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Moduli Space of Model Real Submanifolds

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Abstract. In [6], to a completely nondegenerate germ of a real submanifold of a chosen CR-type (n, K) in a complex space, we assigned a tangent polynomial model of the submanifold. In the present paper, we construct the moduli space $\mathcal{M}(n, K)$ of the family of polynomial models, i.e., the space parametrizing the holomorphically nonequivalent polynomial models. The space thus obtained is used to construct CR-characteristic classes.

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1. INTRODUCTION

To every germ of a completely nondegenerate real submanifold of a complex space, we can assign a polynomial model of the submanifold [6]. This correspondence (germ $\longrightarrow$ model) is holomorphically invariant. In this correspondence, the linear equivalence of model surfaces corresponds to the holomorphic equivalence of germs. The factorization of the family of model surfaces of a chosen type by the action of the linear group gives the moduli space, which is a remarkable object in many aspects. Its geometry represents the geometry of the entire given CR-type in the large. The nontrivial properties of the topological structure of the moduli space enables one to introduce a series of new characteristics (CR-characteristic classes) invariant with respect to smooth isotopies on an arbitrary completely nondegenerate submanifold.

2. EXAMPLE

The situation of the least dimension for which this construction is meaningful is the 6-dimensional real submanifold of a 5-dimensional complex space. Let M be a completely nondegenerate manifold [6], i.e., a smooth (C^∞) submanifold which is given by equations of the form

$$\begin{aligned}
\Im w_1 &= |z|^2 + F_1(z, \bar{z}, \Re w_1, \Re w_2, \Re w_3, \Re w_4),\\
\Im w_2 &= |z|^2 + 2\Re z^2\bar{z} + F_2(z, \bar{z}, \Re w_1, \Re w_2, \Re w_3, \Re w_4),\\
\Im w_3 &= |z|^2 + 2\Im z^2\bar{z} + F_3(z, \bar{z}, \Re w_1, \Re w_2, \Re w_3, \Re w_4),\\
\Im w_4 &= 2\Re(az^3\bar{z}) + b|z|^4 + F_4(z, \bar{z}, \Re w_1, \Re w_2, \Re w_3, \Re w_4)
\end{aligned}$$

in a neighborhood of each point of the submanifold, where a is a complex number, b is a real number, $(a, b) \neq 0$, and the quantity $F_j(tz, t\bar{z}, t^2\Re w_1, t^3\Re w_2, t^3\Re w_3, t^4\Re w_4)$ is $o(t^2)$ for $j = 1$, $o(t^3)$ for $j = 2, 3$, and $o(t^4)$ for $j = 4$ (we shall interpret these conditions below in terms of a certain grading). Note that this manifold has a 1-dimensional complex tangent, has codimension 4, and is generating. The condition that a manifold has a representation of this kind is a nondegeneracy condition indeed. This means that, in the space of values of the 4-jets at a point, the germs satisfying this condition form an open dense set and the complement of the set is a proper algebraic subset. The condition of complete nondegeneracy has the following interpretation: the length of the graded Levi–Tanaka algebra is the minimal possible in this situation, namely, it is equal to four.

The real algebraic surface $Q(M)$ given by

$$\Im w_1 = |z|^2, \qquad \Im w_2 = |z|^2 + 2\Re z^2\bar{z}, \qquad \Im w_3 = |z|^2 + 2\Im z^2\bar{z}, \qquad \Im w_4 = 2\Re(az^3\bar{z}) + b|z|^4,$$

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is referred to as a tangent model surface of the manifold M at the point corresponding to the origin. It is of importance for our purposes that the model surface is not unique and depends on three real parameters. The action of the local holomorphic transformations on the surface M induces a linear action in the three-dimensional space of real polynomials of the above form. An action of $\mathrm{GL}(1,\mathbb{C}) \oplus \mathrm{GL}(1,\mathbb{R})$ on this space is induced and has the following form:

$$\text{if} \quad (re^{i\theta},\rho) \in \mathrm{GL}(1,\mathbb{C}) \oplus \mathrm{GL}(1,\mathbb{R}), \quad \text{then} \quad (re^{i\theta},\rho)(a,b) = r^4\rho^{-1}(ae^{2i\theta},b).$$

