m] = m.$$

The process is terminated at the ℓth step. The condition for terminating the process is that the codimension K occurs in the interval

$$k_2 + \dots + k_{\ell-1} < K \leqslant k_2 + \dots + k_\ell$$

Here the surplus of the codimension, $k = K - (k_2 + \dots + k_{\ell-1})$, can vary in the interval from 1 to k_ℓ. As the result, a model surface of type (n,K) is a surface Q in $\mathbb{C}^n \oplus \mathbb{C}^{k_2} \oplus \dots \oplus \mathbb{C}^{k_{\ell-1}} \oplus \mathbb{C}^k$ given by the relations

$$\begin{aligned} v_2 &= \Phi_2(z,\bar z), \\ &\dots \\ v_{\ell-1} &= \Phi_{\ell-1}(z,\bar z,u_2,\dots,u_{\ell-2}), \\ v_\ell &= \Phi_\ell(z,\bar z,u_2,\dots,u_{\ell-1}), \end{aligned} \tag{1}$$

or simply by the relation $v = \Phi(z,\bar z,u)$. Here the coordinates Φ_j for $j = 2,\dots,\ell-1$ are related to a real basis of the space $\mathcal{C}H_j$ and the coordinates Φ_ℓ are linearly independent elements of $\mathcal{C}H_\ell$. Every completely nondegenerate germ can be represented (by an invertible polynomial change of variables) in the form

$$\begin{aligned} v_2 &= \Phi_2(z,\bar z) + F_{23}(z,\bar z,u) + \dots, \\ &\dots \\ v_{\ell-1} &= \Phi_{\ell-1}(z,\bar z,u_2,\dots,u_{\ell-2}) + F_{\ell-1,\ell}(z,\bar z,u) + \dots, \\ v_\ell &= \Phi_\ell(z,\bar z,u_2,\dots,u_{\ell-1}) + F_{\ell,\ell+1}(z,\bar z,u). \end{aligned}$$

Here $F_{p,q}$ stands for the component of the weight q in the expansion of F_p, i.e., of the term on the right-hand side of the equation for v_p complementary to Φ_p. If we now denote by $F^{(s)}$ the column with the list of entries $(F_{2,s+2}, F_{3,s+3}, \dots, F_{\ell,s+\ell})$, then the equation of the germ can be represented in the form

$$v = \Phi(z,\bar{z},u) + F^{(1)} + F^{(2)} + \dots,$$

which we call the *standard form.*

By definition, the Lie algebra $\operatorname{aut} Q$ of the infinitesimal automorphisms of Q consists of the vector fields with the coefficients holomorphic in a neighborhood of the origin of the form

$$2\operatorname{Re}\left(A\frac{\partial}{\partial z} + B_2\frac{\partial}{\partial w_2} + \dots + B_\ell\frac{\partial}{\partial w_\ell}\right)$$

with the tangency condition

$$\mathcal{L}(f,g) = (\operatorname{Re}(iB + 2\Phi'_z A + \Phi'_u B)|_{w=u+i\Phi} = 0. \tag{3}$$

The Lie algebra corresponding to the zero stabilizer, $\operatorname{aut}_0 Q$, is the Lie subalgebra $\operatorname{aut}_0 Q$ consisting of the fields vanishing at the origin. It turns out that equation (0.3) has polynomial solutions of bounded weight. If one extends the weights introduced above to the vector fields on $\mathbb{C}^{n+K}$ by the condition $[\frac{\partial}{\partial z}] = [\frac{\partial}{\partial \bar{z}}] = -1$ and $[\frac{\partial}{\partial w_j}] = [\frac{\partial}{\partial \bar{w}_j}] = -j$, then the Lie algebra $\operatorname{aut} Q$ becomes a graded Lie algebra. The following assertion holds: along with with every field, every algebra contains every homogeneous graded component of the field. The rigidity of a surface means that the Lie algebra $\operatorname{aut}_0 Q = g_0$ is one-dimensional and is generated by the field

$$2\operatorname{Re}(z\frac{\partial}{\partial z} + 2w_2\frac{\partial}{\partial w_2} + \dots + \ell w_\ell\frac{\partial}{\partial w_\ell}), \tag{4}$$

