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Vitushkin's Germ Theorem for Engel-Type CR Manifolds

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Abstract—We study real analytic CR manifolds of CR dimension 1 and codimension 2 in the three-dimensional complex space. We prove that the germ of a holomorphic mapping between "nonspherical" manifolds can be extended along any path (this is an analog of Vitushkin's germ theorem). For a cubic model surface ("sphere"), we prove an analog of the Poincaré theorem on the mappings of spheres into $\mathbb{C}^2$. We construct an example of a compact "spherical" submanifold in a compact complex 3-space such that the germ of a mapping of the "sphere" into this submanifold cannot be extended to a certain point of the "sphere."

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1. INTRODUCTION

Any embedding of a d-dimensional real smooth manifold M in the complex N-space distinguishes a subspace $T^{\mathrm{CR}}_\xi M = T_\xi M \cap iT_\xi M$ of the tangent space $T_\xi M$ at each point ξ of the manifold. We assume that the embedding is generating, i.e., the complex hull of $T_\xi M$ coincides with the tangent space of the ambient space $\mathbb{C}^N$; in particular, the CR dimension $n = \dim_{\mathbb{C}} T^{\mathrm{CR}}_\xi M$ does not depend on ξ, and $N = d - n$. The number $k = d - 2n = N - n$ is called the CR codimension; it coincides with the codimension of $T^{\mathrm{CR}}_\xi M$ in $T_\xi M$ and the embedding codimension. We refer to the germs of CR dimension n and codimension k as germs of type (n, k). In this paper, we study CR manifolds of type $(1, 2)$, which were also considered in [4–6]. Such manifolds have the remarkable property that, although the Levi form is degenerate in the traditional sense, the generic manifolds can be treated as deformations of a model surface $C \subset \mathbb{C}^3_{(z,w_2,w_3)}$ of the form $\{\operatorname{Im} w_2 = |z|^2, \operatorname{Im} w_3 = 2\operatorname{Re} z^2\bar z\}$ (see [4]). Instead of the nondegeneracy condition, which requires that the sections of $T^{\mathrm{CR}}M$ and their commutators generate the entire TM (which is impossible because the CR dimension is small), we assume that the entire space TM can be obtained from the sections of $T^{\mathrm{CR}}M$ by adding commutators twice. Taking into account the fact that the CR dimension is 1 and the codimension is 2, it is easy to understand that the number of iterations cannot be smaller than two and the above requirement is a condition of general position. A real 4-manifold endowed with a rank-2 distribution $D \subset TM$ is called an Engel manifold [11] if the sections of D together with their commutators of the first and second order generate the entire space TM. We refer to a CR manifold of type $(1, 2)$ with an Engel distribution $D = T^{\mathrm{CR}}M$ as an Engel-type CR manifold.

For Engel-type CR manifolds, just as for nondegenerate hypersurfaces, one can construct analytic normal forms and Cartan connections by analogy with the constructions of Chern–Moser [1] and Tanaka [10] (see [5, 6]).

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Using the Chern–Moser normal form, Vitushkin [7] proved a theorem on the extension to a generic neighborhood of all germs of biholomorphic mappings between two nonspherical strictly pseudoconvex real analytic manifolds. The main result of this paper is an analog of Vitushkin's theorem for Engel-type manifolds. Its proof is based on the construction of a normal form suggested in [5]; for completeness, we cite the necessary statements from [5, 6].

2. NORMAL FORM AND CARTAN CONNECTION

Let $(z,\ w_2 = u_2 + iv_2,\ w_3 = u_3 + iv_3)$ be coordinates in the space $\mathbb{C}^3$. We say that a polynomial in the variables $(z, \bar z, w_2, \bar w_2, w_3, \bar w_3)$ is of weight m if it is multiplied by t^m under the change of variables

$$(z, w_2, w_3) \mapsto (tz, t^2 w_2, t^3 w_3), \tag{1}$$

where $t \in \mathbb{R}^*$. Such polynomials arise on the right-hand sides of the equations of manifolds of the form

$$v_2 = |z|^2 + \dots, \qquad v_3 = 2\,\mathrm{Re}\, z|z|^2 + \dots.$$

Since the change (1) acts on the left-hand side of such an equation as the multiplication by t^2 and t^3, respectively, it is convenient to consider the *degree of homogeneity* of the polynomial on the right-hand side rather than its weight m; the degree of homogeneity is defined as $m-2$ for the polynomial on the right-hand side of the equation for v_2 and as $m-3$ for the polynomial on the right-hand side of the equation for v_3.

