

# CANONICAL CARTAN CONNECTION AND HOLOMORPHIC INVARIANTS ON ENGEL CR MANIFOLDS

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## Abstract

We describe a complete system of invariants for 4-dimensional CR manifolds of CR dimension 1 and codimension 2 with Engel CR distribution by constructing an explicit canonical Cartan connection. The 4 essential invariants arising from the Cartan curvature are geometrically interpreted. We also investigate the relation between the Cartan connection and the normal form of the defining equation of an embedded Engel CR manifold.

## 0. Introduction

The equivalence problem for real hypersurfaces in 2-dimensional complex space with respect to holomorphic mappings became one of the first impressive applications of E. Cartan method of moving frames. Cartan solved this problem in 1932 (see [6]). In modern terminology his approach can be described as follows: Let  $M$  be a manifold with some additional geometric data (in our case the data come from the embedding of a real manifold into a complex manifold). Construct a canonical principal bundle  $\mathcal{G}$  over  $M$  and a distinguished frame field on  $\mathcal{G}$  which depend only on the specified geometric data. This means that for two isomorphic structures  $M_1$  and  $M_2$  a mapping that establishes the isomorphism lifts to a unique mapping of the canonical bundles  $\mathcal{G}_1$  and  $\mathcal{G}_2$  that maps the distinguished frames at  $\mathcal{G}_1$  to the distinguished frames at  $\mathcal{G}_2$ . Hence, the initial equivalence problem reduces to an equivalence problem for manifolds with distinguished frames. Such structure is called *absolute parallelism* and its geometry is well-understood (see, e.g., [16]). Cartan connections were constructed for nondegenerate CR-manifolds of hypersurface type (see [23], [10]), for 2-codimensional CR-manifolds of CR-dimension 2 of hyperbolic and elliptic types ([14], [21], [22] [5], [24]). For 5-dimensional uniformly degenerate CR-manifolds

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of hypersurface type a parallelism (with respect to a subgroup of the structure group) was obtained in [11] (see also [12]).

A Cartan connection is a pair of a canonical bundle and a distinguished frame field that satisfies certain invariance conditions (see Section 3 for the precise definitions). Those conditions are very useful in the study of automorphisms of manifolds with given geometric structure.

In this paper we will describe a Cartan connection for some class of CR manifolds that are related to third order ODE.

**Definition 1.** *A CR manifold is a smooth manifold  $M$  equipped with a distribution  $D \subset TM$  of even rank  $2n$  and a smooth field  $J$  of endomorphisms  $J_a : D_a \rightarrow D_a$  with  $J_a^2 = -\text{id}$ . A CR manifold is supposed to satisfy the following integrability condition: The complex distribution spanned by  $X - iJX$  for local sections  $X$  of  $D$  is involutive.*

*The number  $n$  is called CR dimension of  $M$  and  $k = \dim M - 2n$  is called the codimension of  $M$ .*

CR manifolds appear naturally as embedded real submanifolds of complex manifolds. The distribution  $D$  is then defined as  $D = TM \cap iTM$  and  $J$  is the restriction of the complex structure in the ambient complex manifold to  $D$ .

S.S. Chern and J. Moser [10] and N. Tanaka [23] constructed canonical Cartan connections for Levi-nondegenerate real hypersurfaces (1-codimensional CR manifolds) of arbitrary dimension. Similar results for so-called hyperbolic and elliptic CR manifolds of codimension two in  $\mathbb{C}^4$  were obtained by G. Schmalz and J. Slovák [21, 22] (see also [14, 5, 24]). These geometries fit into the general concept of *parabolic geometry* introduced by Ch. Fefferman [15] (see also [25], [4]). This is due to the following features they share:

- (i) **Nondegeneracy.** The algebraic bracket (see Subsection 7.1)

$$\mathcal{L}_a : D_a \otimes D_a \rightarrow T_a/D_a$$

that assigns to a pair of vectors in  $D_a$  the projection to  $T_a/D_a$  of the commutator of two local sections of  $D$  that extend the two vectors is called *Levi form*. A CR manifold is *Levi nondegenerate* if the image of the Levi form coincides with  $T_a/D_a$  and if the null-space of the Levi form is trivial.

- (ii) **Uniformity.** A CR manifold is called *strongly uniform* (cp. [18]) if the Levi forms at different base points are equivalent with respect to linear mappings.
- (iii) **Semi-simplicity of the structure group.** At any point  $a \in M$  the Levi form  $\mathcal{L}_a$  can be used as a product of a graded Lie algebra  $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = D_a \oplus T_a/D_a$ . The structure group  $G$  is the group of automorphisms of  $\mathfrak{g}_-$  that preserve  $J_a$  at  $D_a$ . For strongly uniform CR manifolds the structure groups at different points are isomorphic.

The above nondegeneracy condition turns out to be too strong and it immediately excludes CR manifolds whose codimension  $k$  is bigger than  $n^2$ .

In this paper we will study CR manifolds of codimension  $k = 2$  and CR dimension  $n = 1$  which is the simplest case when  $k > n^2$ . Instead of the nondegeneracy condition we will assume a weaker condition introduced by V. Beloshapka in [2] that is based on the following observation: For given CR-dimension  $n$  the dimension of the subspace of  $T_aM$  that is generated by brackets of local sections of  $D$  of order  $\leq r$  have precise upper bounds  $k_{n,r}$ . The generalized non-degeneracy requires that these upper bounds are being attained until the complete tangent space is generated.

**Definition 2.** *Let  $M$  be a CR-manifold of CR-dimension  $n$  and codimension  $k \geq n^2$ . By  $D_a^{(r)}$  denote the subspace of  $T_aM$  generated by brackets of local sections of  $D$  of order  $\leq r$ . Then  $M$  is called non-degenerate at  $a$  if there exists  $r_0$  such that  $T_a = D_a^{(r_0)}$  and  $\dim D^{(r)} = k_{n,r}$  for  $r < r_0$ .*

For a 4-dimensional manifold with a rank 2 distribution  $D$  this means that  $TM$  is spanned by  $D$  and commutators of first and second orders of local sections of  $D$ . Such manifolds are called *Engel manifolds* (cp. [13]). In Section 1 we show that Engel CR manifolds are strongly uniform in an appropriate sense but their structure group is not semisimple. It follows that the corresponding geometry is not parabolic.

Nevertheless, as the main result of this paper we obtain the following

**Theorem 1.** *Engel CR manifolds admit a canonical Cartan connection. In particular, there are four (out of 30) components of the Cartan curvature that constitute a complete system of CR-invariants.*

We prove this theorem in Section 4 by an explicit and transparent construction. We will derive a complete system of invariants that consists of four 1-dimensional components. A geometric interpretation of these invariants will be given in Section 5.

In Section 6 we investigate the relation between the invariants from the Cartan connection and a normal form earlier constructed by the authors [3].

## 1. Engel manifolds and their Levi-Tanaka algebras

Let  $M$  be an Engel manifold. Then we have a canonical flag of distributions  $D \subset D' \subset D'' = TM$ .

For two local sections  $X, Y$  of  $D$  defined in a neighbourhood of  $a \in M$  consider their bracket at  $a$  followed the natural projection

$$\pi : T_aM \rightarrow T_aM/D_a.$$

The result takes values in  $D'_a/D_a$  and depends only on  $X(a)$  and  $Y(a)$  (see Subsection 7.1). Thus, we obtain an algebraic bracket

$$\mathcal{L}_a^1 : D_a \times D_a \rightarrow D'_a/D_a.$$

Analogously, we define a bilinear form

$$\tilde{\mathcal{L}}_a^1 : D_a \times D'_a \rightarrow T_a M/D'_a$$

as the commutator of vector fields followed by the natural projection. Since  $\tilde{\mathcal{L}}_a^1(D_a, D_a) = 0$  this bracket lifts to a bracket

$$\mathcal{L}_a^2 : D_a \times D'_a/D_a \rightarrow T_a M/D'_a.$$

For fixed  $a$  denote  $D_a$  by  $\mathfrak{g}_{-1}$ ,  $D'_a/D_a$  by  $\mathfrak{g}_{-2}$ , and  $T_a M/D'_a$  by  $\mathfrak{g}_{-3}$ . Then

$$\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$$

forms a graded nilpotent Lie algebra whose product is defined by  $\mathcal{L}_a^1$  and  $\mathcal{L}_a^2$ . This Lie algebra  $\mathfrak{g}_-$  is called the *Levi-Tanaka algebra* of  $M$  at  $a$ .