i.e., the one-dimensional space generated by (0.4) is the intersection of the kernel of the operator $\mathcal{L}$ with the subspace of fields vanishing at the origin. Let us treat $\mathcal{L}$ as a linear operator taking the space V of families of $n+K$ formal power series in (z,w) to the space $\mathcal{F}$ formed by families of K formal power series in $(z,\bar{z},u)$. Let us represent the space $\mathcal{F}$ as the direct sum $\operatorname{Im} V \oplus \mathcal{N}$ of the image of the operator $\mathcal{L}$ and a direct complement to the image. When choosing the completion, we satisfy the following conditions. First, we carry out the choice in every weight component separately, and, second, we assume that some monomials vanish. It is clear that the complement thus chosen is invariant with respect to the one-parameter subgroup of transformations of the form $z \mapsto tz, w_2 \mapsto t^2 w_2, \dots, w_\ell \mapsto t^2 w_\ell$, where $t \in \mathbb{R}^*$; this subgroup is a stabilizer of a rigid model surface.

After this, by using the standard arguments of [2], we obtain the following assertion.

Proposition 1. (1) *Any equation of a completely nondegenerate germ whose tangent model surface is rigid can be reduced to the normal form $v = \Phi(z,\bar{z},u) + N(z,\bar{z},u)$, where $N \in \mathcal{N}$, by a formal change of variables.*

(2) *This reduction $(z \mapsto f(z,w),\ w \mapsto g(z,w))$, is unique up to choice of a single real parameter $t = \frac{\partial f^1}{\partial z^1}(0,0)$.*

(3) *If $(z \mapsto f(z,w),\ w \mapsto g(z,w))$ is a mapping of a normal form into another normal form, then $f(z,w) = tz$ and $g(z,w) = (t^2 w_2, \dots, t^\ell w_\ell)$.*

(4) *The only normal form of an equation of a model surface is $v = \Phi(z,\bar{z},u)$.*

Assume that a normal equation of a germ of a surface is

$$\begin{aligned} v_2 &= \Phi_2(z,\bar{z}) + N_{23}(z,\bar{z},u) + N_{24} + \dots, \\ &\dots \\ v_{\ell-1} &= \Phi_{\ell-1}(z,\bar{z},u_2,\dots,u_{\ell-2}) + N_{\ell-1,\ell}(z,\bar{z},u) + \dots, \\ v_\ell &= \Phi_\ell(z,\bar{z},u_2,\dots,u_{\ell-1}) + N_{\ell,\ell+1}(z,\bar{z},u) + \dots, \end{aligned} \tag{5}$$

or, in other words,

$$v = \Phi(z, \bar{z}, u) + N^{(2)} + N^{(3)} + \dots.$$

If the germ is equivalent to a model surface, then this means that all components of the normal equation vanish; by analogy with the type $(1,1)$, we refer to such a germ as a *spherical* germ. It is clear that, if a germ of a completely nondegenerate real-analytic surface at some point of this surface turned out to be spherical, then it is spherical at all other points as well. If this is not the case and if m is the index of the first nonzero component $N^{(m)}$, then we say that the germ is *aspherical* and that m is the *asphericity order* at the given point. Obviously, the entire one-parameter family of normal equations of a germ has the same asphericity order, i.e., this is an invariant characteristic. One can make the following remark concerning the dependence of the components $N^{(j)}$ on the equations of the original germ.

Remark 2. (1) The weight j-jet of a normalized equation, $\{N^{(2)}, \dots, N^{(j)}\}$, depends only on the weight j-jet of the original equation $\{F^{(1)}, F^{(2)}, \dots, F^{(j)}\}$ and a real parameter t, where the dependence on the jet is real-analytic and the dependence on the parameter t is polynomial.

(2) The change of variables of the form $z \mapsto tz, \quad w \mapsto (t^2 w_2, \dots, t^\ell w_\ell)$, replaces a normal equation of the germ $v = \Phi(z, \bar{z}, u) + N^{(2)} + N^{(3)} + \dots$ by the equation

$$v = \Phi(z, \bar{z}, u) + t^2 N^{(2)} + t^3 N^{(3)} + \dots$$

If a manifold is aspherical, then one can impose an additional condition at a point at which the asphericity order does not exceed m. Denote by $\|N\|_m$ the sum of absolute values of all coefficients of all components of the m-jet $\{N^{(1)}, N^{(2)}, \dots, N^{(j)}\}$, and now, using the remaining degree of freedom, assume that the condition $\|N\|_m = 1$ is satisfied. We refer to this normal form as a *special normal form of order m.* A special normal form is defined uniquely up to a transformation $(z \mapsto tz,\ w_2 \mapsto t^2 w_2, \dots, w_\ell \mapsto t^\ell w_\ell)$, where t is either 1 or (-1). To get rid of the $\mathbf{Z}_2$-symmetry, we must formulate a condition that excludes the germs for which the transform