We define the amount of analyticity for a germ of an analytic function as the maximal positive number ε such that the power series representing this germ converges in the polydisk of radius ε and the absolute value of the sum of this series is bounded by the constant $\frac{1}{\varepsilon}$ on the polydisk of radius $\frac{\varepsilon}{2}$. Let M_ξ be a germ of an analytic manifold; suppose that it is spherical of order at most m.

Theorem 1. *Any completely nondegenerate germ of a four-dimensional real analytic manifold in $\mathbb{C}^3$ can be reduced by a holomorphic transformation to the normal form*

$$v_2 = |z|^2 + N^2(z, \bar z, u_2, u_3), \qquad v_3 = 2\,\mathrm{Re}\, z^2 \bar z + N^3(z, \bar z, u_2, u_3),$$

where the power series $N^2 = \sum_{k,\ell} N^2_{k,\ell}(u_2, u_3) z^k \bar z^\ell$ satisfies the conditions

$$N^2_{k,0} = 0, \quad k \ge 0, \qquad N^2_{1,1} = 0, \qquad N^2_{2,1} = 0, \qquad N^2_{3,1}|_{u_3=0} = 0, \qquad \mathrm{Im}\, N^2_{4,2}|_{u_3=0} = 0$$

and the power series $N^3 = \sum_{k,\ell} N^3_{k,\ell}(u_2, u_3) z^k \bar z^\ell$ satisfies the conditions

$$N^3_{k,0} = 0, \quad k \ge 0, \qquad N^3_{k,1} = 0, \quad k \ge 1, \qquad \mathrm{Im}\, N^3_{4,2}|_{u_3=0} = 0.$$

Such a reduction is unique up to a transformation of the form (1), *and the amount of analyticity of both the normalizing transformation and the normalized surface is bounded below by a positive constant that depends only on the amount of analyticity of the initial surface.*

We say that an Engel-type germ M_ξ is spherical if it is equivalent to the germ of the cubic C. The sphericity order of a germ is defined as the number $m > 0$ such that (N^2_m, N^3_m) is the first nonzero component of homogeneity m in the normal form of its equation. Thus, a germ is spherical if and only if it is spherical of infinite order. The condition that the order of a germ of an analytic manifold at a given point is not smaller than a certain number is an analytic condition that determines a system of embedded analytic subsets of the initial manifold M. If the manifold is nonspherical, then all these subsets are proper subsets of M.

We define the m-measure of nonsphericity of an embedded germ as the maximum number $\delta > 0$ such that $\delta \le \|J_m N\| \le \frac{1}{\delta}$, where $\|J_m N\|$ is the norm of the m-jet of the normal form of the embedded germ, parameterized by the identity automorphism $T(1,0,0,0)$ and treated as a vector of coefficients.

For nonspherical germs, we can employ the remaining degree of freedom in the choice of normal coordinates and apply an additional reduction of the form (1) so as to obtain $\|J_m N\| = 1$. Such a normal form is said to be *special.*

Theorem 2. (a) *If M_ξ is a nonspherical germ in the normal form, then it can be reduced to the special normal form described above by a mapping of the form* (1).