It is clear that the Levi-Tanaka algebras at different points of  $M$  are isomorphic. In this sense Engel manifolds are strongly uniform.

It is well-known [13] that Engel manifolds admit normal coordinates  $x, y, p, q$  such that

$$D = \text{Ann}(dp - q dx, dy - p dx).$$

Thus any two Engel manifolds are locally isomorphic.

Since  $\mathfrak{g}_{-2}$  and  $\mathfrak{g}_{-3}$  are 1-dimensional, there is a unique direction  $Y_a$  in  $D_a = \mathfrak{g}_{-1}$  at each point  $a$  such that  $\mathcal{L}_a^2(Y_a, \mathfrak{g}_{-2}) \equiv 0$ . Thus, there is a canonical line bundle  $D^0$  in  $D$  that consists of the vectors that annihilate  $\mathcal{L}_a^2$  (in the first argument).

In normal coordinates  $D^0$  is spanned by  $Y = \frac{\partial}{\partial q}$  and  $D$  is spanned by  $Y$  and an additional direction field of the form

$$X = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p} + B \frac{\partial}{\partial q},$$

where  $B$  is a function of  $x, y, p, q$ .

A choice of a particular direction field  $X$  (i.e., of  $B$ ) is an additional structure on  $M$ . An Engel manifold with direction field  $X$  can be interpreted as a third order ODE. In fact, the flow of  $\xi$  consists of curves  $x(t), y(t), p(t), q(t)$  with  $p = \frac{dy}{dx}$ ,  $q = \frac{dp}{dx}$  and  $\frac{dq}{dx} = B(x, y, p, q)$ . Therefore,  $y'''(x) = B(x, y, y', y'')$ .

The geometry of Engel manifolds with direction field is equivalent to the geometry of third order ODE with respect to contact transformations and has been studied by E. Cartan [7], S.S. Chern [9], H. Sato and A.Y. Yoshikawa [20] and by S. Neut [19]. H. Sato and A.Y. Yoshikawa showed that this geometry fits into the scheme of parabolic geometry

with structure group  $\mathrm{Sp}(4, \mathbb{R})$  (the group of contact automorphisms of  $y''' = 0$ ).

Moreover, Engel manifolds with direction field are examples of multi-contact structures as introduced by M. Cowling, F. de Mari, A. Korányi and H.R. Reimann in [8], see also [17].

Now let  $M$  be an Engel CR manifold, i.e. an Engel manifold with a smooth endomorphism field  $J_a : D_a \rightarrow D_a$  such that  $J_a^2 = -\mathrm{id}$ . Note that the so-defined CR structure is automatically integrable because the CR dimension is 1. Now, let  $Y$  be a smooth local section of  $D^0$ . Then  $X = JY$  represents a complementary direction field to  $Y$  in  $D$ . Thus, the CR structure encodes a third order ODE. Observe that the CR structure cannot be completely recovered from the ODE since  $J$  couples the scale of  $X$  with a choice of a scale of  $Y$ .

## 2. Prolongation of the Levi-Tanaka algebra and the homogeneous model

It is convenient to fix a basis  $V_x, V_y, V_2, V_3$  of the Levi-Tanaka algebra  $\mathfrak{g}_-$  such that  $V_y \in \mathfrak{g}_{-1}$  and

$$(1) \quad \begin{aligned} V_x &= JV_y \in \mathfrak{g}_{-1} & V_2 &= [V_x, V_y] \in \mathfrak{g}_{-2} \\ V_3 &= [V_x, V_2] \in \mathfrak{g}_{-3} & [V_y, V_2] &= 0. \end{aligned}$$

$V_y$ , and hence  $V_x, V_2, V_3$  are determined up to scale by the above conditions.

Let  $\hat{G}_0$  be the group of automorphisms of the graded Lie algebra  $\mathfrak{g}_-$  and let  $\hat{\mathfrak{g}}_0$  be the Lie algebra of  $\hat{G}_0$ . Then the first step of the prolongation procedure is a choice of a subalgebra  $\mathfrak{g}_0 \subset \hat{\mathfrak{g}}_0$ .

The algebra  $\hat{\mathfrak{g}}_0$  can be identified with the Lie algebra of all endomorphisms  $C$  of  $\mathfrak{g}_{-1}$  that preserve the distinguished direction  $V_y$ . Such an endomorphism  $C$  acts on  $\mathfrak{g}_{-2}$  by multiplication with  $\mathrm{tr} C$  and on  $\mathfrak{g}_{-3}$  by multiplication with  $2 \mathrm{tr} C - \lambda_Y$  where  $\lambda_Y$  is the eigenvalue of  $C$  to the eigenvector  $V_y$ .

For Engel CR geometry  $C \in \mathfrak{g}_0$  has to preserve the direction  $JV_y = V_x$  with the same eigenvalue as  $V_y$ , hence  $\mathfrak{g}_0 = \mathbb{R}$  consists of endomorphisms  $\lambda \mathrm{id}$ .

The components  $\mathfrak{g}_\ell$  with  $\ell > 0$  are recursively determined as linear mappings from  $\mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{\ell-1}$  that extend to derivations on  $\mathfrak{g}_-$ . A simple computation shows that  $\mathfrak{g}_1$  and consequently  $\mathfrak{g}_\ell$  with  $\ell > 1$  are trivial.

We remark that a different choice of  $\mathfrak{g}_0$  leads to different geometries: for  $\mathfrak{g}_0 = \hat{\mathfrak{g}}_0$  the prolongation will be infinite and it corresponds to infinitesimal automorphisms of Engel manifolds without additional structure. If we choose  $\mathfrak{g}_0$  as the Lie algebra of endomorphisms of  $\mathfrak{g}_{-1}$  that preserve  $V_y$  and an additional vector  $V_x$  (but with possibly different eigenvalues)

the resulting prolongation is  $\mathfrak{sp}(4, \mathbb{R})$  which corresponds to contact geometry of 3rd order ODE. Pointwise geometry of 3rd order ODE results from a non-maximal prolongation with the same choice of  $\mathfrak{g}_0$ .

The homogeneous model for the construction of the Cartan connection can be obtained by exponentiating the Levi-Tanaka algebra  $\mathfrak{g}_-$ . The (simply connected) Lie group  $G_-$  of  $\mathfrak{g}_-$  has the structure of an Engel CR manifold. Let  $(x, y, u_2, u_3)$  be coordinates with respect to the basis  $V_x, V_y, V_2, V_3$  in  $\mathfrak{g}_-$  and in  $G_- = \exp \mathfrak{g}_-$ . Then the Lie algebra product, defined by the Baker-Campbell-Hausdorff formula is

$$(x, y, u_2, u_3) \circ (\alpha, \beta, \mu_2, \mu_3) = (x + \alpha, y + \beta, u_2 + \mu_2 + \frac{1}{2}(x\beta - y\alpha), u_3 + \mu_3 + \frac{1}{2}(x\mu_2 - u_2\alpha) + \frac{1}{12}(x - \alpha)(x\beta - y\alpha)).$$

The left-invariant vector fields are

$$\begin{aligned} \hat{V}_x &= d_0 L_g V_x = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial u_2} - \left( \frac{u_2}{2} + \frac{xy}{12} \right) \frac{\partial}{\partial u_3} \\ \hat{V}_y &= d_0 L_g V_y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial u_2} + \frac{x^2}{12} \frac{\partial}{\partial u_3} \\ \hat{V}_2 &= d_0 L_g V_2 = \frac{\partial}{\partial u_2} + \frac{x}{2} \frac{\partial}{\partial u_3} \\ \hat{V}_3 &= d_0 L_g V_3 = \frac{\partial}{\partial u_3}. \end{aligned}$$

The distribution  $D$  at  $g \in G_-$  is spanned by  $\hat{V}_x, \hat{V}_y$  and the complex structure on  $D$  is defined by  $J\hat{V}_y = \hat{V}_x$  and (consequently)  $J\hat{V}_x = -\hat{V}_y$ .

