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Real submanifolds in complex space: polynomial models, automorphisms, and classification problems

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Abstract. This is a survey of results on the local theory of real submanifolds of a complex space. Most of the results included here were obtained in Vitushkin's seminar at Moscow State University over the last fifteen years. The most important achievement is a technique for computing automorphisms, invariants, and classifications of real submanifolds, which includes as a main step the construction of a 'good' model surface (an analogue of an osculating paraboloid in classical differential geometry).

Contents

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§ 1. Introduction

Vitushkin's 1985 survey [64] deals mainly with non-degenerate real hypersurfaces. The present survey can be viewed as a continuation of that paper. Our main subject is the results on local theory of real submanifolds obtained by participants of Vitushkin's seminar in the last ten to fifteen years, when manifolds of codimension greater than one have been intensively studied and an efficient technique has been created for computing automorphisms, constructing invariants, and so on. (We refer to this technique as the model-surface method.)

This topic is of interest not only in multidimensional complex analysis, but also in differential geometry and partial differential equations. The geometric thread includes the work of E. Cartan, Tanaka, Chern, and others (see [20], [58], [21], and [42]). In modern terminology, this is the reduction technique for G-structures. The main (although by no means the only) successful application of the method to real submanifolds pertains to hypersurfaces. The approach has also been applied to manifolds of codimension greater than one by Tanaka in his classical papers as well as by Garrity–Mizner [33] and Ezhov–Isaev–Schmalz [25]. The analytic thread is due to Tress, Poincaré, Riquier, Moser, Webster, Pinchuk, and others (see [59], [47], [36], [21], [65], and [46]). Of contemporary authors contributing to this direction, we mention Baouendi, Rothschild, Stanton, Ebenfelt, and Zaitsev ([5], [6], [54], [8], [9]) as well as Sukhov ([56], [57]). Our approach, which will be discussed below, is also closely related to the analytical thread.

Almost all results presented in this survey have been obtained by a unified method. A key point in applying the method is the construction of a 'good model'. (We shall assign an exact meaning to this term.) The main result is the model-surface method itself rather than any of its specific applications.

A real submanifold of the complex space is an object arising naturally in multidimensional complex analysis. Hypersurfaces, that is, submanifolds of real codimension one, are the topological boundaries of domains. Submanifolds of higher codimension arise as the skeletons of topological boundaries, or their Shilov boundaries.

If a smooth real manifold M is embedded in the complex space $\mathbb{C}^N$, then there is an interplay between the smooth structure of the manifold and the complex structure of the ambient space. This interplay gives rise to local biholomorphic invariants, the automorphism group of the manifold germ proves to be finite-dimensional or even trivial, two germs chosen at random turn out to be non-equivalent, and so on. The coarsest local invariants are related to the 1-jet of the germ. The first indication of the interaction between the two above-mentioned structures is the existence of a non-trivial complex part in the tangent space of the manifold. If the real dimension of the manifold is sufficiently large (namely, is greater than the complex dimension of the space), then the complex part is necessarily non-trivial. The pair (n, k), where n is the dimension of the complex tangent (which may vary from point to point) and k is the codimension, will be called the *manifold type*. It is determined by the 1-jet of the manifold germ at a point and is a local biholomorphic invariant. For example, it does not permit one to map the complex line onto the two-dimensional real plane biholomorphically.

We mainly pay attention to local properties of a real submanifold of the complex space. Three interrelated issues—automorphisms, invariants, and classification—are the focus of our attention.

We need the following notation. If M is a smooth surface in $\mathbb{C}^N$, $\xi \in M$, and M_ξ is the germ of M at ξ, then by $\operatorname{aut} M_\xi$ we denote the Lie algebra of germs at ξ of real vector fields tangent to M_ξ with holomorphic coefficients. If $z = (z_1, \dots, z_N)$ are coordinates in $\mathbb{C}^N$, then

$$\operatorname{aut} M_\xi = \left\{ X(z) = 2\operatorname{Re}\left(f_1(z)\frac{\partial}{\partial z_1} + \dots + f_N(z)\frac{\partial}{\partial z_N} \right) \right\},$$

where the restriction of X to M_ξ is a field germ tangent to M_ξ and $(f_1(z), \dots, f_n(z))$ are holomorphic function germs at ξ. This is the Lie algebra of infinitesimal holomorphic automorphisms. These vector fields generate a holomorphic action on M_ξ. The corresponding local group $\operatorname{Aut} M_\xi$ is the image of $\operatorname{aut} M_\xi$ under the exponential map and acts on M_ξ by transformations biholomorphic at ξ. In the following, the algebra and the corresponding group are called the germ algebra and the germ group, respectively.