(b) *Two such reductions can differ only by a change of the form $(\pm z, w_2, \pm w_3)$; i.e., two nonspherical germs M_ξ and $\widetilde{M}_{\tilde{\xi}}$ are holomorphically equivalent if and only if their right-hand sides in the special normal coordinates N and $\widetilde{N}$ satisfy at least one of the following two relations: $N(z,\bar z,u_2,u_3) = \widetilde{N}(z,\bar z,u_2,u_3)$ or $N(-z,-\bar z,u_2,-u_3) = \widetilde{N}(z,\bar z,u_2,u_3)$.*

(c) *If ε is the amount of analyticity of an initial germ M_ξ (as in Theorem* 1), *which is of spherical order at most m, and δ is its m-measure of nonsphericity, then the amount of analyticity of the germ of the mapping that reduces M_ξ to its mth-order special normal form depends only on ε and δ.*

In [6], we constructed a Cartan connection. Among the 17 components of the curvature of the connection, we selected four components such that if these four components vanish identically, then the other 13 components also vanish identically and the manifold is therefore equivalent to the model surface C. It turns out that these four principal curvature components can be interpreted geometrically in terms of the integrability of certain distributions of planes. If the equation of the manifold has the normal form

$$\operatorname{Im} w_1 = |z|^2 + A_1 \operatorname{Re} z^2\bar z^3 + A_2 \operatorname{Im} z^2\bar z^3 + \dots,$$

$$\operatorname{Im} w_2 = \operatorname{Re} z^2\bar z + B_1 \operatorname{Re} z^4\bar z + B_2 \operatorname{Re} z^2\bar z^3 + B_3 \operatorname{Im} z^2\bar z^3 + B_4 \operatorname{Re} z^5\bar z + B_5 \operatorname{Im} z^5\bar z + B_6 \operatorname{Re} z^4\bar z^2 + B_7 \operatorname{Im} z^4\bar z^2 + B_8|z|^6 + \dots,$$

where the dots stand for the terms of higher degree of homogeneity, then the principal curvature components at the origin are calculated as

$$R^x_{y2}(0) = 2B_1 - B_2, \qquad R^y_{y2}(0) = -3B_3,$$

$$R^x_{x3}(0) = 2B_1 - 5B_2, \qquad R^y_{x3}(0) = 4A_1 + 5B_4 - 2B_6 - 6B_8.$$

This gives rise to a natural analog of umbilicity, which is defined for real hypersurfaces: an Engel-type manifold is said to be umbilic at the origin if

$$B_1 = B_2 = B_3 = 0, \qquad 4A_1 + 5B_4 - 2B_6 - 6B_8 = 0.$$

Just as for hypersurfaces, the umbilicity in a neighborhood of the origin means that the manifold is spherical.

3. AN ANALOG OF THE POINCARÉ THEOREM

In 1907, Poincaré [2] proved that the germ of an invertible holomorphic mapping of a sphere in $\mathbb{C}^2$ to another sphere in $\mathbb{C}^2$ is a linear-fractional transformation, which can be extended to a biholomorphic mapping between the balls bounded by these spheres. This theorem was generalized in 1974 by Alexander [3], who proved it for spheres in $\mathbb{C}^N$ with $N \ge 2$.

A sphere is a model surface of type $(n,1)$. For the real algebraic surface $C = \{\operatorname{Im} w_2 = |z|^2, \operatorname{Im} w_3 = 2\operatorname{Re} z^2\bar z\}$, which is a model surface for manifolds of type $(1,2)$, an analog of the Poincaré theorem is valid (see Theorem 3 below).

The 5-dimensional Lie group $\operatorname{Aut} C$ of quadratic triangular transformations of the form

$$\begin{aligned}
&z \mapsto \lambda(z+p),\\
&w_2 \mapsto \lambda^2\bigl(w_2 + 2i\bar{p}z + (q_2 + i|p|^2)\bigr),\\
&w_3 \mapsto \lambda^3\bigl(w_3 + 4(\operatorname{Re} p)(w_2+q_2) + 2i(2|p|^2+\bar{p}^2)z + 2i\bar{p}z^2 + (q_3 + 2i\operatorname{Re} p^2\bar{p})\bigr),
\end{aligned} \tag{2}$$