The homogeneous CR manifold  $G_-$  can be embedded into  $\mathbb{C}^3$  with coordinate functions  $(z, w_2, w_3)$  which satisfy the equation

$$\begin{aligned} (\hat{V}_y + i\hat{V}_x)f &= \\ \left( \frac{\partial}{\partial y} + i \frac{\partial}{\partial x} + \frac{x - iy}{2} \frac{\partial}{\partial u_2} + \left( -\frac{i u_2}{2} + \frac{x(x - iy)}{12} \right) \frac{\partial}{\partial u_3} \right) f &= 0. \end{aligned}$$

An obvious solution is  $z = \frac{x - iy}{2}$ . After introducing  $z$  as a complex coordinate the equation takes the form

$$\left( \frac{\partial}{\partial \bar{z}} + iz \frac{\partial}{\partial u_2} + \left( \frac{u_2}{2} + \frac{iz(z + \bar{z})}{6} \right) \frac{\partial}{\partial u_3} \right) f = 0.$$

We find the solutions  $w_2 = u_2 - i|z|^2$  and  $w_3 = u_3 - \frac{i(z^2 \bar{z} - z \bar{z}^2)}{6} - \frac{u_2 \bar{z}}{2}$ .

The equation of the resulting embedding is

$$\operatorname{Im} w_2 = -|z|^2, \quad \operatorname{Im} w_3 = \frac{u_2 \operatorname{Im} z}{2} = \frac{\operatorname{Im}(w_2 z)}{2} + \frac{\operatorname{Re}(z|z|^2)}{2}.$$

This is clearly equivalent to the standard cubic that was introduced by Beloshapka in [1]:

$$Q : \quad \text{Im } w_2 = |z|^2, \quad \text{Im } w_3 = \text{Re } z |z|^2.$$

The Tanaka prolongation  $\mathfrak{g}$  of  $\mathfrak{g}_-$  can be interpreted as the Lie algebra of the group  $G$  of all CR automorphisms of  $Q$ . In fact,  $G = G_- \times \mathbb{R}^*$  where  $G_-$  acts transitively on  $Q = G_-$  by right multiplication and  $\mathbb{R}^*$  acts by

$$z \rightarrow tz, \quad w_2 \rightarrow t^2 w_2, \quad w_3 \rightarrow t^3 w_3.$$

Notice that these CR automorphisms extend holomorphically to  $\mathbb{C}^3$ . The same is true for the CR automorphisms that correspond to  $G_-$ : The CR automorphism that corresponds to the right multiplication by  $(p, q_2, q_3) \in Q = G_-$  extends to

$$\begin{pmatrix} z \\ w_2 \\ w_3 \end{pmatrix} \mapsto \begin{pmatrix} z + p \\ w_2 + q_2 + 2i z \bar{p} \\ w_3 + q_3 + i(2|p|^2 + \bar{p}^2)z + i\bar{p}z^2 + (p + \bar{p})w_2 \end{pmatrix}.$$

The structure group  $G$  has a matrix representation by  $5 \times 5$  matrices of the form

$$g = \begin{pmatrix} t^3 & t^2(z + \bar{z}) & i t^2 \bar{z} & i t(2|z|^2 + \bar{z}^2) & w_3 \\ 0 & t^2 & 0 & 2i t \bar{z} & w_2 \\ 0 & 0 & t^2 & 2tz & z^2 \\ 0 & 0 & 0 & t & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The Maurer-Cartan form  $\omega = g^{-1}dg$  of the Lie group  $G$  in matrix notation is

$$\omega = \begin{pmatrix} \frac{3dt}{t} & \frac{dz+d\bar{z}}{t} & \frac{id\bar{z}}{t} & 0 & \frac{du_3 - zdu_2 - \bar{z}du_2 - \frac{1}{2}z^2d\bar{z} + \frac{1}{2}\bar{z}^2dz}{t^3} \\ 0 & \frac{2dt}{t} & 0 & \frac{2id\bar{z}}{t} & \frac{du_2 + izd\bar{z} - i\bar{z}dz}{t^2} \\ 0 & 0 & \frac{2dt}{t} & \frac{2dz}{t} & 0 \\ 0 & 0 & 0 & \frac{dt}{t} & \frac{dz}{t} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $u_2 = \text{Re } w_2$  and  $u_3 = \text{Re } w_3$ . The forms

$$\begin{aligned} \Phi_0 &= \frac{dt}{t} & \Phi_x &= \frac{2dx}{t} & \Phi_y &= -\frac{2dy}{t} \\ \Phi_2 &= \frac{du_1 + izd\bar{z} - i\bar{z}dz}{t^2} & &= \frac{du_1 - 2ydx + 2xdy}{t^2} \\ \Phi_3 &= \frac{du_2 - zdu_1 - \bar{z}du_1 - \frac{1}{2}z^2d\bar{z} + \frac{1}{2}\bar{z}^2dz}{t^3} \\ &= \frac{du_2 - 2xdu_1 + 2xydx - (x^2 - y^2)dy}{t^3} \end{aligned}$$

constitute a basis of left invariant 1-forms on  $G$ . The structure equation  $d\omega + \frac{1}{2}[\omega, \omega] = 0$ , or direct verification, yields the structure formulae

$$\begin{aligned} d\Phi_0 &= 0, & d\Phi_x &= \Phi_x \wedge \Phi_0, & d\Phi_y &= \Phi_y \wedge \Phi_0, \\ d\Phi_2 &= 2\Phi_2 \wedge \Phi_0 - \Phi_x \wedge \Phi_y, & d\Phi_3 &= 3\Phi_3 \wedge \Phi_0 - \Phi_x \wedge \Phi_2. \end{aligned}$$

### 3. Preliminaries on Cartan connections

Our aim is to construct a canonical Cartan connection for Engel CR manifolds based on the cubic model. This means that for a general Engel CR manifold  $M$  we have to find “curved” analogues of the principal  $\mathbb{R}^*$ -bundle  $G \rightarrow G/H = C$  and the  $\mathfrak{g}$ -valued Maurer-Cartan form. More precisely, we will construct a principal  $\mathbb{R}^*$ -bundle  $\mathcal{G} \rightarrow M$  and a  $\mathfrak{g}$ -valued 1-form  $\omega : T\mathcal{G} \rightarrow \mathfrak{g}$  that satisfy the properties

- (i) for any  $p \in \mathcal{G}$  the mapping  $\omega_p : T_p\mathcal{G} \rightarrow \mathfrak{g}$  is isomorphic
- (ii) the fundamental vector field of the principal action is mapped to  $V_0$
- (iii)  $R_h^*\omega = \text{Ad}(h^{-1})\omega$  for any  $h \in H$

The curvature tensor of the Cartan connection is defined as

$$K = d\omega + \frac{1}{2}[\omega, \omega].$$

Differentiation of (iii) implies

$$\omega([X, Y]) = [\omega(X), \omega(Y)]$$

when  $X$  is the fundamental vector field of the principal action. This means that  $K$  is a horizontal form, i.e. it vanishes if one of the arguments is vertical.