This action can be treated as the following action of $U(1)$ on $\mathbb{R}P^2$:

$$(\Re a : \Im a : b) \mapsto (\Re(e^{2i\theta})a : \Im(e^{2i\theta}a) : b).$$

One can readily see that the real rational function $|a|^2/b^2$ separates the generic orbits of this action, and therefore (see [1]), this function is the generator of the field of rational invariants of this action. This function defines a regular mapping from $\mathbb{R}P^2$ to $\mathbb{R}P^1$ given by

$$(\Re a : \Im a : b) \mapsto (|a|^2 : b^2).$$

The image of this mapping is the entire space $\mathbb{R}P^1$. For this reason, we refer to this space as the *moduli space of model surfaces* for the manifolds of this type and denote it by $\mathcal{M}$. The points of this space, which is topologically a surface, are in a natural one-to-one correspondence with the family of orbits of our action.

If the moduli space of the model surfaces is topologically nontrivial, then we can use it to define the corresponding *CR-characteristic classes* on a manifold of type $(1,4)$. If M is a completely nondegenerate manifold of type $(1,4)$, ξ is a point of M, M_ξ is a germ of M at the point ξ, and $Q(M_\xi)$ is a tangent model surface of M_ξ, then the composition of the mappings

$$\xi \in M \mapsto M_\xi \mapsto Q(M_\xi) \mapsto \text{a point of the manifold } \mathcal{M}$$

canonically defines a "Gaussian" mapping

$$\Gamma\colon M \mapsto \mathcal{M}(n,K).$$

The one-dimensional cohomology group of the circle $H^1(\mathcal{M}(1,4),\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and, if ω is the generator of the group, then the form Ω induced by the mapping Γ on M is invariant with respect to the holomorphic mappings, and the corresponding one-dimensional cohomology class of M is invariant in the class of smooth isotopies in the enveloping five-dimensional complex space. The geometric meaning of this invariant is clear, namely, this is the degree of the above "Gaussian" mapping.

3. CORRESPONDENCE "GERM ⟶ MODEL"

Before treating the general case, let us briefly recall the construction of the correspondence "germ ⟶ model" (for details, see [6]). Let M_ξ be a germ of a smooth real generating submanifold in a complex space, let n be the complex dimension of the complex tangent, let K be the real codimension, and let ℓ be the length of the Levi–Tanaka graded algebra. In this case, $n+K$ is the dimension of the enveloping complex space. The pair (n,K) is referred to as the *type*. For completely nondegenerate manifolds, the type determines the length uniquely. A model surface of an arbitrary type (n,K) is constructed by using a recurrent process. At the first step, one assigns the weight 1 to the variable $z \in \mathbb{C}^n$. This is continued by a process of evaluating of some sequence of data. Here is the list of data with the index m: the direct decomposition of the space of polynomials of weight m into the sum of spaces of harmonic forms $\mathcal{H}_m$ and its complementary space of normalized polynomials of weight m, $\mathcal{CH}_m$, the dimension k_m of the space $\mathcal{CH}_m$, and the space $\mathbb{C}^{k_m}$ with the coordinates $w_m = u_m + iv_m$ of weight m. The process is terminated at the ℓth step, where ℓ is the length of the Levi–Tanaka algebra. The condition for the termination of the process is that the codimension K falls into the interval

$$k_2 + \cdots + k_{\ell-1} < K \leqslant k_2 + \cdots + k_\ell.$$

Here the *excessive codimension* $k = K - (k_2 + \cdots + k_{\ell-1})$ can vary from 1 to k_ℓ. Finally, a model surface of type (n, K) is a real algebraic surface $Q = Q(M_\xi)$ in $\mathbb{C}^n \oplus \mathbb{C}^{k_2} \oplus \cdots \oplus \mathbb{C}^{k_{\ell-1}} \oplus \mathbb{C}^k$ given by the relations

$$\begin{aligned} v_2 &= \Phi_2(z, \bar{z}), \\ &\cdots\cdots\cdots \\ v_j &= \Phi_j(z, \bar{z}, u_2, \ldots, u_{j-1}), \\ &\cdots\cdots\cdots \\ v_{\ell-1} &= \Phi_{\ell-1}(z, \bar{z}, u_2, \ldots, u_{\ell-2}). \\ v_\ell &= \Phi_\ell(z, \bar{z}, u_2, \ldots, u_{\ell-1}), \end{aligned} \tag{1}$$