§ 2. Demonstration of the method: a hypersurface in $\mathbb{C}^2$

The study of real submanifolds started from the case of codimension one (hypersurfaces). A relevant bibliography can be found in [64]. Here we only note that the first paper concerning hypersurfaces was written in 1907 by Poincaré [47]. It deals with the lowest-dimensional case of a three-dimensional hypersurface in $\mathbb{C}^2$.

A hypersurface in $\mathbb{C}^2$ is a surface of type $(1,1)$. Let use present the technique in this simplest situation.

2.1. The choice of a model surface. A hypersurface Γ can be represented in a neighbourhood of a point $\xi \in \Gamma$ as the zero set of a smooth function ρ satisfying $\operatorname{grad} \rho(\xi) \neq 0$. By an affine change of variables one can ensure that $\xi = (0,0)$ and $\operatorname{grad} \rho(\xi) = (0, i)$. Let $(z, w = u + iv)$ be coordinates in $\mathbb{C}^2$; then, solving the equation $\rho(z, \overline{z}, u, v) = 0$ for v, we obtain an equation for the hypersurface germ at zero in the form $\Gamma_0 = \{v = F(z, \overline{z}, u)\}$, where the function F and its first derivatives vanish at the origin. Furthermore, the z-plane $\{w = 0\}$ is the complex tangent to the hypersurface at zero, and the entire tangent plane is $\{v = 0\}$. If the surface is real-analytic, then F is a convergent power series. If the surface is only smooth, then we can consider the formal series. We introduce a gradation in the space of series in the variables $(z, \overline{z}, u)$ by assigning the weights $[z] = 1$, $[\overline{z}] = 1$, and $[u] = 2$. Then the equation of the germ can be written as $v = 2\operatorname{Re}(Az^2) + hz\overline{z} + \cdots$, where the dots stand for terms of weight ≥ 3. This equation can be rewritten in the form $2\operatorname{Im}(w + 2iAz^2) = hz\overline{z} + \cdots$, or, after the simple quadratic-triangular coordinate transformation $z \to z$, $w \to w + 2iAz^2$, in the form

$$v = hz\overline{z} + \cdots . \tag{1}$$

We note that the one-dimensional Hermitian form $hz\overline{z}$ is the Levi form of Γ_0 at zero. If it is non-degenerate (that is, $h \neq 0$), then the change of variables $z \to z$, $w \to hw$ gives

$$v = z\overline{z} + \cdots . \tag{2}$$

Our model surface is the quadratic hypersurface $Q=\{v=|z|^2\}$ (the tangent quadric).

We have not restricted ourselves to 1-jets, since all hyperplanes have an infinite-dimensional automorphism group. We deal only with the Hermitian term in the 2-jet, since all other terms can be eliminated by changing the variables and dropping higher-weight components. We shall see that if one wishes to obtain a finite-dimensional germ group, it indeed suffices to consider 2-jets.

2.2. Automorphisms of the model surface. We start by computing the algebra $\operatorname{aut} Q$ of infinitesimal automorphisms of Q. Writing out the tangency condition, we arrive at the following description of this algebra:

$$\operatorname{aut} Q=\left\{X(z,w)=2\operatorname{Re}\left(f(z,w)\frac{\partial}{\partial z}+g(z,w)\frac{\partial}{\partial w}\right)\right\},$$

where the functions f and g are holomorphic in a neighbourhood of the origin and satisfy the functional equation

$$\operatorname{Im} g(z,u+i|z|^2)=2\operatorname{Re}(f(z,u+i|z|^2)\overline{z}). \tag{3}$$