The curvature function on  $\mathcal{G}$  is defined by

$$\kappa(p)(X, Y) = K(\omega|_p^{-1}X, \omega|_p^{-1}Y).$$

It that takes values in the 2-cochains  $\mathcal{C}^2(\mathfrak{g}_-, \mathfrak{g})$ . The  $\mathfrak{g}$ -module  $\mathcal{C}^2(\mathfrak{g}_-, \mathfrak{g})$  splits into the subspaces of homogeneity  $i = -1, \dots, 5$ . Let  $\partial$  be the cochain operator. Then the  $\partial\kappa$  can be expressed by the so-called Bianchi identity (see [4])

$$\partial\kappa(X, Y, Z) = - \sum_{cycl} \kappa(\kappa_-(X, Y), Z) + \omega^{-1}(X)\kappa(Y, Z),$$

where  $X, Y, Z \in \mathfrak{g}_-$ ,  $\kappa_-$  is the  $\mathfrak{g}_-$ -part of  $\kappa$  and the sum runs over all cyclic permutations of  $(X, Y, Z)$ . It follows that the component  $\partial\kappa^{(i)}$  of homogeneity  $i$  can be expressed by lower components and therefore the components of degree  $i$  are determined by the lower components up to  $\ker \partial = \mathcal{Z}^2(\mathfrak{g}_-, \mathfrak{g})$ . Thus, the “essential” part of the curvature takes values in  $\mathcal{Z}^2(\mathfrak{g}_-, \mathfrak{g})$ .



We will see below that there is a unique Cartan connection whose  $\partial$ -exact part (with respect to some complementary subspace) of the curvature function vanishes. Our choice of this complementary subspace is based on the fact that the Lie algebra  $\mathfrak{g}$  splits into 1-dimensional subspaces. This induces a splitting of  $\mathcal{C}^2(\mathfrak{g}_-, \mathfrak{g})$  into 1-dimensional components. We choose a basis for the complement that involves a minimal number of those 1-dimensional components. This leads to the simplest expressions for the curvatures (see Subsection 7.2 for the choice adopted in our construction).

Since we have fixed a basis  $(V_0, V_x, V_y, V_2, V_3)$  of  $\mathfrak{g}$ , the connection form  $\hat{\Phi}$  amounts to a distinguished coframe  $\hat{\Phi}^0, \hat{\Phi}^x, \hat{\Phi}^y, \hat{\Phi}^2, \hat{\Phi}^3$  on  $\mathcal{G}$  or to its dual frame  $\hat{V}_0, \hat{V}_x, \hat{V}_y, \hat{V}_2, \hat{V}_3$ . Notice that  $\hat{V}_0 = \hat{\Phi}^{-1}(V_0)$ ,  $\hat{V}_x = \hat{\Phi}^{-1}(V_x)$ ,  $\hat{V}_y = \hat{\Phi}^{-1}(V_y)$ ,  $\hat{V}_2 = \hat{\Phi}^{-1}(V_2)$ ,  $\hat{V}_3 = \hat{\Phi}^{-1}(V_3)$ .

The curvature components

$$K_{\beta\gamma}^\alpha := K^\alpha(\hat{V}_\beta, \hat{V}_\gamma)$$

equal  $-2k_{\beta\gamma}^\alpha$ , with

$$k_{\beta\gamma}^\alpha = \hat{\Phi}^\alpha([\hat{V}_\beta, \hat{V}_\gamma]) - \Phi^\alpha[V_\beta, V_\gamma]$$

where  $\Phi^\alpha[V_\beta, V_\gamma]$  are the structure constants of  $\mathfrak{g}$ . This shows that the curvature can be interpreted as the aberration of the commutator relations between the frame vector fields from the commutator relations between their images under  $\hat{\Phi}$  in  $\mathfrak{g}$ .

#### 4. Cartan Connection of Engel CR manifolds (Proof of Theorem 1)

First, we define the Cartan bundle  $\mathcal{G} := D^0$ , i.e.  $\mathcal{G}$  is the line bundle of all distinguished vectors  $tY$ . It is clearly a  $\mathbb{R}^*$ -principal bundle. Notice that

$$\mathcal{G} \oplus J\mathcal{G} = D \subset D' \subset TM.$$

Now we construct a canonical frame field on  $T\mathcal{G}$ . First of all we have the fundamental vector field  $\hat{V}_0$  that is induced from the principal action of  $R^*$ . Consider the two mappings  $\tilde{V}_x, \tilde{V}_y : \mathcal{G} \rightarrow TM$ , defined by

$$\tilde{V}_x(Y) = JY, \quad \tilde{V}_y(Y) = Y$$

If a connection form  $\hat{\Phi}^0$  at  $\mathcal{G}$  with  $\hat{\Phi}^0(\hat{V}_0) = 1$  is given, these mappings lift to vector fields on  $\mathcal{G}$

$$\begin{aligned} \hat{V}_x &= \tilde{V}_x + a_{x0}\hat{V}_0 \\ \hat{V}_y &= \tilde{V}_y + a_{y0}\hat{V}_0 \end{aligned}$$

by the condition  $\hat{\Phi}^0(\hat{V}_\alpha) = 0$ . We try to determine  $\hat{\Phi}^0$  and the missing vector fields  $\hat{V}_2, \hat{V}_3$  by making the commutator bracket relations as “close

as possible" to the corresponding relations for the cubic model, i.e. we annihilate as many curvature components as possible.

For computation we fix a local section  $Y$  on  $\mathcal{G}$  and the frame  $T_x = JY$ ,  $T_y = Y$ ,  $T_2 = [T_x, T_y]$ ,  $T_3 = [T_x, T_2]$  on  $M$ . Denote by  $\varphi^x, \varphi^y, \varphi^2, \varphi^3$  the dual coframe. We may choose the scale of  $T_y$  in such a way that

$$[T_y, T_2] \in D.$$

In fact,

$$[\tau T_y, [\tau T_x, \tau T_y]] \equiv \tau^3 [T_y, T_2] + 3\tau^2 (T_y \tau) T_2 \quad \text{mod } D.$$

Thus,  $\tau$  has to satisfy a differential equation  $T_y(\log \tau) = \frac{1}{3}\varphi^2([T_y, T_2])$ .

This choice and the Jacobi identity imply

$$\begin{aligned} [T_x, T_y] &= T_2 \\ [T_x, T_2] &= T_3 \\ [T_y, T_2] &= \phi_{y2}^x T_x + \phi_{y2}^y T_y \\ [T_x, T_3] &= \phi_{x3}^x T_x + \phi_{x3}^y T_y + \phi_{x3}^2 T_2 + \phi_{x3}^3 T_3 \\ [T_y, T_3] &= [T_x, [T_y, T_2]] = (T_x \phi_{y2}^x) T_x + (T_x \phi_{y2}^y) T_y + \phi_{y2}^y T_2 \\ [T_2, T_3] &= [[T_x, T_3], T_y] + [T_x, [T_y, T_3]] = \\ &\quad (-T_y \phi_{x3}^x + T_x^2 \phi_{y2}^x - \phi_{y2}^x \phi_{x3}^2 - \phi_{x3}^3 T_x \phi_{y2}^x) T_x + \\ &\quad + (-T_y \phi_{x3}^y + T_x^2 \phi_{y2}^y - \phi_{y2}^y \phi_{x3}^2 - \phi_{x3}^3 T_x \phi_{y2}^y) T_y + \\ &\quad + (\phi_{x3}^x + 2T_x \phi_{y2}^y - T_y \phi_{x3}^2 - \phi_{x3}^3 \phi_{y2}^y) T_2 + \\ &\quad + (\phi_{y2}^y - T_y \phi_{x3}^3) T_3 \end{aligned}$$