or simply $v = \Phi(z, \bar{z}, u)$. Here the coordinates of Φ_j for $j = 2, \ldots, \ell - 1$ are an (arbitrary) real basis of the space $\mathcal{C}H_j$ and the coordinates of Φ_ℓ are linearly independent elements of $\mathcal{C}H_\ell$. Any completely nondegenerate germ can be reduced by an invertible polynomial change of variables to the form

$$\begin{aligned} v_2 &= \Phi_2(z, \bar{z}) + O(3), \\ &\cdots\cdots\cdots \\ v_{\ell-1} &= \Phi_{\ell-1}(z, \bar{z}, u_2, \ldots, u_{\ell-2}) + O(\ell), \\ v_\ell &= \Phi_\ell(z, \bar{z}, u_2, \ldots, u_{\ell-1}) + O(\ell + 1), \end{aligned}$$

where $O(m)$ is the sum of monomials of weights not less than m. If two model surfaces, i.e., surfaces of the form (1), with some families of right-hand sides $\Phi = (\Phi_2, \ldots, \Phi_\ell)$ and $\tilde{\Phi} = (\tilde{\Phi}_2, \ldots, \tilde{\Phi}_\ell)$ are holomorphically equivalent, then they are linearly equivalent, and this linear transformation is of the form

$$z \mapsto Cz, \quad w_2 \mapsto \rho_2 w_2, \quad \ldots, \quad w_\ell \mapsto \rho_\ell w_\ell,$$

where $C \in \mathrm{GL}(n, \mathbb{C}), \rho_2 \in \mathrm{GL}(k_2, \mathbb{R}), \ldots, \rho_\ell \in \mathrm{GL}(k_\ell, \mathbb{R})$ and the relations

$$\Phi_j(Cz, \overline{Cz}, \rho_2 u_2, \ldots, \rho_{j-1} u_{j-1}) = \rho_j \tilde{\Phi}_j(z, \bar{z}, u_2, \ldots, u_{j-1}) \tag{2}$$

hold for any $j = 2, \ldots \ell$.

Notation. Let us use the notation of the form $\langle A^3 B^2 C \rangle$ by assuming that, in this case, this form (whose dimension is clear from the context) homogeneous of degree three with respect to A, of degree two with respect to B, and of first degree with respect to C with linearly independent coordinates. If the coordinates of the form constitute a basis of the corresponding space, we write $[A^3 B^2 C]$.

Let us look at model surfaces for several lower values of ℓ.

For $\ell = 2$, these are well-known surfaces of the form

$$\Im w_2 = \langle z\bar{z} \rangle,$$

where $w_2 \in \mathbb{C}^K$. These surfaces serve the codimensions in the interval $1 \leqslant K \leqslant n^2$. This case differs from the others because the nondegeneracy condition here includes not only the linear independence of the coordinate Hermitian forms but additionally assumes that the kernel is trivial (if $e \neq 0$, then $\langle z, \bar{e} \rangle$ is not identically zero).

For $\ell = 3$,

$$\Im w_2 = [z\bar{z}], \qquad \Im w_3 = 2\Re \langle z^2 \bar{z} \rangle.$$

Here the dimension of w_2 is equal to n^2, $w_3 \in \mathbb{C}^{K-k_2}$, $k_3 = n^2(n+1)$, and the interval of codimensions is $k_2 + 1 \leqslant K \leqslant k_2 + k_3$.

For $\ell = 4$,

$$\Im w_2 = [z\bar{z}], \qquad \Im w_3 = 2\Re[z^2\bar{z}], \qquad \Im w_4 = 2\Re\big(\langle z^3\bar{z}\rangle + \langle z^2\bar{z}^2\rangle\big),$$

where $w_j \in \mathbb{C}^{k_j}$, $j = 2, 3$, $w_4 \in \mathbb{C}^k$, $k_4 = n^2(n+1)(7n+11)/12$, $1 \leqslant k \leqslant k_4$, and

$$k_2 + k_3 + 1 \leqslant K \leqslant k_2 + k_3 + k_4.$$

For $\ell = 5$,

$$\Im w_2 = [z\bar{z}], \quad \Im w_3 = 2\Re[z^2\bar{z}], \quad \Im w_4 = 2\Re[z^3\bar{z}] + [z^2\bar{z}^2], \quad \Im w_5 = 2\Re(\langle z^4\bar{z}\rangle + \langle z^3\bar{z}^2\rangle + \langle z^2\bar{z}\, u_2\rangle),$$