where $\lambda \in \mathbb{R}^*$, $p \in \mathbb{C}$, $q_2 \in \mathbb{R}$, and $q_3 \in \mathbb{R}$, acts transitively on the surface C. The representation of elements of the group by four-tuples of parameters (λ, p, q_2, q_3) corresponds to the decomposition of the group into the semidirect product $\mathbb{R}^* \ltimes \mathbb{C} \ltimes \mathbb{R}^2$. We denote the corresponding transformation by $T(\lambda, p, q_2, q_3)$. The subgroup $\operatorname{Aut}_0 C$ consisting of transformations $T(1, p, q_2, q_3)$ acts on C transitively without fixed points and can be identified with C by associating each transformation $T(1, p, q_2, q_3)$ with the point $\xi = (p, q_2 + i|p|^2, q_3 + 2i\operatorname{Re} p^2\bar{p}) \in C$.

Theorem 3. *Let ξ_1 and ξ_2 be two points in C and ϕ be the germ of an invertible holomorphic mapping from the germ C_{ξ_1} to the germ C_{ξ_2}. Then, ϕ can be extended to a quadratic triangular isomorphism of $\mathbb{C}^3$ that has the form* (2) *and belongs to the group* $\operatorname{Aut} C$.

Proof. This theorem follows directly from the uniqueness of the reduction of C to the normal form [5]. Indeed, since the group $\operatorname{Aut} C$ is transitive, we may assume that ξ_1 and ξ_2 coincide with the origin. Hence, ϕ is a reduction of C to the normal form and, by Corollary 5 from [5], $\phi = T(\lambda, 0, 0, 0)$, which proves the theorem. □

4. AN ANALOG OF VITUSHKIN'S GERM THEOREM

In 1978, Pinchuk proved the following remarkable theorem, whose statement is very similar to that of the Poincaré theorem. Let Γ_1 and Γ_2 be two compact strictly pseudoconvex real analytic nonspherical hypersurfaces in $\mathbb{C}^N$. Then, the germ of any invertible holomorphic mapping from Γ_1 to Γ_2 can be extended along all paths in Γ_1. For simply connected hypersurfaces, this means that the germ can be extended to a biholomorphic equivalence of the hypersurfaces and, accordingly, of the domains bounded by them. Pinchuk's proof employed global constructions (such as the Fefferman metric) and essentially used the structure of $\mathbb{C}^N$. Somewhat later, Vitushkin proved a local version of this assertion by a delicate analysis of the power series that define the mapping. In this paper, we formulate and prove a similar assertion for CR manifolds of type $(1, 2)$.

According to Theorem 8 from [5], the amount of analyticity of the germ of a mapping that reduces M_ξ to the special normal form of order m depends only on the amount of analyticity of the initial germ and its m-measure of nonsphericity.

Theorem 4. *Let M^1 and M^2 be two nonspherical real analytic submanifolds of type* $(1, 2)$ *in three-dimensional complex manifolds X^1 and X^2, and let K^1 and K^2 be compact subsets of M^1 and M^2, respectively. Let ξ_1 and ξ_2 be points in K^1 and K^2, respectively, and ϕ be the germ of an invertible holomorphic mapping from $M^1_{\xi_1}$ to $M^2_{\xi_2}$. Then, the amount of analyticity of ϕ is a positive quantity depending only on K^1 and K^2.*

Proof. The order of sphericity of a connected real analytic manifold is either bounded on each compact subset or infinite everywhere. In the case under consideration, the former possibility is realized. Suppose that all orders are bounded by m. Then, to construct the special normal form at any point, we can use the special normal form of order m. For the compact set K^1, there exist a positive constant ε_1 that bounds the amount of analyticity of the manifold at all points of this compact set and a positive constant δ_1 that bounds the m-measure of nonsphericity at all points of the compact set K^1. These parameters also bound the amount of analyticity of the inverse mapping. Let $(\varepsilon_2, \delta_2)$ be a similar pair of constants chosen for K^2. The mapping ϕ is the composition of the reduction to the special normal form of order m and its inverse. Thus, the amount of analyticity of the germ of ϕ is a positive constant depending on the set $(\varepsilon_1, \delta_1, \varepsilon_2, \delta_2)$. The theorem is proved. □

This theorem has a number of immediate corollaries. All of them are related to four-dimensional nonspherical real analytic submanifolds of complex 3-manifolds and to locally invertible holomorphic mappings.