Here  $\phi_{\beta\gamma}^\alpha = \varphi^\alpha([T_\beta, T_\gamma])$ . Further consequences of the Jacobi identity which will be used in calculations are

$$(2) \quad \begin{aligned} 0 &= [T_x, [T_2, T_3]] + [T_2, [T_3, T_x]] \\ 0 &= [T_y, [T_2, T_3]] + [T_2, [T_3, T_y]] + [T_3, [T_y, T_2]]. \end{aligned}$$

By  $t$  we denote the fibre coordinate in  $\mathcal{G}$  with respect to the trivialization induced by the fixed section  $Y$ , i.e. for  $W \in \mathcal{G}$  the coordinate  $t(W) = \varphi^y(W)$ . Then

$$\begin{aligned} \hat{V}_0 &= t \frac{\partial}{\partial t} \\ \hat{V}_x &= t T_x + a_{x0} \hat{V}_0 \\ \hat{V}_y &= t T_y + a_{y0} \hat{V}_0 \\ \hat{V}_2 &= a_{22} T_2 + a_{2x} T_x + a_{2y} T_y + a_{20} \hat{V}_0 \\ \hat{V}_3 &= a_{33} T_3 + a_{32} T_2 + a_{3x} T_x + a_{3y} T_y + a_{30} \hat{V}_0 \end{aligned}$$

The dual coframe is then

$$\begin{aligned}\hat{\Phi}^0 &= \frac{dt}{t} + b_{0x}\varphi_x + b_{0y}\varphi_y + b_{02}\varphi_2 + b_{03}\varphi_3 \\ \hat{\Phi}^x &= \frac{1}{t}\varphi_x + b_{x2}\varphi_2 + b_{x3}\varphi_3 \\ \hat{\Phi}^y &= \frac{1}{t}\varphi_y + b_{y2}\varphi_2 + b_{y3}\varphi_3 \\ \hat{\Phi}^2 &= b_{22}\varphi_2 + b_{23}\varphi_3 \\ \hat{\Phi}^3 &= b_{33}\varphi_3\end{aligned}$$

with  $b_{33} = \frac{1}{a_{33}}$ ,  $b_{22} = \frac{1}{a_{22}}$ , etc.

The invariance condition (iii) of a Cartan connection means

$$[\hat{V}_0, \hat{V}_j] = -|j| \hat{V}_j.$$

Here  $|j| = j$  for  $j = 0, 2, 3$  and  $|j| = 1$  for  $j = x, y$ . It follows that  $a_{j*}(m, t) = t^{|j|}\alpha_{j*}(m)$  and  $b_{j*}(m, t) = t^{-|j|}\beta_{j*}(m)$ . In particular,

$$\hat{\Phi}_0 = \frac{dt}{t} + \varpi$$

where  $\varpi$  is a 1-form on  $M$ . (More precisely,  $\varpi$  is the pullback of a 1-form at  $M$ .)

Now we will investigate all curvature components. By our construction all components of homogeneity  $-1$  and  $0$  vanish automatically:

$$\begin{aligned}k_{xy}^3 &= \hat{\Phi}^3([\hat{V}_x, \hat{V}_y]) = 0 \\ k_{y2}^3 &= \hat{\Phi}^3([\hat{V}_y, \hat{V}_2]) = 0\end{aligned}$$

By setting  $a_{22} = t^2$  and  $a_{33} = t^3$  we annihilate

$$\begin{aligned}k_{xy}^2 &= \hat{\Phi}^2([\hat{V}_x, \hat{V}_y] - \hat{V}_2) = b_{22}(t^2 - a_{22}) \\ k_{x2}^3 &= \hat{\Phi}^3([\hat{V}_x, \hat{V}_2] - \hat{V}_3) = b_{33}(t^3 - a_{33}).\end{aligned}$$

Next we consider the components of homogeneity 1. The subspace of closed cochains is 5-dimensional and coincides with the subspace of coboundaries (see Subsection 7.2 for the explicit calculation of the

cohomologies). Thus vanishing of curvature of homogeneity 1 yields

$$\begin{aligned}
k_{xy}^y &= \hat{\Phi}^y([\hat{V}_x, \hat{V}_y] - \hat{V}_2) = a_{x0} - \frac{a_{2y}}{t} \\
k_{xy}^x &= \hat{\Phi}^x([\hat{V}_x, \hat{V}_y] - \hat{V}_2) = -a_{y0} - \frac{a_{2x}}{t} \\
\hat{\Phi}^2([\hat{V}_x, \hat{V}_2] - \hat{V}_3) &= 2a_{x0} + \frac{a_{2y}}{t} - \frac{a_{32}}{t^2} \\
\hat{\Phi}^2([\hat{V}_y, \hat{V}_2]) &= 2a_{y0} - \frac{a_{2x}}{t} \\
\hat{\Phi}^3([\hat{V}_x, \hat{V}_3]) &= 3a_{x0} + \frac{a_{32}}{t^2} + t\phi_{x3}^3 \\
\hat{\Phi}^3([\hat{V}_y, \hat{V}_3]) &= 3a_{y0}
\end{aligned}$$

We find

$$\begin{aligned}
a_{32} &= -\frac{t^3}{2}\phi_{x3}^3, \\
a_{2y} &= -\frac{t^2}{6}\phi_{x3}^3, \\
a_{x0} &= -\frac{t}{6}\phi_{x3}^3, \\
a_{2x} &= 0, \\
a_{y0} &= 0
\end{aligned}$$

Due to the Bianchi identity the curvature of homogeneity 2 takes values in  $\mathcal{Z}^{(2)}$  which has dimension 6. It splits into an exact and a non-exact component. Below, we used a bar to separate the exact part at

the left hand side from the non-exact part at the right hand side.

$$\begin{aligned}
 k_{xy}^0 &= \left| \frac{1}{6}t^2T_y\phi_{x3}^3 - a_{20} \right| \\
 k_{x2}^x &= \left| -a_{20} - \frac{1}{t}a_{3x} \right| \\
 k_{x2}^y &= \left| \frac{1}{18}t^2(\phi_{x3}^3)^2 - \frac{1}{6}t^2T_x\phi_{x3}^3 - \frac{1}{t}a_{3y} \right| \\
 k_{y2}^x &= \left| t^2\phi_{y2}^x \right| \\
 k_{y2}^y &= \left| \frac{1}{6}t^2T_y\phi_{x3}^3 - a_{20} \right| - \frac{1}{3}t^2T_y\phi_{x3}^3 + t^2\phi_{y2}^y \\
 k_{x3}^2 &= \left| \frac{1}{t}a_{3y} + \frac{1}{6}t^2T_x\phi_{x3}^3 - \frac{1}{18}t^2(\phi_{x3}^3)^2 \right| + \frac{11}{36}t^2(\phi_{x3}^3)^2 - \frac{2}{3}t^2T_x\phi_{x3}^3 + t^2\phi_{x3}^2 \\
 k_{y3}^2 &= \left| -\frac{1}{t}a_{3x} - \frac{1}{6}t^2T_y\phi_{x3}^3 \right| - \frac{1}{3}t^2T_y\phi_{x3}^3 + t^2\phi_{y2}^y \\
 k_{23}^3 &= \left| 3a_{20} - \frac{1}{t}a_{3x} - \frac{2}{3}t^2T_y\phi_{x3}^3 \right| - \frac{1}{3}t^2T_y\phi_{x3}^3 + t^2\phi_{y2}^y
 \end{aligned}$$

The functions  $a_{3x}$  and  $a_{3y}$  and  $a_{20}$  will be determined by equating the exact part of homogeneity 2 to zero

$$\begin{aligned}
 a_{3x} &= -\frac{t^3}{6}T_y\phi_{x3}^3 \\
 a_{3y} &= \frac{t^3}{18}(\phi_{x3}^3)^2 - \frac{t^3}{6}T_x\phi_{x3}^3 \\
 a_{20} &= \frac{t^2}{6}T_y\phi_{x3}^3.
 \end{aligned}$$