There is an easy-to-use technique for solving this equation in the class of formal power series. We use expansions of the form

$$f(z,u+i|z|^2)=\sum_{m=0}^{\infty}\frac{i^m}{m!}\Delta^m f(z,u),$$

where $\Delta f=|z|^2\dfrac{\partial}{\partial u}f$ $(\Delta^0=\mathrm{Id})$, and also

$$f(z,u)=\sum_{m=0}^{\infty}f_m(u)z^m.$$

The original equation splits into relations of given bidegrees in $(z,\overline{z})$. Thus we obtain an infinite system of ordinary differential equations for the sequence $(f_0(u),g_0(u),f_1(u),g_1(u),\dots)$ of unknown functions. However, writing out the components of bidegrees $(2,0),(3,0),\dots$, we find that $g_2=g_3=\dots=0$, while the components of bidegrees $(3,1),(4,1),\dots$ give $f_3=f_4=\dots=0$. Then one writes out the relations of bidegrees $(0,0),(1,0),(1,1),(2,1),(2,2),(3,2),(3,3)$ for the remaining five functions g_0,g_1,f_0,f_1,f_2. The solutions of the resulting system of ordinary linear differential equations are polynomials of relatively low order; moreover, the equations so far unused impose no additional restrictions on the solution. As a result, we find that the algebra consists of vector fields with coefficients of the form

$$f=p+\lambda z+aw+2i\overline{a}z^2+rzw,\qquad g=q+2i\overline{p}z+2(\operatorname{Re}\lambda)w+2i\overline{a}zw+rw^2,$$

where (p,λ,a) are complex parameters, (q,r) are real parameters, and $\dim\operatorname{aut} Q=8$.

We introduce a gradation in the algebra of vector fields by the conditions

$$[z]=1,\ [\overline{z}]=1,\ [w]=2,\ [\overline{w}]=2,\ \left[\frac{\partial}{\partial z}\right]=-1,\ \left[\frac{\partial}{\partial \overline{z}}\right]=-1,\ \left[\frac{\partial}{\partial w}\right]=-2,\ \left[\frac{\partial}{\partial \overline{w}}\right]=-2.$$

This gradation is consistent with the Lie bracket: if g_m is the subspace of fields of weight m, then $[g_m, g_n] \subseteq g_{m+n}$. It follows from our computations that $\operatorname{aut} Q$ has the following structure:

$$\operatorname{aut} Q = g_{-2} + g_{-1} + g_0 + g_1 + g_2,$$

where

$$\begin{aligned}
g_{-2} &= \left\{2\operatorname{Re}\left(q\frac{\partial}{\partial w}\right)\right\},\\
g_{-1} &= \left\{2\operatorname{Re}\left(p\frac{\partial}{\partial z} + 2i\overline{p}z\frac{\partial}{\partial w}\right)\right\},\\
g_0 &= \left\{2\operatorname{Re}\left(\lambda z\frac{\partial}{\partial z} + 2(\operatorname{Re}\lambda)w\frac{\partial}{\partial w}\right)\right\},\\
g_1 &= \left\{2\operatorname{Re}\left((aw + 2i\overline{a}z^2)\frac{\partial}{\partial z} + 2i\overline{a}zw\frac{\partial}{\partial w}\right)\right\},\\
g_{-2} &= \left\{2\operatorname{Re}\left(rzw\frac{\partial}{\partial z} + rw^2\frac{\partial}{\partial w}\right)\right\}.
\end{aligned}$$

To construct one-parameter transformation groups corresponding to the algebra of vector fields, one must solve the corresponding differential equations. If $(z,w) \to (Z(t,z,w), W(t,z,w))$ is the desired group corresponding to the field $2\operatorname{Re}\left(f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}\right)$, then the functions (Z,W) satisfy the system of ordinary differential equations and the initial conditions

$$Z' = f(Z,W), \quad W' = g(Z,W), \quad Z(0,z,w) = z, \quad W(0,z,w) = w.$$

By solving these equations for the fields in the five weight components, we find that

$$\begin{aligned}
&g_{-2} \text{ corresponds to } z \to z,\ w \to w + tq;\\
&g_{-1} \text{ corresponds to } z \to z,\ w \to w + 2i\overline{p}zt + i|p|^2t^2;\\
&g_0 \text{ corresponds to } z \to z\exp(\lambda t),\ w \to w\exp(2\operatorname{Re}(\lambda t));\\
&g_1 \text{ corresponds to } z \to \frac{z + awt}{1 - 2i\overline{a}zt - i|a|^2wt^2},\ w \to \frac{w}{1 - 2i\overline{a}zt - i|a|^2wt^2};\\
&g_2 \text{ corresponds to } z \to \frac{z}{1 - rwt},\ w \to \frac{w}{1 - rwt}.
\end{aligned}$$

Solving the systems corresponding to the weights -2, -1, 0, and 2 involves no difficulties. These are equations with separated variables. The system corresponding to the first component can be solved with the help of the change of variables $Z = 1/\widetilde{Z}$, $W = \widetilde{W}/\widetilde{Z}$.