Corollary 5. (a) *If both submanifolds M^1 and M^2 are compact, then the amount of analyticity of the germ of any mapping from $M^1_{\xi_1}$ to $M^2_{\xi_2}$ is bounded below by a constant depending only on the pair of manifolds.*

(b) *All mappings from a compact submanifold M^1 to a compact submanifold M^2 admit holomorphic extensions to a common neighborhood of M^1.*

(c) *The global automorphism group of a compact submanifold M^1 is a compact Lie group of holomorphic transformations of M^1 in the topology of uniform convergence.*

Examples of compact submanifolds of type $(1, 2)$ are given in Section 5. Corollary 5 establishes a relationship between the local theory discussed in this paper and the global theory. Note that (c) also implies that if the automorphism group is noncompact, then the manifold is spherical. An analogous assertion is valid for compact strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$ [7]: If the automorphism group of such a surface is noncompact, then the surface is spherical. A similar result was obtained in [9].

Corollary 6. *Let ϕ be the germ of a holomorphic mapping from a submanifold M^1 to a submanifold M^2.*

(a) *If M^2 is compact, then ϕ can be extended along all paths in M^1.*

(b) *If M^2 is compact and M^1 is simply connected, then ϕ can be extended to a holomorphic locally invertible mapping from M^1 to M^2.*

(c) *If both M^1 and M^2 are compact and simply connected, then ϕ can be extended to a holomorphic equivalence between M^1 and M^2.*

The proof of the germ theorem in the preceding section does not essentially differ from that of Vitushkin's theorem, but it is technically much simpler. The main reason for this is that the automorphism group of the model surface C, which parameterizes the mappings of Engel-type CR manifolds, is much more meager than the automorphism group of a hyperquadric, which parameterizes the mappings of nondegenerate hypersurfaces.

5. COMPACT REALIZATIONS OF THE CUBIC AND AN EXAMPLE OF AN INEXTENDIBLE MAPPING

As mentioned above, the group $\operatorname{Aut} C$ of holomorphic automorphisms of C consists of transformations of the form (2), which act on the ambient space $\mathbb{C}^3$.

It is seen from (2) that if $\lambda \neq \pm 1$, then such a transformation has precisely one fixed point. Any such transformation is conjugate in the group $\operatorname{Aut} C$ to a transformation with fixed point at the origin. Such a transformation is $T(\lambda, 0, 0, 0)$. If $\lambda = -1$ and $p = q_2 = 0$, then the fixed points in the ambient space form the line $(z = 0,\ w_2,\ w_3 = -\frac{q_3}{2})$, where w_2 is an arbitrary complex number. If w_2 is real, these points belong to C. If $\lambda = 1$ or $\lambda = -1$ and $p \neq 0$ or $q_2 \neq 0$, then there are no fixed points.

Now, consider the dimensions of the orbits of the action of the 5-dimensional real Lie group $\operatorname{Aut} C$. The cubic C is a singular 4-dimensional orbit. A generic orbit is a 5-dimensional hypersurface. This can be shown as follows. Note that the action of $\operatorname{Aut} C$ on $\mathbb{C}^3$ commutes with the projection onto $\mathbb{C}^2_{(z,w_2)}$; therefore, the action of $\operatorname{Aut} C$ on $\mathbb{C}^2$ is well defined. As is known, this action has three orbits: the hypersurface $\{\operatorname{Im} w_2 = |z|^2\}$ and the two domains on both sides of it. Each orbit of the action in $\mathbb{C}^3$ must cover an orbit of the action in $\mathbb{C}^2$; the orbits that cover domains are at least 4-dimensional, but the presence of transformations of the form $T(1,0,0,q_3)$ in the group under consideration allows us to assert that the orbits have the form of a direct product in which one of the factors is the real line along w_3. Therefore, the orbits are at least 5-dimensional, and the dimension of the group prevents them from being more than 5-dimensional.