$$(z \mapsto -z,\ w_2 \mapsto w_2,\ w_3 \mapsto -w_3, \dots, w_\ell \mapsto (-1)^\ell w_\ell)$$

is an automorphism from our family of "generic" germs. The germs with this symmetry are the germs whose equations contain only components $N^{(j)}$ with even indices j. These germs are said to be *even.* We refer to the index m of the first nonzero odd component as the it oddity order. For a germ which is not even, one can introduce an additional condition on the normal form. Namely, we can assume that the argument of the coefficient at the least monomial (in the sense of lexicographical order) of the least nonzero odd component be in the half-interval $[0, \pi)$. This condition removes the last ambiguity from the procedure constructing a normal form. In other words, we have two special normal forms, one of which does not use the oddity whereas the other does use. We introduce no special term, and the normal form in question will be always clear from the context. Let us formulate the following obvious assertion.

Proposition 3. (1) *Every aspherical germ which is not even can be reduced to a special normal form of some order by a formal change of variables.*

(2) *This reduction is unique.*

(3) *If both the asphericity order and the oddity order of a germ do not exceed m, then the m-jet of the equation of this germ in the special normal form of order m analytically depends on the m-jet of the original equation of the germ.*

If we have a real-analytic completely nondegenerate manifold M which is aspherical of order m and has no even points (i.e., the manifold is not even at any point), then we can now choose a holomorphically invariant frame of the tangent bundle. To this end, it suffices to write out the germ M_ξ at every point $\xi \in M$ in the coordinates that coincide with the special normal coordinates $(z = x + iy,\ w = u + iv)$ up to the order m and consider the standard frame of the tangent space at the center of the germ, i.e.,

$$E = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_K} \right\}.$$

One can add that the family

$$NE = \Big\{ \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_K} \Big\}$$

is a holomorphically invariant frame of the normal bundle. The dependence of the frames on the point is analytic. Thus, the following assertion holds.

Theorem (on the reduction). (1) *Let M be a real-analytic completely nondegenerate manifold, let M be aspherical, and let M be not even at any point. In this case, there is a holomorphic invariant frame of the tangent bundle on M* analytically depending on the point *and a similar frame of the normal bundle. In particular, the CR-structure can be reduced to an $\{\mathbf{e}\}$-structure on any manifold of this kind.*

(2) *If the condition concerning the absence of even points is not imposed, then one can claim that there is a two-sheeted covering over M whose fiber is the invariant pair of frames of the tangent space.*

As is well known, the fact that the tangent bundle is trivial is a certain condition concerning the topological structure of the manifold.

Let us now apply the invariant frame constructed above to the extension problem for holomorphic mappings, in the spirit of the Vitushkin theorem of [1]. Our point of view at this problem is as follows. The experience shows that there are two sources of difficulties here. The first of them is related to the desire to encircle both the singular manifolds with rich symmetry groups and generic manifolds that have no symmetries at all by some unified approach. The other source is typical for an analytic approach with the help of normal forms and coordinates, and the corresponding problems are related to convergence of some formal series [2]. We exclude the first circle of problems by considering "generic" manifolds only (however, the $\mathbf{Z}_2$-symmetry is admissible) and the other by using the formal normal form only, without studying any convergence problems for this form.

Before formulating the main result, we give the following definition. A number $\varepsilon > 0$ is referred to as the *amount* (or *supply*) *of analyticity* of a function or a mapping Φ at a point ξ if Φ has a representation as the sum of a power series in a polydisk centered at ξ and of radius ε and does not exceed $1/\varepsilon$ on the closed polydisk of radius $\varepsilon/2$. By the *amount* (or *supply*) *of analyticity* of a surface M at a point ξ we mean the amount of analyticity of a function F such that $M = \{v - F(z, \bar{z}, u) = 0\}$ in a neighborhood of a point ξ at which the differential of F vanishes. Our version of the theorem on a germ is as follows.