where $w_j \in \mathbb{C}^{k_j}$, $j = 2, 3, 4$, $w_5 \in \mathbb{C}^k$, $k_5 = n^2(n+1)(15n^2 + 11n + 10)/12$, and

$$k_2 + k_3 + k_4 + 1 \leqslant K \leqslant k_2 + k_3 + k_4 + k_5.$$

4. CONSTRUCTION OF THE MODULI SPACE

Denote by $\mathcal{CH} = \mathcal{CH}(n, K)$ the space

$$\mathcal{CH}_2^{k_2} \oplus \cdots \oplus \mathcal{CH}_{\ell-1}^{k_{\ell-1}} \oplus \mathcal{CH}_\ell^k,$$

where the symbol V^m stands for the direct sum of m copies of the linear space V. In this case, the right-hand side of the equations of all possible model surfaces of type (n, K) are elements of $\mathcal{CH}$. To be more precise, when taking the nondegeneracy condition into account, these are elements of an open dense subset $\mathcal{CH}^*$ of $\mathcal{CH}$. Two model surfaces are holomorphically equivalent if and only if the right-hand sides (which are elements of the space $\mathcal{CH}^*$) belong to the same orbit of the following action of the group $\mathrm{GL}(n, \mathbb{C}) \oplus \mathrm{GL}(k_2, \mathbb{R}) \oplus \cdots \oplus \mathrm{GL}(k_{\ell-1}, \mathbb{R}) \oplus \mathrm{GL}(k, \mathbb{R})$:

$$\begin{gathered}
\Phi_2(z, \bar{z}) \mapsto \rho_2^{-1}\Phi_2(Cz, \overline{Cz}),\\
\cdots\cdots\cdots\\
\Phi_j(z, \bar{z}, u_2, \dots, u_{j-1}) \mapsto \rho_j^{-1}\Phi_j(Cz, \overline{Cz}, \rho_2 u_2, \dots, \rho_{j-1}u_{j-1}),\\
\cdots\cdots\cdots\\
\Phi_\ell(z, \bar{z}, u_2, \dots, u_{\ell-1}) \mapsto \rho^{-1}\Phi_\ell(Cz, \overline{Cz}, \rho_2 u_2, \dots, \rho_{\ell-1}u_{\ell-1}).
\end{gathered} \tag{3}$$

However, the action of $\mathrm{GL}(k_j, \mathbb{R})$ for $j = 2, \dots, \ell - 1$ on the bases of the space $\mathcal{C}H_j$ is free. Therefore, when distinguishing a basis in each of these spaces and assuming that the distinguished basis remains fixed, we see that, for a chosen C, relations (2) are uniquely solvable with respect to $\rho_2, \dots, \rho_{\ell-1}$. Denoting by $\tau_j(C)$ the unique solution of the relation

$$\Phi_j(Cz, \overline{Cz}, \rho_2 u_2, \dots, \rho_{j-1}u_{j-1}) = \rho_j \Phi_j(z, \bar{z}, u_2, \dots, u_{j-1})$$

with respect to ρ_j, we see that the mapping $C \mapsto \tau_j(C)$ is a representation of $\mathrm{GL}(n, \mathbb{C})$ into $\mathrm{GL}(k_j, \mathbb{R})$. It becomes clear now that the orbits of the action (3) are in a natural correspondence with the orbits of the action of $\mathrm{GL}(n, \mathbb{C}) \oplus \mathrm{GL}(k, \mathbb{R})$ of the form

$$\Phi_\ell(z, \bar{z}, u_2, \dots, u_{\ell-1}) \mapsto \rho^{-1}\Phi_\ell(Cz, \overline{Cz}, \tau_2(C)u_2, \dots, \tau_{\ell-1}(C)u_{\ell-1}) \tag{4}$$

on the space $(\mathcal{C}H_\ell^k)^*$, i.e., the space of families of k linearly independent elements $\mathcal{C}H_\ell$.

To take one more step, note that, when considering k-dimensional subspaces of $\mathcal{C}H_\ell$ spanned by k linearly independent elements of $\mathcal{C}H_\ell$ instead of the families of these elements by themselves, our orbits can naturally be identified with the orbits of the $\mathrm{GL}(n, \mathbb{C})$-action induced by (4) on the real Grassmann manifold $Gr_k(\mathcal{C}H_\ell)$. Finally, the real scalar extensions belong to the kernel of the action (4), and therefore, one can pass from the group $\mathrm{GL}(n, \mathbb{C})$ to the subgroup G of the matrices C such that $|\det C| = 1$.