The essential curvatures of homogeneity 2 are

$$\begin{aligned}
 R_{y2}^x &= t^2\phi_{y2}^x \\
 R_{y2}^y &= t^2\phi_{y2}^y - \frac{t^2}{3}T_y\phi_{x3}^3 \\
 R_{x3}^2 &= t^2\phi_{x3}^2 + \frac{11t^2}{36}(\phi_{x3}^3)^2 - \frac{2t^2}{3}T_x\phi_{x3}^3
 \end{aligned}$$

The curvature component of homogeneity 3 splits into a 1-dimensional exact part, a 1-dimensional non-exact closed part and a 5-dimensional

non-closed part. Below, we used two bars to separate the three components.

$$\begin{aligned}
k_{x2}^0 &= B_3 \left\| \right. \\
k_{y2}^0 &= \left\| \frac{1}{4} \hat{V}_x R_{y2}^x + \frac{1}{4} \hat{V}_y R_{y2}^y \right. \\
k_{x3}^x &= B_3 \left\| -\frac{3}{2} \hat{V}_x R_{y2}^y + \frac{1}{2} \hat{V}_y R_{x3}^2 \right. \\
k_{x3}^y &= \left\| R_{x3}^y \right. \\
k_{y3}^x &= \left\| \frac{3}{4} \hat{V}_x R_{y2}^x - \frac{1}{4} \hat{V}_y R_{y2}^y \right. \\
k_{y3}^y &= B_3 \left\| + \hat{V}_x R_{y2}^y \right. \\
k_{23}^2 &= 2B_3 \left\| + \frac{1}{2} \hat{V}_x R_{y2}^y - \frac{1}{2} \hat{V}_y R_{x3}^2 \right. ,
\end{aligned}$$

with

$$B_3 := -a_{30} + \frac{1}{3} t^3 T_x T_y \phi_{x3}^3 - \frac{1}{6} t^3 T_y T_x \phi_{x3}^3 - \frac{1}{18} t^3 \phi_{x3}^3 T_y \phi_{x3}^3.$$

The only essential curvature of homogeneity 3 is

$$R_{x3}^y = \phi_{x3}^y t^3 + \frac{t^3}{12} \phi_{x3}^3 T_x \phi_{x3}^3 - \frac{t^3}{6} T_x^2 \phi_{x3}^3 + \frac{5t^3}{216} (\phi_{x3}^3)^3 + \frac{t^3}{6} \phi_{x3}^3 \phi_{x3}^2$$

Vanishing of the exact part yields

$$a_{30} = \frac{t^3}{3} T_x T_y \phi_{x3}^3 - \frac{t^3}{6} T_y T_x \phi_{x3}^3 - \frac{t^3}{18} \phi_{x3}^3 T_y \phi_{x3}^3.$$

In homogeneity 4 and 5 the cochain operator  $\partial$  has no kernel. Therefore, according to the Bianchi identity, all curvature of these homogeneities can be expressed in terms of curvature of lower homogeneity:

$$\begin{aligned}
 k_{x3}^0 &= -\frac{1}{2}\hat{V}_x^2 R_{y2}^y - \frac{1}{3}\hat{V}_y \hat{V}_x R_{x3}^2 + \frac{1}{2}\hat{V}_x \hat{V}_y R_{x3}^2 + \frac{1}{3}\hat{V}_y R_{x3}^y \\
 k_{y3}^0 &= \frac{1}{4}\hat{V}_x^2 R_{y2}^x + \frac{1}{4}\hat{V}_x \hat{V}_y R_{y2}^y \\
 k_{23}^x &= -\frac{1}{2}\hat{V}_x \hat{V}_y R_{y2}^y + \frac{1}{2}\hat{V}_x^2 R_{y2}^x + \frac{3}{2}\hat{V}_y \hat{V}_x R_{y2}^y - \frac{1}{2}\hat{V}_y^2 R_{x3}^2 - R_{y2}^x R_{x3}^2 \\
 k_{23}^y &= \frac{1}{2}\hat{V}_x^2 R_{y2}^y - \frac{1}{3}\hat{V}_y \hat{V}_x R_{x3}^2 + \frac{1}{2}\hat{V}_x \hat{V}_y R_{x3}^2 - \frac{2}{3}\hat{V}_y R_{x3}^y - R_{y2}^y R_{x3}^2 \\
 k_{23}^0 &= \frac{1}{2}\hat{V}_x^3 R_{y2}^x + 3\hat{V}_x \hat{V}_y \hat{V}_x R_{y2}^y - \frac{3}{2}\hat{V}_y \hat{V}_x^2 R_{y2}^y + \frac{1}{2}\hat{V}_y \hat{V}_x \hat{V}_y R_{x3}^2 \\
 &\quad - \hat{V}_x \hat{V}_y^2 R_{x3}^2 - \frac{1}{2}\hat{V}_x^2 \hat{V}_y R_{y2}^y - R_{y2}^x \hat{V}_x R_{x3}^2 - R_{x3}^2 \hat{V}_x R_{y2}^x \\
 &\quad + \frac{1}{2}R_{y2}^y \hat{V}_y R_{x3}^2 - \frac{3}{2}R_{y2}^y \hat{V}_x R_{y2}^y + R_{y2}^x R_{x3}^y
 \end{aligned}$$

Here we used the Jacobi identity (2). This proves Theorem 1.

**Corollary 1.** *An Engel CR manifold is locally equivalent to the cubic C if and only if the invariants  $R_{y2}^x, R_{y2}^y, R_{x3}^2, R_{x3}^y$  vanish identically.*

The structure equations take the form

$$\begin{aligned}
 d\hat{\Phi}^3 &= 3\hat{\Phi}^3 \wedge \hat{\Phi}^0 - \hat{\Phi}^x \wedge \hat{\Phi}^2 + K^3 \\
 d\hat{\Phi}^2 &= 2\hat{\Phi}^2 \wedge \hat{\Phi}^0 - \hat{\Phi}^x \wedge \hat{\Phi}^y + K^2 \\
 d\hat{\Phi}^y &= \hat{\Phi}^y \wedge \hat{\Phi}^0 + K^y \\
 d\hat{\Phi}^x &= \hat{\Phi}^x \wedge \hat{\Phi}^0 + K^x \\
 d\hat{\Phi}^0 &= K^0 = \pi^*(d\varpi)
 \end{aligned}$$

The curvature form  $K^0$  is a pullback of a 2-form on  $M$ . Its vanishing is equivalent to the integrability of the horizontal distribution defined as the null distribution of the connection form

$$\hat{\Phi}^0 = \frac{dt}{t} + \frac{\phi_{x3}^3}{6}\varphi_x - \frac{T_y \phi_{x3}^3}{6}\varphi_2 - \frac{2T_x T_y \phi_{x3}^3 - T_y T_x \phi_{x3}^3}{6}\varphi_3.$$

## 5. Distinguished frames and interpretation of the curvature

The Cartan connection constructed above induces a family of distinguished frames at any point  $a$  of the underlying manifold itself. Let  $\pi : \mathcal{G} \rightarrow M$  be the projection of the Cartan bundle and  $\hat{a} \in \pi^{-1}(a)$ . Then for any choice of  $\hat{a}$  we define for  $\alpha = x, y, 2, 3$

$$\tilde{V}_\alpha = d\pi|_{\hat{a}} \hat{V}_\alpha$$

Another choice of  $\hat{a}' = t\hat{a}$  leads to the frame

$$\tilde{V}'_x = t\tilde{V}_x, \quad \tilde{V}'_y = t\tilde{V}_y, \quad \tilde{V}'_2 = t^2\tilde{V}_2, \quad \tilde{V}'_3 = t^3\tilde{V}_3.$$