By setting $t = 1$ in the resulting expressions, we obtain the values of the exponential map on vectors lying in the respective weight components of the algebra. The subalgebra $g_- = g_{-2} + g_{-1}$ generates the subgroup

$$\operatorname{Aut}_- Q = \bigl\{z \to z + p,\ w \to w + 2i\overline{p}z + (q + i|p|^2)\bigr\},$$

which acts transitively on Q. Thus, Q is affine homogeneous. The subalgebra g_0 generates the subgroup

$$\operatorname{Aut}_0 Q = \bigl\{z \to \Lambda z,\ w \to |\Lambda|^2 w\bigr\}$$

of linear automorphisms preserving the origin. The subalgebra $g_+ = g_1 + g_2$ generates the subgroup

$$\operatorname{Aut}_+ Q = \left\{z \to \frac{z + awt}{1 - (2i\overline{a}z + (r + i|a|^2)w)},\ w \to \frac{w}{1 - (2i\overline{a}z + (r + i|a|^2)w)}\right\}$$

of linear-fractional automorphisms of Q. This subgroup is singled out by the condition that the origin is preserved and the Jacobian matrix of an automorphism at the origin has unit diagonal entries ($\Lambda = 1$).

2.3. A bound for the dimension of the automorphism group of the germ. The fact that the model surface Q plays a distinguished role in the class of non-degenerate hypersurfaces is most convincingly illustrated by the estimate

$$\dim \operatorname{Aut} \Gamma_\xi \leqslant \dim \operatorname{Aut} Q,$$

which holds for an arbitrary non-degenerate smooth hypersurface germ in $\mathbb{C}^2$. The forthcoming argument is a version of the implicit function theorem for formal series with an estimate of the number of parameters.

The implicit function theorem states that a non-linear equation has a unique solution whenever its linear part does. In our case, the linear part of a non-linear relation coincides with the already familiar equation determining the infinitesimal automorphism algebra of Q.

Let us consider a somewhat more general situation. Suppose that a map of the form

$$f = z + f_2 + \cdots, \qquad g = w + g_3 + \cdots \tag{4}$$

takes a hypersurface germ

$$\operatorname{Im} w = |z|^2 + F(z, \overline{z}, \operatorname{Re} w) \tag{5}$$

to a hypersurface germ

$$\operatorname{Im} w = |z|^2 + \widetilde{F}(z, \overline{z}, \operatorname{Re} w) \tag{6}$$

of the same form, where the components F and $\widetilde{F}$ are of weight ≥ 3. The fact that (4) takes (5) to (6) can be represented in the form of the identity

$$\operatorname{Im}(w + g_3 + \cdots) = |z + f_2 + \cdots|^2 + \widetilde{F}(z + f_2 + \cdots, \overline{z} + \overline{f}_2 + \cdots, u + \operatorname{Re} g_3 + \cdots),$$

where $w = u + i(|z|^2 + \widetilde{F}(z, \overline{z}, \operatorname{Re} w))$.

By taking the component of weight $(m+1)$ in this relation, we obtain

$$\operatorname{Re}(ig_{m+1} + 2f_m\overline{z}) = F_{m+1} - \widetilde{F}_{m+1} + \cdots,$$

where $w = u + i|z|^2$ on the left-hand side and the dots on the right-hand side stand for terms depending on f_j, g_{j+1}, F_{j+1}, and $\widetilde{F}_{j+1}$ with $j < m$.