The action described above is a quite traditional object of investigation in classical invariant theory [1]. Let $\mathcal{F}$ be the field of rational functions on $Gr_k(\mathcal{C}H_\ell)$ and let $\mathcal{F}^G$ be the subfield of functions invariant with respect to the above action of G. These fields are finitely generated over $\mathbb{R}$. Moreover, the field $\mathcal{F}^G$ contains a finite set of generators $R_1, \dots, R_s$ separating the generic objects. These real rational functions define a rational mapping of $Gr_k(\mathcal{C}H_\ell)$ into $\mathbb{R}P^s$. We can consider the image of this mapping [2]. This very manifold will be referred to as the moduli space and denoted by $\mathcal{M}(n, K)$.

As well as in the example treated at the beginning, we can define the canonical "Gaussian" mapping. If M is a completely nondegenerate manifold, ξ is a point of M, M_ξ is a germ of M at the point ξ, and $Q(M_\xi)$ is a tangent model surface to M_ξ, then the Gaussian mapping $\Gamma : M \mapsto \mathcal{M}(n,K)$ is defined as the composition

$$\xi \in M \mapsto M_\xi \mapsto Q(M_\xi) \mapsto \text{a point of the manifold } \mathcal{M}.$$

The canonical mapping Γ enables one to transfer any covariant constructions from $\mathcal{M}(n,K)$ to M and, in particular, to transfer the integral cohomology of $\mathcal{M}$. The cohomology of $\mathcal{M}$, transferred to M by using G, define "*CR-characteristic classes*" on any smooth completely nondegenerate manifold. These classes are obviously invariant with respect to the smooth isotopies that do not violate the complete nondegeneracy.

5. PROPERTIES OF THE MODULI SPACE

The dimension of the moduli space is determined by the dimension of the generic orbit, which in turn is the difference between the dimension of the group and the dimension of the stabilizer of a generic point [1]. The dimension of the group $\mathrm{GL}(n,\mathbb{C}) \oplus \mathrm{GL}(k,\mathbb{R})$ is obviously equal to $2n^2 + k^2$. This group acts on the space of dimension $k_\ell k$. The dimension of the orbit of this action is equal to the dimension of the group minus the dimension of the stabilizer of any point of the orbit. The dimension of the stabilizer is equal to the dimension of the Lie algebra of the stabilizer. The equations defining this Lie algebra are linear homogeneous equations whose coefficients linearly depend on the coefficients of the leading form Φ_ℓ. Thus, a system of embedded algebraic subsets of the space $\mathcal{CH}_\ell$ arises, and this system stratifies the space with respect to the dimension of the stabilizer. The value of the stabilizer is minimal on an open dense subset of $\mathcal{CH}_\ell$. Denote this quantity by DS and refer to it as the *dimension of the stabilizer in general position*. In this case, one can claim that an open dense subset of $\mathcal{CH}_\ell$ is fibered into orbits of dimension $2n^2 + k^2 - DS$. This family of orbits obviously depends on $k_\ell k - (2n^2 + k^2 - DS)$ parameters. One can readily evaluate the quantity DS. If $k = k_\ell$, then $DS = 2n^2$ for any ℓ. If $\ell = 2$, then DS depends on $k = 1, \dots, n^2$ as follows. If $k = 1, n^2 - 1$, then $DS = n^2 + 1$; if $k = 2, n^2 - 2$, then the case $n = 1$ follows from excluding considerations due to the condition $k \leqslant n^2$; if $n = 2$, then $DS = 4$ and, if $n \geqslant 3$, then $DS = n + 1$. If $k = 3, \dots, n^2 - 3$, then $DS = 2$. If now $\ell > 2$ and $k < k_\ell$, then $DS = 1$. We finally obtain the following assertion.