Theorem (on a germ). *Let M and $\tilde{M}$ be two rigid completely nondegenerate aspherical real-analytic submanifolds of complex spaces X and $\tilde{X}$, respectively, let K and $\tilde{K}$ be compact subsets of M and $\tilde{M}$, respectively, let ξ and $\tilde{\xi}$ be points of K and $\tilde{K}$, respectively, and let ψ be the germ of an invertible holomorphic mapping from M_ξ to $\tilde{M}_{\tilde{\xi}}$ that takes ξ to $\tilde{\xi}$. In this case, the amount of analyticity of the mapping ψ is a positive quantity depending on the objects M, K, $\tilde{M}$, and $\tilde{K}$ only.*

Proof. The asphericity order of points on the compactum K is bounded by some quantity m because otherwise there is a point at which the germ of the manifold (and thus the entire manifold as well) is spherical. If the manifold M is aspherical of order m, then, since the quantity $\|N\|_m(\xi)$ is bounded on K above and below by positive constants, then the same holds for the absolute values of the factor t which is used when transforming a normal form to a special normal form. Thus, the amount of analyticity of the invariant pair of frames,

$$\Theta^+ = (\Theta_1^+, \dots, \Theta_{2n+K}^+), \qquad \Theta^- = (\Theta_1^-, \dots, \Theta_{2n+K}^-),$$

at an arbitrary point of K depends on M and K only. Symmetrically, one can claim the same concerning the invariant pair of frames on $\tilde{K}$,

$$\tilde{\Theta}^+ = (\tilde{\Theta}_1^+, \dots, \tilde{\Theta}_{2n+K}^+), \qquad \tilde{\Theta}^- = (\tilde{\Theta}_1^-, \dots, \tilde{\Theta}_{2n+K}^-).$$

Let us construct a certain real-analytic mapping ϕ of the first germ to the other for each of the four pairs of the form $(\Theta, \tilde{\Theta})$, where Θ is one of the two frames on the first manifold and $\tilde{\Theta}$

is one of the two frames on the other by proceeding as follows. Set $\psi(\xi) = \tilde{\xi}$. If $x(t_1)$ and $\tilde{x}(t_1)$ are integral curves of the fields Θ_1 and $\tilde{\Theta}_1$ passing at $t_1 = 0$ through ξ and $\tilde{\xi}$, respectively, then $\psi(x(t_1)) = \tilde{x}(t_1)$. Further, let $x(t_1, t_2)$ and $\tilde{x}(t_1, t_2)$ be integral curves of the fields Θ_2 and $\tilde{\Theta}_2$ such that $x(t_1, 0) = x(t_1)$ and $\tilde{x}(t_1, 0) = \tilde{x}(t_1)$, respectively, then we set $\psi(x(t_1, t_2)) = \tilde{x}(t_1, t_2)$. etc. After the last $(2n + K)$th step we see that the mapping ψ is well defined on a full neighborhood of the point ξ and invertibly takes this neighborhood onto a full neighborhood of the point $\tilde{\xi}$. Here the amounts of analyticity of the direct and inverse mappings depend only on the amounts of analyticity of the pairs of frames. We now note that, if the germs M_ξ and $\tilde{M}_{\tilde{\xi}}$ are holomorphically equivalent, then on any set where is coincides with a holomorphic mapping it must coincide with one of the four mappings constructed above. However, a real-analytic mapping holomorphic in a small neighborhood is holomorphic wherever it is analytic. This completes the proof.

One can derive a lot of corollaries to this theorem, as in [6]. In all these corollaries one speaks of a completely nondegenerate rigid aspherical real-analytic submanifolds of complex spaces and of germs of holomorphic locally invertible mappings.

Corollaries. (1) *If both the submanifolds M and $\tilde{M}$ are compact, then the amount of analyticity of the germ of any mapping from M_ξ into $\tilde{M}_{\tilde{\xi}}$ is bounded below by a positive constant depending on the pair of manifolds only.*

(2) *The group of global automorphisms of a compact submanifold M is compact Lie group* (*in the natural topology of uniform convergence on compacta*) *of holomorphic transformations of M.*

(3) *If $\tilde{M}$ is compact then the mapping ψ can be extended along all paths on M.*

(4) *If $\tilde{M}$ is compact and M is simply connected, then ψ can be extended to a holomorphic locally invertible mapping from M into $\tilde{M}$.*

(5) *If both M and $\tilde{M}$ are compact and simply connected, then the mapping ψ can be extended to a holomorphic equivalence between M and $\tilde{M}$.*

Every situation in which some of these conclusions fails shows that the above $\{\mathbf{e}\}$-structure was subjected to some degeneration. Under the asphericity assumption, the reason is clear, namely, the cause is in the loss of complete nondegeneracy. This means that we deal with a singularity. The singularities of our $\{\mathbf{e}\}$-structure form a topic of a separate consideration.

If the tangent bundle of the variety is nontrivial, then singularities of this kind necessarily occur.

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