For given F and $\widetilde{F}$, these relations permit one to compute the polynomial components (f_m, g_{m+1}) recursively by solving systems of linear algebraic equations whose right-hand sides depend only on components computed earlier. The dimension of the solution set of a non-homogeneous system of linear equations does not exceed the dimension of the solution set of the homogeneous system. But the homogeneous equations are just the equations (3), which determine the algebra $\operatorname{aut} Q$. Hence the number of free parameters specifying a map in the class of formal power series of the form (4) does not exceed the dimension of the subalgebra $\operatorname{aut}_+ Q$, which is equal to three in this case. (The parameters are a and r.) Thus, the number of parameters specifying an arbitrary map of these germs does not exceed the dimension of the entire algebra $\operatorname{aut} Q$, which is equal to eight. By applying this estimate to a self-map of the germ, we obtain the desired bound for the dimension of the germ automorphism group.

In fact, this computation goes beyond estimating the dimension. We have obtained more, namely, we have indicated a system of parameters uniquely determining a map of one non-degenerate germ into another. In particular, a parametrization of automorphisms of an arbitrary germ has been obtained. Indeed, consider the parameters (p, q, Λ, a, r) determining an automorphism of Q and forming a part of the 2-jet of the automorphism. This system of parameters uniquely determines a self-map of the germ; furthermore, the parameters may be related by additional conditions (see [10]), which results in a dimension drop. This dimension is zero in general position.

2.4. What is a good model? Let us list the main stages of the study carried out in the preceding subsection.

1. The choice of a good model surface.
2. The computation of the algebra and the group for the model surface.
3. The parametrization of maps of germs by the germ algebra of the model surface.

Let us state once more the properties of a good model surface Q.

1. *Universality*: an arbitrary non-degenerate hypersurface germ in $\mathbb{C}^2$ is equivalent to a germ of the form (2).

2. *Finite dimension*: (a) the group of holomorphic automorphisms of Q is finite-dimensional; (b) every hypersurface specified by equations of degree less than two has an infinite-dimensional automorphism group.

3. *Homogeneity*: the hyperquadric Q is homogeneous; that is, its holomorphic automorphisms act on Q transitively. In this case, the homogeneity is provided by affine automorphisms.

4. *Symmetry*: (a) the hyperquadric is the most symmetric non-degenerate hypersurface in that the dimension of the germ group of a non-degenerate hypersurface does not exceed the dimension of the germ group of the tangent hyperquadric;

(b) the algebra of the model surface parametrizes the family of maps of one non-degenerate germ into another.

5. *Algebraic properties*: (a) the Lie algebra of holomorphic vector fields on a non-degenerate hyperquadric is an algebra of polynomial vector fields of bounded degree; in our case, the degrees of the coefficients do not exceed two; (b) the automorphism group of a non-degenerate hyperquadric is a Lie subgroup of the group of linear-fractional transformations of $\mathbb{C}^2$.

In the following, we say that some surface is a good model surface for germs of some given type if some version of the above-mentioned properties holds for this surface.

There are also properties specific to surfaces of type $(1,1)$. Let us mention two of these properties.

(1) The model surface is unique: the same quadric Q is a good model for an arbitrary non-degenerate germ of type $(1,1)$.

(2) The model surface is the image of the standard sphere under a linear-fractional transformation. Indeed, the change of variables

$$\widetilde{z}=\frac{z}{w-i}\,, \qquad \widetilde{w}=\frac{i(w+i)}{w-i}$$

takes $Q=\{\operatorname{Im}\widetilde{w}=|\widetilde{z}|^2\}$ to $\{|z|^2+|w|^2=1\}$. The point $(0,i)$ corresponds to the intersection of Q with projective infinity.

§ 3. Hypersurfaces in $\mathbb{C}^N$

Non-degenerate hypersurfaces in spaces of arbitrary dimension have been studied in a number of papers (see [58], [21], and [64]).

An argument similar to that carried out for $\mathbb{C}^2$ shows that the equation of a smooth hypersurface germ in the space $\mathbb{C}^{n+1}$ with coordinates ($z=(z_1,\dots,z_n)$, $w=u+iv$) can be represented in the form

$$v=\langle z,\overline{z}\rangle+\cdots\,, \tag{7}$$

where $\langle z,\overline{z}\rangle$ is a Hermitian form (the Levi form) and the dots stand for terms of weight $\geqslant 3$. (The weights are defined by the same formulae $[z]=1$, $[\overline{z}]=1$, and $[u]=2$.) In this case, the model surface is the hyperquadric $Q=\{v=\langle z,\overline{z}\rangle\}$. It will be called the tangent quadric to the hypersurface (7). The group $\operatorname{Aut}Q$ is finite-dimensional if and only if the Hermitian form $\langle z,\overline{z}\rangle$ is non-degenerate. If this is the case, then we also say that the hypersurface germ and the quadric are non-degenerate. If the form is degenerate, then one of the variables, say z_j, is absent in the representation of the form as a sum of squares. Then an arbitrary transformation of the form $z_j\to f(z_j)$, where $f'(0)\neq 0$ (the other variables are not affected) is an automorphism, and the group $\operatorname{Aut}Q$ is infinite-dimensional. If the form is non-degenerate, then Q is a good model surface, that is, the following typical assertions hold.