Proposition 1. *If $\ell = 2$ and $n = 1$ or $n = 2$, then $\mathcal{M}$ is a point and the dimension is zero.*
If $\ell = 2$, $n \geqslant 3$, and $k = 1, n^2 - 1, n^2$, then $\mathcal{M}$ is a point and the dimension is zero.
If $\ell = 2$, $n \geqslant 3$, and $k = 2, n^2 - 2$, then $\dim \mathcal{M} = n - 3$.
If $\ell = 2$ and $k = 3, \dots, n^2 - 3$, then $\dim \mathcal{M} = n^2(k-2) + 2 - k^2$.
If $\ell \geqslant 3$ and $k < k_\ell$, then $\dim \mathcal{M} = k_\ell k + 1 - (2n^2 + k^2)$.
If $k = k_\ell$, then $\mathcal{M}$ is a point and the dimension is zero.

If one fixes the value of n and considers the value of K as increasing from 1 step by step, then the types of manifolds can be grouped into some *periods* indexed by the values of ℓ. Along with the representation of a model surface as a point in the Grassmann manifold of k-dimensional subspaces of the space of forms of the leading degree $\mathcal{CH}_\ell$, one can consider the dual representation, namely, the representation in the form of a point in the Grassmann manifold of $(k_\ell - k)$-dimensional planes in the space of linear functions on $\mathcal{CH}_\ell$, i.e., in the dual space. This representation is constructed on the basis of the canonical correspondence (a k-dimensional subspace) $\longrightarrow$ the $(k_\ell - k)$-dimensional subspace of linear functions annihilating the k-dimensional subspace.

As a result, every period is partitioned into "symmetric" pairs of types,

$$(n, k_2 + \dots + k_{\ell-1} + k) \leftrightarrow (n, k_2 + \dots + k_{\ell-1} + (k_\ell - k)).$$

When identifying a space with its dual space, the moduli spaces of symmetric types are identified. Thus, the following assertion holds.

Proposition 2. $\mathcal{M}(n, k_2 + \cdots + k_{\ell-1} + k) \cong \mathcal{M}(n, k_2 + \cdots + k_{\ell-1} + (k_\ell - k))$.

A period is terminated whenever the excessive codimension k attains the maximal value $k = k_\ell$ in the framework of the current period and the value of K is at the same time equal to $k_2 + \cdots + k_\ell$. It is natural to refer to this type as the *last* type of the period. The model surface for the last type is unique, and $\mathcal{M}(n, k_2 + \cdots + k_\ell)$ is a point (see Proposition 1).

6. SOME OTHER EXAMPLES

Let us consider the value $n = 1$ and take several lower values of K. Our "periodic table" is opened by the period with index $\ell = 2$. This period consists of the unique type $(1, 1)$. For $K = 1$, the complete nondegeneracy of a manifold coincides with its Levi nondegeneracy, which means that, in a neighborhood of any point ξ of such a manifold, this manifold can be represented by local equations of the form

$$\Im w_2 = |z|^2 + O(3)$$

(we assume that the weights of the monomials are $[z] = 1$ and $[w_2] = 2$). This means that any surface of this kind can locally be represented as a perturbation of a unique model surface $\Im w_2 = |z|^2$, and the moduli space is certainly a point. The next period with $\ell = 3$ consists of two types, $(1, 2)$ and $(1, 3)$. The model surfaces

$$\Im w_2 = |z|^2, \quad \Im w_3 = |z|^2 \Re z$$

and

$$\Im w_2 = |z|^2, \quad \Im w_3^1 = |z|^2 \Re z, \quad \Im w_3^2 = |z|^2 \Im z$$

are also unique. The period with $\ell = 4$ consists of three types $(1, 4)$, $(1, 5)$, and $(1, 6)$. The first of these types was already considered above, and, as we know, $\mathcal{M}(1, 4) = S^1$. The type $(1, 5)$ is symmetric to the type $(1, 4)$ and, by Proposition 2, the space $\mathcal{M}(1, 5)$ is also the circle S^1. The type $(1, 6)$ is the last type in the period, and $\mathcal{M}(1, 6)$ is a point.