Thus, the Cartan connection induces a complete splitting of the tangent spaces of  $M$ . The essential curvatures  $R_{y2}^x, R_{y2}^y, R_{x3}^y, R_{x3}^2$ , can be interpreted as the  $\tilde{V}_x$  component of  $[\tilde{V}_y, \tilde{V}_2]$ , the  $\tilde{V}_2$  component of  $[\tilde{V}_y, \tilde{V}_3]$  and the  $\tilde{V}_y$  and  $\tilde{V}_2$  components of  $[\tilde{V}_x, \tilde{V}_3]$ , respectively. This proves the following geometric interpretation of the curvatures

**Corollary 2.** *Vanishing of  $R_{y2}^x$  is equivalent to the integrability of the distribution spanned by  $\tilde{V}_y, \tilde{V}_2$ , vanishing of  $R_{x3}^y$  and  $R_{x3}^2$  is equivalent to the integrability of the distribution spanned by  $\tilde{V}_x, \tilde{V}_3$ . If  $\tilde{V}_x R_{y2}^x \equiv 0$  then  $R_{y2}^y \equiv 0$  is equivalent to the integrability of the distribution spanned by  $\tilde{V}_y, \tilde{V}_3$ .*

The splitting of  $TM$  can be understood in terms of  $M$  only in the following special case: Choose  $T_x, T_y$  as above. The remaining freedom is a rescaling of  $T_x, T_y$  by a multiplier  $f$  with  $T_y f = 0$ . Suppose there is a solution of

$$(3) \quad \frac{Xf}{f} = X(\log f) = \frac{\phi_{x3}^3}{3}.$$

This determines all partial derivatives of  $f$ , thus  $f$  is determined up to a multiplicative constant. In this case  $\tilde{V}_x = tfT_x$ ,  $\tilde{V}_y = tfT_y$ ,  $\tilde{V}_2 = [\tilde{V}_x, \tilde{V}_y]$ ,  $\tilde{V}_3 = [\tilde{V}_x, \tilde{V}_2]$ . All formulae simplify significantly. In particular  $\hat{\Phi}^0 = \frac{dt}{t}$  and  $d\hat{\Phi}^0 = 0$ . Vice versa, let  $d\hat{\Phi}^0 = 0$ . Then  $d\varpi = 0$  and locally  $\varpi = d\psi$  for some function  $f$ . But then  $T_y\psi = 0$  and  $T_x\psi = \frac{\phi_{x3}^3}{6}$  and, hence,  $f = \exp(2\psi)$  solves equation (3). It follows

**Corollary 3.** *On an Engel CR manifold  $M$  there exists locally a unique family of vector fields  $tV_x, tV_y, t^2V_2, t^3V_3$  ( $t \in \mathbb{R}^*$ ) on  $M$  such that*

$$\begin{aligned} [V_x, V_y] &= V_2 \\ [V_x, V_2] &= V_3 \\ [V_y, V_2] &\in D \\ [V_x, V_3] &\in D' \end{aligned}$$

*if and only if  $d\varpi = 0$  which is equivalent to  $K^0 = 0$ .*

Obviously, the families  $(tV_x, tV_y, tV_2, tV_3)$  and  $(t\tilde{V}_x, t\tilde{V}_y, t\tilde{V}_2, t\tilde{V}_3)$  coincide.

## 6. Relation to the normal form

In [3] the authors constructed a normal form for embedded real-analytic Engel CR manifolds. For any such manifold  $M$  there exist



local coordinates  $(z, w_1, w_2)$  in a neighbourhood of a point  $a \in M$  in the ambient space such that the equation of  $M$  takes the form

$$\begin{aligned} \operatorname{Im} w_1 &= |z|^2 + A_1 \operatorname{Re} z^2 \bar{z}^3 + A_2 \operatorname{Im} z^2 \bar{z}^3 + \dots \\ \operatorname{Im} w_2 &= \operatorname{Re} z^2 \bar{z} + B_1 \operatorname{Re} z^4 \bar{z} + B_2 \operatorname{Re} z^2 \bar{z}^3 + B_3 \operatorname{Im} z^2 \bar{z}^3 + B_4 \operatorname{Re} z^5 \bar{z} \\ &\quad B_5 \operatorname{Im} z^5 \bar{z} + B_6 \operatorname{Re} z^4 \bar{z}^2 + B_7 \operatorname{Im} z^4 \bar{z}^2 + B_8 |z|^6 + \dots \end{aligned}$$

where the dots indicate terms of higher homogeneity.

The expressions of the invariants in terms of the normal form are

$$\begin{aligned} R_{y_2}^x(0) &= (2B_1 - B_2)t^2 \\ R_{y_2}^y(0) &= -3B_3t^2 \\ R_{x_3}^2(0) &= (2B_1 - 5B_2)t^2 \\ R_{x_3}^y(0) &= (4A_1 + 5B_4 - 2B_6 - 6B_8)t^3 \end{aligned}$$

The distinguished frames defined in Section 5 at the reference point  $a$  are

$$\frac{t}{2} \frac{\partial}{\partial x}, \quad -\frac{t}{2} \frac{\partial}{\partial y}, \quad t^2 \frac{\partial}{\partial u_2}, \quad t^3 \frac{\partial}{\partial u_3} - t^3(B_1 - B_2) \frac{\partial}{\partial y}.$$

Notice, that the normal coordinates themselves provide a family of distinguished frames at the reference point, namely (for the most natural choice)

$$t \frac{\partial}{\partial x}, \quad t \frac{\partial}{\partial y}, \quad t^2 \frac{\partial}{\partial u_2}, \quad t^3 \frac{\partial}{\partial u_3}.$$

Since  $B_1 - B_2$  is an invariant of homogeneity 2 the frame from the Cartan connection and the frame from the normal form are equivalent.

It is natural to introduce the notion of umbilicity. In analogy to the definition of umbilic points of hypersurface given in [10] we call a point  $a$  of an Engel CR-manifold umbilic if  $R_{y_2}^x(a) = R_{y_2}^y(a) = R_{x_3}^2(a) = R_{x_3}^y(a) = 0$ . Then Corollary 1 can be reformulated as

**Corollary 1'.** *An Engel CR manifold is locally equivalent to the cubic  $C$  if and only if all its points are umbilic.*

In terms of the normal form umbilicity is equivalent to vanishing of  $B_1, B_2, B_3$  and  $4A_1 + 5B_4 - 2B_6 - 6B_8$ .

## 7. Appendix

**7.1. Algebraic brackets.** Let  $D \subset TM$  be a distribution of rank  $n$  on a  $n + k$ -dimensional manifold  $M$  and let  $\mathcal{L}_a$  be the bilinear mapping

$$\mathcal{L}_a : (X, Y) \rightarrow \pi([X, Y]_a)$$

where  $X, Y$  are local sections of  $D$ ,  $a \in M$  and  $\pi$  is the natural projection  $T_a M \rightarrow T_a M / D_a$ . Then  $\mathcal{L}_x(X, Y)$  depends only on  $X_a$  and  $Y_a$ . Indeed,

let  $\theta$  be an  $\mathbb{R}^k$  valued 1-form with  $\ker \theta = D$ . Then  $\theta$  identifies  $T_a M/D_a$  with  $\mathbb{R}^k$  and, thus  $\mathcal{L}_a$  with

$$\theta_a([X, Y]) = -2d\theta_a(X, Y) + X\theta(Y) - Y\theta(X).$$

Since  $\theta(X) = \theta(Y) = 0$ , we conclude  $\theta_a([X, Y]) = -2d\theta_a(X_a, Y_a)$ .