1. *Universality*: an arbitrary hypersurface germ in $\mathbb{C}^{n+1}$ is equivalent to a germ of the form (7).

2. *Finite dimension*: (a) the group of holomorphic automorphisms of a generic hyperquadric is finite-dimensional; (b) the group of holomorphic automorphisms of Q is finite-dimensional if and only if the form $\langle z, \overline{z}\rangle$ is non-degenerate; (c) every hypersurface specified by equations of degree less than two has an infinite-dimensional automorphism group.

3. *Homogeneity*: the hyperquadric Q is homogeneous; that is, its holomorphic automorphisms act transitively on Q. The homogeneity is provided by affine automorphisms.

4. *Symmetry*: (a) the hyperquadric is the most symmetric non-degenerate hypersurface in that the dimension of the germ group of a non-degenerate hypersurface does not exceed the dimension of the germ group of the tangent hyperquadric; (b) the algebra of the model surface parametrizes the family of maps of one non-degenerate germ into another.

5. *Algebraic properties*: (a) the Lie algebra of holomorphic vector fields on a non-degenerate hyperquadric is an algebra of polynomial vector fields of bounded degree; in our case, the degrees of the coefficients do not exceed two; (b) the automorphism group of a non-degenerate hyperquadric is a Lie subgroup of the group of linear-fractional transformations of $\mathbb{C}^{n+1}$.

The algebra $\operatorname{aut} Q$ has the following structure (the gradation is the same): $\operatorname{aut} Q = g_{-2} + g_{-1} + g_0 + g_1 + g_2$. Moreover, the components admit the following explicit description:

$$\begin{aligned}
g_{-2} &= \left\{ 2\operatorname{Re}\left(q\frac{\partial}{\partial w}\right)\right\},\\
g_{-1} &= \left\{ 2\operatorname{Re}\left(p\frac{\partial}{\partial z} + 2i\langle z, \overline{p}\rangle \frac{\partial}{\partial w}\right)\right\},\\
g_0 &= \left\{ 2\operatorname{Re}\left(\lambda z\frac{\partial}{\partial z} + \rho w \frac{\partial}{\partial w}\right)\right\},\\
g_1 &= \left\{ 2\operatorname{Re}\left((aw + 2i\langle z, \overline{a}\rangle z)\frac{\partial}{\partial z} + 2i\langle z, \overline{a}\rangle w \frac{\partial}{\partial w}\right)\right\},\\
g_2 &= \left\{ 2\operatorname{Re}\left(rwz\frac{\partial}{\partial z} + rw^2 \frac{\partial}{\partial w}\right)\right\},
\end{aligned}$$

where $q, r \in \mathbb{R}$, $p, a \in \mathbb{C}^n$, and the $n \times n$ matrix λ is related to the real number ρ by the formula $2\operatorname{Re}\langle \lambda z, \overline{z}\rangle = \rho\langle z, \overline{z}\rangle$. The subalgebra $g_- = g_{-2} + g_{-1}$ generates the subgroup

$$\operatorname{Aut}_- Q = \bigl\{ z \to z + p,\ w \to w + 2i\langle z, \overline{p}\rangle + (q + i\langle p, \overline{p}\rangle)\bigr\},$$

which acts transitively on Q and is known as the Heisenberg group. The subalgebra $g_0 + g_1 + g_2$ generates the subgroup of automorphisms of Q preserving the origin (the stabilizer subgroup). The subalgebra g_0 generates the subgroup $\operatorname{Aut}_0 Q$ that is the connected component of the group $\operatorname{LAut} Q$ of all linear automorphisms of Q preserving the origin. To obtain a description of this group, let us consider a more general situation. Suppose that a map of the form