The next period ($\ell = 5$) contains six types, from $(1, 7)$ to $(1, 12)$. Locally, the equation of a completely nondegenerate manifold of type $(1, 7)$ can be represented in the form

$$\begin{gathered} \Im w_2 = |z|^2 + O(3), \\ \Im w_3^1 = |z|^2 \Re z + O(4), \quad \Im w_3^2 = |z|^2 \Im z + O(4), \\ \Im w_4^1 = 2\Re(z^3 \bar{z}) + O(5), \quad \Im w_4^2 = 2\Im(z^3 \bar{z}) + O(5), \quad \Im w_4^3 = |z|^4 + O(5), \\ \Im w_5 = 2\Re(a z^4 \bar{z} + b^3 \bar{z}^2 + c u_2 z^2 \bar{z}) + O(6), \end{gathered}$$

where the lower index of a coordinate is equal to its weight, a, b, c are complex numbers, and $(a, b, c) \neq 0$. The group $\mathrm{GL}(1, \mathbb{C}) \oplus \mathrm{GL}(1, \mathbb{R})$ acts on a real six-dimensional space of parameters as follows:

$$(re^{i\theta}, \rho)(a, b, c) = \frac{r^5}{\rho}(e^{3i\theta} a, e^{i\theta} b, e^{i\theta} c).$$

If we pass to $\mathbb{R}P^5$ with the coordinates $(\Re a : \Im a : \Re b : \Im b : \Re c : \Im c)$, then the above action is reduced to the action of $U(1)$ of the form

$$(e^{i\theta})(a, b, c) = (e^{3i\theta} a, e^{i\theta} b, e^{i\theta} c).$$

The generic orbits are separated by the pair of complex rational functions a/b^3 and c^3/b^3. These functions define a mapping of $\mathbb{R}P^5$ into $\mathbb{C}P^2$ of the form

$$(a, b, c) \mapsto (a : b^3 : c^3).$$

For the moduli space $\mathcal{M}(1, 7)$, we can take the image of this mapping, i.e., $\mathbb{C}P^2$. Proposition 2 enables one to write $\mathcal{M}(1, 11) = \mathcal{M}(1, 7) = \mathbb{C}P^2$.

In what follows, we restrict ourselves to the evaluation of the dimensions up to the periods with $\ell = 2, 3, 4, 5, 6, 7$. The rows of the table are ordered by the parameter K, which ranges from 1 to 35, and the periods are separated by double vertical lines.

$\mathbf{n = 1}$.

K	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\dim M$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\dim \mathcal{M}$	0	0	0	1	1	0	4	7	8	7	4	0	6	11	14	15	14	11

K	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$\dim M$	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
$\dim \mathcal{M}$	6	0	13	25	35	43	49	53	55	55	53	49	43	35	25	13	0

It is clear from the table that the value of $\dim \mathcal{M} = 25$ exceeds the value of $\dim M = 24$ for the first time at $K = 22$, i.e., for the submanifold $\mathbb{C}^{23}$ of type $(1, 22)$, and this relation is preserved during almost all of the seventh period. For the subsequent periods, only the last types form exclusions.

Here are two periods $\ell = 2$ and $\ell = 3$ for $\mathbf{n = 2}$.

K	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim M$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\dim \mathcal{M}$	0	0	0	0	4	13	20	25	28	29	28	25	20	13	4	0

The value $\dim \mathcal{M} = 13$ exceeds the value $\dim M = 10$ for the first time at $K = 6$, i.e., for the submanifold $\mathbb{C}^8$ of type $(2, 6)$.

And here are the same two periods $\ell = 2$ and $\ell = 3$ for $\mathbf{n = 3}$.

K	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\dim M$	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$\dim \mathcal{M}$	0	0	2	4	4	2	0	0	0	18	51	82	111	138	163	186	207

K	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$\dim M$	24	25	26	27	28	29	30	31	32	33	34	35	36	37
$\dim \mathcal{M}$	226	243	258	271	282	291	298	303	306	307	306	303	298	291

K	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$\dim M$	38	39	40	41	42	43	44	45	46	47	48	49	50	51
$\dim \mathcal{M}$	282	271	258	243	226	207	186	163	138	111	82	51	18	0

The value $\dim \mathcal{M} = 18$ exceeds the value $\dim M = 16$ for the first time at $K = 10$, i.e., for the submanifold $\mathbb{C}^{13}$ of type $(3, 10)$.

The quantity k_ℓ is a polynomial in n of degree not less than ℓ. Thus, it becomes clear that, for the overwhelming majority of types, the dimension of $\mathcal{M}$ is significantly larger than the dimension of the manifold M itself, which is equal to $2n + k$. This seemingly means that, in this case, the generic manifold M can be uniquely recovered (up to holomorphic equivalence) from its Gaussian image, i.e., the coincidence of these manifolds is a criterion for the holomorphic equivalence (the necessity is obvious). General position must be understood as the condition that the rank of the Gaussian mapping is maximal. For all holomorphically homogeneous manifolds, i.e., for manifolds with transitive action of the group of holomorphic symmetries, the Gaussian image is the point, i.e., the Gaussian mapping has zero rank. In particular, the model surfaces by themselves have this property.