Example 1. Consider the subgroup H of $\operatorname{Aut} C$ generated by the four transformations $T(1,\omega_1,0,0)$, $T(1,\omega_2,0,0)$, $T(1,0,q_2^1,q_3^1)$, and $T(1,0,q_2^2,q_3^2)$, where the complex numbers ω_1 and ω_2 are linearly independent over $\mathbb{R}$ and the vectors (q_2^1,q_3^1) and (q_2^2,q_3^2) are independent as vectors in $\mathbb{R}^2$. This is a discrete subgroup of holomorphic transformations of $\mathbb{C}^3$ with a completely discontinuous action. The homogeneous space given by the quotient of $\mathbb{C}^3$ modulo H is a complex manifold with the topological structure of $(S^1)^4\times\mathbb{R}^2$. After the factorization, the cubic C becomes a compact spherical real 4-submanifold $\widehat{C}$ with the topological structure of the 4-torus $(S^1)^4$. The universal covering of $\widehat{C}$ is, obviously, the cubic C itself, which has the topological type of $\mathbb{R}^4$. Any biholomorphic mapping between two such manifolds (with different sets of parameters) can be lifted to an automorphism of the cubic. It is easy to show that such an automorphism ensures the linear equivalence of the lattices. Thus, almost all of them are holomorphically inequivalent.

Example 2. As the second example, consider the quotients of $\mathbb{C}^3\setminus\{0\}$ and $C\setminus\{0\}$ modulo the infinite cyclic subgroup generated by the transformation $T(\lambda,0,0,0)$, where $\lambda>1$. The expression $\tau=|z|^6+|w_2|^3+|w_3|^2$ is a conformal invariant of this group. The space $\mathbb{C}^3\setminus\{0\}$ is foliated by the level surfaces of the function τ, which are topologically equivalent to the 5-sphere S^5. Since $T(\lambda,0,0,0)(\tau)=\lambda^6\tau$, it follows that the "spherical layer" $\{1\le\tau<\lambda^6\}$ is a fundamental domain, and we obtain a compact complex manifold of the topological type $S^5\times S^1$. The cubic C can be regarded as the graph of a mapping from $\mathbb{C}\times\mathbb{R}^2$ to $\mathbb{R}^2$. The fundamental domain of the cubic after factorization is the graph over the spherical layer of the form $1\le|z|^6+|u_2|^3+|u_3|^2<\lambda^6$. Thus, the factored cubic $\widehat{C}$ has the topological type of $S^3\times S^1$, and its universal covering is a punctured cubic, i.e., $\mathbb{R}^4\setminus\{0\}$. It is easy to see that the quotient manifolds corresponding to different values of the parameter $\lambda>1$ are holomorphically inequivalent.

A "counterexample" to the germ theorem similar to the example of Burns and Shnider [8] is very simple. Consider the germ of the identity mapping at any point different from the origin, e.g., at $(0,0,1)$, as a mapping from C to $\widehat{C}$. The line joining this point with the origin is mapped to a cycle on $\widehat{C}$ that is traversed infinitely many times, which prevents the germ from being extended along this path to the origin.

Burns and Shnider realized the quotient manifold as a compact surface in the initial complex linear space, whereas the manifold which we consider here is embedded in the quotient of $\mathbb{C}^3\setminus\{0\}$ by the action of $T(\lambda,0,0,0)$.

Example 3. The last, third, example is the quotient of $C\setminus\{(z=0,\ \operatorname{Im} w_2=0,\ w_3=0)\}$ by the group $\mathbb{Z}_2$ generated by the transformation $T(-1,0,0,0)$. As a result, we obtain the *nonorientable* spherical submanifold $\mathbb{C}^3\setminus\{(z=0,\ w_2\in\mathbb{C},\ w_3=0)\}$ factored by $T(-1,0,0,0)$.

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