$$f = \Lambda z + f_2 + \cdots, \qquad g = \rho w + g_3 + \cdots \tag{8}$$

takes a hypersurface germ

$$\operatorname{Im} w = \langle z, \overline{z} \rangle_1 + F_1(z, \overline{z}, \operatorname{Re} w) \tag{9}$$

to a hypersurface germ

$$\operatorname{Im} w = \langle z, \overline{z} \rangle_2 + F_2(z, \overline{z}, \operatorname{Re} w) \tag{10}$$

of the same form, where the components F_1 and F_2 are of weight ≥ 3. Rewriting the condition that (8) takes (9) to (10) in the form of an identity and isolating components of weight 1 and 2 in this identity, we find that the linear part of the map has the form

$$\begin{pmatrix} \Lambda & * \\ 0 & \rho \end{pmatrix},$$

and moreover, $\langle \Lambda z, \overline{\Lambda z} \rangle_2 = \rho \langle z, \overline{z} \rangle_1$. This computation has several corollaries:

(1) the action of holomorphic maps on the Levi form reduces to the action of the Lie group $GL(n, \mathbb{C}) \oplus \mathbb{R}^*$ by the formula $\langle z, \overline{z} \rangle \to \rho \langle \Lambda^{-1} z, \overline{\Lambda^{-1} z} \rangle$;

(2) two model hypersurfaces Q_1 and Q_2 are holomorphically equivalent if and only if they are linearly equivalent;

(3) the group $\operatorname{LAut} Q$ of linear automorphisms of Q preserving the origin has the form $z \to \Lambda z$, $w \to \rho w$, where $\langle \Lambda z, \overline{\Lambda z} \rangle = \rho \langle z, \overline{z} \rangle$, and the algebra g_0 is the Lie algebra of this group.

The algebra $g_+ = g_1 + g_2$ generates the group $\operatorname{Aut}_+ Q$ of non-linear automorphisms of Q that preserve the origin and have linear part with the block form

$$\begin{pmatrix} \mathrm{Id}_n & * \\ 0 & 1 \end{pmatrix}.$$

The group $\operatorname{Aut}_+ Q$ consists of linear-fractional transformations of $\mathbb{C}^{n+1}$ of the form

$$\begin{aligned} z &\to \frac{z + aw}{1 - (2i\langle z, \overline{a} \rangle + (r + i\langle a, \overline{a} \rangle) w)}, \\ w &\to \frac{w}{1 - (2i\langle z, \overline{a} \rangle + (r + i\langle a, \overline{a} \rangle) w)}. \end{aligned}$$

Let us write out the dimensions:

$$\begin{gathered} \dim g_{-2} = \dim g_2 = 1, \qquad \dim g_{-1} = \dim g_1 = 2n, \\ \dim \operatorname{LAut} Q = \dim g_0 = n^2 + 1, \qquad \dim \operatorname{Aut}_- Q = \dim \operatorname{Aut}_+ Q = 2n + 1, \\ \dim \operatorname{Aut} Q = (n+2)^2 - 1. \end{gathered}$$

For $n > 1$ there are finitely many pairwise non-equivalent model hypersurfaces. If (μ, ν) is the signature of a non-degenerate form, then the only invariant is $|\mu - \nu|$. A model hypersurface is equivalent to the hypersphere only if the form is sign definite.

For $n=2$ there are two non-equivalent models, $v=|z_1|^2+|z_2|^2$ and $v=|z_1|^2-|z_2|^2$. Despite the existence of non-equivalent hyperquadrics, the dimensions of the groups are the same for all classes.

The hyperquadric is not the only homogeneous hypersurface in the complex space. The list of such surfaces in $\mathbb{C}^2$ was obtained in 1932 by E. Cartan [20]. It includes separate hypersurfaces as well as families depending on one real parameter. Cartan's list is also remarkable in that all hypersurfaces are described as level lines of elementary functions.

In a series of papers (see [40] and other papers), Loboda attempted to construct a classification of homogeneous hypersurfaces in $\mathbb{C}^3$ and gave a long list of such surfaces. The work is apparently close to completion. Loboda's list shares the properties of Cartan's list: the moduli space is finite-dimensional (two-dimensional families), and the surfaces are represented via elementary functions.