The moduli space can be equipped with some metric, after which an invariant metric is induced on any generic manifold of the given type.

The manifolds of codimension two, i.e., submanifolds of type $(n,2)$ in the space $\mathbb{C}^{n+2}$, fall within the lowest period $\ell = 2$, beginning with $n = 2$. A complete classification of model surfaces of this kind was carried out in [5]. A model surface of this type is of the form

$$\Im w_1 = \langle z, \bar{z}\rangle_1, \qquad \Im w_2 = \langle z, \bar{z}\rangle_2,$$

where $\langle z, \bar{z}\rangle_1$ and $\langle z, \bar{z}\rangle_2$ are two linearly independent Hermitian forms in $\mathbb{C}^n$ having a trivial common kernel and (H_1, H_2) are the matrices of these forms. In this case, the moduli space $\mathcal{M}(n,2)$ is in a natural correspondence with the projective classes of the real hyperelliptic curves. The projectively nonequivalent families of n distinct points of $\mathbb{C}P^1$-characteristic values $(t_1 : t_2) \in \mathbb{C}P^1$, i.e., roots of the equation $\det(t_1 H_1 + t_2 H_2) = 0$, parametrize the space $\mathcal{M}(n,2)$. In particular, $\mathcal{M}(2,2)$ and $\mathcal{M}(3,2)$ are points. In the case of $(4,2)$, we obtain a family of four characteristic values. If ν is this double ratio (for some indexing) and if

$$j = \frac{(\nu^2 - \nu + 1)^2}{\nu^2(\nu - 1)^2}$$

is the j-invariant, then, expressing j in terms of the elements H_1 and H_2, we obtain a unique generator of the field of rational invariants and $\mathcal{M}(4,2)$ is $\mathbb{R}P^1$, i.e., the circle.

For arbitrary types (n,K) of the initial period $\ell = 2$, i.e., for $K \leqslant n^2$, a model surface is determined by a family of K Hermitian matrices of order n, $(H_1, \dots, H_K)$. In this case, the moduli space turns out to correspond to projective classes of real-algebraic hypersurfaces of degree n in $\mathbb{R}P^{K-1}$, namely,

$$\det(t_1 H_1 + \dots + t_K H_K) = 0.$$

For instance, if $n = K = 3$, then these are projective classes of real cubic curves in $\mathbb{C}P^2$. As is known [3], a generic curve of this kind can be reduced to the Weierstrass normal form

$$t_2^2 t_3 = t_1^3 + p t_1 t_3^2 + q t_3^3.$$

Thus, expressing p and q in terms of the elements (H_1, H_2, H_3), we obtain two generators of the field of invariants.

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REFERENCES

1. É. B. Vinberg and V. I. Popov, “Invariant Theory,” in: *Algebraic geometry* 4*, Itogi Nauki i Tekhniki* **55** (Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989), pp. 137–314.
2. J. Harris, *Algebraic Geometry. A First Course* (Springer, New York, 1995, 2005).
3. V. V. Prasolov and Yu. P. Solov’ev, *Elliptic Functions and Algebraic Equations* (Faktorial, Moscow, 1997); English transl. in: V. Prasolov and Yu. Solovyev, *Elliptic Functions and Elliptic Integrals*, Translations of Mathematical Monographs **170** (American Mathematical Society, Providence, 1997).
4. E. N. Shananina, “Models of CR-Manifolds of Type $(1,K)$ for $3 \leqslant K \leqslant 7$ and Their Automorphisms,” Mat. Zametki **67** (3), 452–459 (2000) [Math. Notes **67** (3–4), 382–388 (2000)].
5. S. N. Shevchenko, “Description of the Algebra of Infinitesimal Automorphisms of Quadrics of Codimension Two and Their Classification,” Mat. Zametki **55** (5), 142–153 (1994).
6. V. K. Beloshapka, “A Universal Model for a Real Submanifold,” Mat. Zametki **75** (4), 507–522 (2004) [Math. Notes **75** (3–4), 475–488 (2004)].