**7.2. The space of 2-cocycles.** Let  $V_a$  a basis of  $\mathfrak{g}$  and  $\Phi^a$  the dual basis, where  $a$  runs over  $\{0, x, y, 2, 3\}$ . Then a 2-cochain can be written as  $\Phi = \sum \phi_{\alpha\beta}^a \Phi^\alpha \wedge \Phi^\beta \otimes V_a$  (the indices  $\alpha, \beta$  run over  $\{x, y, 2, 3\}$  and  $\phi_{\alpha\beta}^a = -\phi_{\beta\alpha}^a$ ). Thus, the space of 2-cochains  $\mathcal{C}^2(\mathfrak{g}_-, \mathfrak{g})$  has dimension 30. The subspace  $\mathcal{Z}^2$  of  $\partial$ -closed cochains is 17-dimensional and is characterized by the following conditions (in order of their homogeneity):

$$\begin{aligned} \phi_{xy}^x - \phi_{y2}^2 + \phi_{y3}^3 &= 0 \\ \phi_{x2}^x + \phi_{y2}^y + 5\phi_{xy}^0 - \phi_{23}^3 &= 0 & \phi_{y3}^2 + 3\phi_{xy}^0 - \phi_{23}^3 &= 0 \\ \phi_{y3}^y - \phi_{x3}^x &= 0 & \phi_{y2}^0 &= 0 & \phi_{y3}^x &= 0 \\ \phi_{23}^2 - 2\phi_{x3}^x &= 0 & \phi_{x2}^0 + \phi_{x3}^x &= 0 \\ \phi_{y3}^0 &= 0 & \phi_{23}^x &= 0 & \phi_{x3}^0 &= 0 & \phi_{23}^y &= 0 \\ \phi_{23}^0 &= 0 \end{aligned}$$

The subspace  $\mathcal{B}^2$  of  $\partial$ -exact cochains is described by the additional equations

$$\begin{aligned} \phi_{y2}^x &= 0 & \phi_{x2}^y + \phi_{x3}^2 &= 0 & \phi_{y2}^y - \phi_{xy}^0 &= 0 \\ \phi_{x3}^y &= 0 \end{aligned}$$

Thus, the cohomology  $H^2(\mathfrak{g}_-, \mathfrak{g})$  is 4-dimensional and can be represented by the following complementary subspaces to  $\mathcal{B}^2(\mathfrak{g}_-, \mathfrak{g})$  in  $\mathcal{Z}^2(\mathfrak{g}_-, \mathfrak{g})$ : in homogeneity 2 we choose the subspace spanned by

$$\begin{aligned} \Phi^y \wedge \Phi^2 \otimes V_x, & \quad \Phi^x \wedge \Phi^3 \otimes V_2 \\ \Phi^y \wedge \Phi^2 \otimes V_y + \Phi^y \wedge \Phi^3 \otimes V_2 + \Phi^2 \wedge \Phi^3 \otimes V_3, \end{aligned}$$

in homogeneity 3 we choose the subspace spanned by

$$\Phi^x \wedge \Phi^3 \otimes V_y.$$

**7.3. The curvatures.** Here we list the curvatures by their homogeneity

$$\begin{aligned}
 (-1) : \quad & k_{xy}^3 = 0 \\
 (0) : \quad & k_{y2}^3 = 0, \quad k_{xy}^2 = 0, \quad k_{x2}^3 = 0 \\
 (1) : \quad & k_{xy}^y = 0, \quad k_{xy}^x = 0, \quad k_{x2}^2 = 0, \\
 & k_{y2}^2 = 0, \quad k_{x3}^3 = 0, \quad k_{y3}^3 = 0 \\
 (2) : \quad & k_{xy}^0 = 0, \quad k_{x2}^x = 0, \quad k_{x2}^y = 0 \\
 & k_{y2}^x = R_{y2}^x, \quad k_{y2}^y = R_{y2}^y, \quad k_{x3}^2 = R_{x3}^2 \\
 & k_{y3}^2 = R_{y3}^y, \quad k_{23}^3 = R_{y2}^y \\
 (3) : \quad & k_{x2}^0 = 0, \quad k_{y2}^0 = \frac{1}{4}\hat{V}_x R_{y2}^x + \frac{1}{4}\hat{V}_y R_{y2}^y, \quad k_{x3}^x = -\frac{3}{2}\hat{V}_x R_{y2}^y + \frac{1}{2}\hat{V}_y R_{x3}^2 \\
 & k_{x3}^y = R_{x3}^y, \quad k_{y3}^x = \frac{3}{4}\hat{V}_x R_{y2}^x - \frac{1}{4}\hat{V}_y R_{y2}^y, \quad k_{y3}^y = \hat{V}_x R_{y2}^y \\
 & k_{23}^2 = \frac{1}{2}\hat{V}_x R_{y2}^y - \frac{1}{2}\hat{V}_y R_{x3}^2 \\
 (4) : \quad & k_{x3}^0 = -\frac{1}{2}\hat{V}_x^2 R_{y2}^y - \frac{1}{3}\hat{V}_y \hat{V}_x R_{x3}^2 + \frac{1}{2}\hat{V}_x \hat{V}_y R_{x3}^2 + \frac{1}{3}\hat{V}_y R_{x3}^y \\
 & k_{y3}^0 = \frac{1}{4}\hat{V}_x^2 R_{y2}^x + \frac{1}{4}\hat{V}_x \hat{V}_y R_{y2}^y \\
 & k_{23}^x = -\frac{1}{2}\hat{V}_x \hat{V}_y R_{y2}^y + \frac{1}{2}\hat{V}_x^2 R_{y2}^x + \frac{3}{2}\hat{V}_y \hat{V}_x R_{y2}^y - \frac{1}{2}\hat{V}_y^2 R_{x3}^2 - R_{y2}^x R_{x3}^2 \\
 & k_{23}^y = \frac{1}{2}\hat{V}_x^2 R_{y2}^y - \frac{1}{3}\hat{V}_y \hat{V}_x R_{x3}^2 + \frac{1}{2}\hat{V}_x \hat{V}_y R_{x3}^2 - \frac{2}{3}\hat{V}_y R_{y2}^y - R_{y2}^y R_{x3}^2 \\
 (5) : \quad & k_{23}^0 = \frac{1}{2}\hat{V}_x^3 R_{y2}^x + 3\hat{V}_x \hat{V}_y \hat{V}_x R_{y2}^y - \frac{3}{2}\hat{V}_y \hat{V}_x^2 R_{y2}^y + \frac{1}{2}\hat{V}_y \hat{V}_x \hat{V}_y R_{x3}^2 - \\
 & \quad - \hat{V}_x \hat{V}_y^2 R_{x3}^2 - \frac{1}{2}\hat{V}_x^2 \hat{V}_y R_{y2}^y - R_{y2}^x \hat{V}_x R_{x3}^2 - \\
 & \quad - R_{x3}^2 \hat{V}_x R_{y2}^x + \frac{1}{2}R_{y2}^y \hat{V}_y R_{x3}^2 - \frac{3}{2}R_{y2}^y \hat{V}_x R_{y2}^y + R_{y2}^x R_{x3}^y
 \end{aligned}$$

The essential curvatures evaluate as

$$\begin{aligned}
 R_{y2}^x &= t^2 \phi_{y2}^x \\
 R_{y2}^y &= t^2 \phi_{y2}^y - \frac{t^2}{3} T_y \phi_{x3}^3 \\
 R_{x3}^2 &= t^2 \phi_{x3}^2 + \frac{11t^2}{36} (\phi_{x3}^3)^2 - \frac{2t^2}{3} T_x \phi_{x3}^3 \\
 R_{x3}^y &= t^3 \phi_{x3}^y + \frac{t^3}{12} \phi_{x3}^3 T_x \phi_{x3}^3 - \frac{t^3}{6} T_x^2 \phi_{x3}^3 + \frac{5t^3}{216} (\phi_{x3}^3)^3 + \frac{t^3}{6} \phi_{x3}^3 \phi_{x3}^2
 \end{aligned}$$

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