§ 4. A quadratic model of a higher-codimensional germ

4.1. Construction of the model. Now let M_ξ be a smooth surface germ of arbitrary dimension $k \geqslant 1$ in $\mathbb{C}^N$, that is, the set of common zeros of several smooth function germs vanishing at ξ. Thus, $M_\xi=\{\zeta\in\mathbb{C}^N : \rho_1(\zeta)=\dots=\rho_k(\zeta)=0\}$. The smoothness of the germ is guaranteed by the linear independence of the gradients $(\operatorname{grad}\rho_1(\xi),\dots,\operatorname{grad}\rho_k(\xi))$ viewed as vectors in the space $\mathbb{R}^{2N}$. However, even at the level of 1-jets one can encounter a certain relation, which we wish to eliminate. Thus, we require the gradients to be linearly independent over the field of complex numbers. In this case, the germ is referred to as generating, and if n is the dimension of the complex tangent space, then $n+k=N$ and the real dimension of the germ is $2n+k$.

Thus, let M_ξ be a generating surface germ of type (n,k) in $\mathbb{C}^{n+k}$, where $n\geqslant 1$, $k\geqslant 1$. By choosing appropriate coordinates $(z=(z_1,\dots,z_n),\ w=(w_1,\dots,w_k))$, one can reduce the germ equations to the form $\operatorname{Im} w = F(z,\overline{z},\operatorname{Re} w)$, where F has no free or linear terms. Introducing weights of the coordinates and performing a quadratic transformation by analogy with the case of a hypersurface, we obtain the germ equation in the form

$$\operatorname{Im} w = \langle z,\overline{z}\rangle + O(3). \tag{11}$$

The surface $Q=\{\operatorname{Im} w=\langle z,\overline{z}\rangle\}$ will be called the tangent quadric of M_ξ by analogy with the case of hypersurfaces. The main distinction from that case is that now the Hermitian form $\langle z,\overline{z}\rangle=(\langle z,\overline{z}\rangle_1,\dots,\langle z,\overline{z}\rangle_k)$ is vector-valued.

The first problem that arises here is finding a criterion for the automorphism group of the quadric to be finite-dimensional. There are two obvious cases in which this group fails to be finite-dimensional. First, the form may have a non-zero kernel. Suppose that there is a vector $e\neq 0$ such that $\langle e,\overline{z}\rangle=0$ for all $z\in\mathbb{C}^n$. We perform a linear change of variables in $\mathbb{C}^n$ taking e to the first vector $(1,0,\dots,0)$ of the standard basis. After that, the form does not contain the variable z_1 in any of the coordinates. An arbitrary change of variables of the form $z_1\to f(z_1)$, where $f'(0)\neq 0$ (the remaining variables are not affected), is now an automorphism, and so the group $\operatorname{Aut} Q$ is infinite-dimensional. Second, the coordinate Hermitian forms may be linearly dependent. For example, let $\langle z,\overline{z}\rangle_1$ be a linear combination of the other forms. Then after an appropriate linear change of the variable w the form $\langle z,\overline{z}\rangle_1$ is equal to zero. The change of variables $g_1 \to g(w_1)$ (the remaining variables are not affected), where g is an arbitrary locally invertible power series with real coefficients, is now an automorphism, and the group $\operatorname{Aut} Q$ is again infinite-dimensional. It turns out that there are no other cases in which the group can be infinite-dimensional. For this reason these two conditions are taken as the definition of non-degeneracy of a vector-valued Hermitian form [11].

Definition 1. We say that a Hermitian form $\langle z,\overline{z}\rangle = (\langle z,\overline{z}\rangle_1,\dots,\langle z,\overline{z}\rangle_k)$ and the corresponding quadric are non-degenerate if the following conditions hold:

(a) if $\langle e,\overline{z}\rangle = 0$ for all z, then $e = 0$;
(b) the forms $(\langle z,\overline{z}\rangle_1,\dots,\langle z,\overline{z}\rangle_k)$ are linearly independent.

If $k = 1$, then condition (a) is just ordinary non-degeneracy and condition (b) follows from (a). We note that condition (b) cannot hold for $k > n^2$; thus, a generic quadric is non-degenerate only if the codimension lies in the range $k = 1,\dots,n^2$.

It turns out that a quadric of type (n,k) is a good model of a germ of the same type for $k \leqslant n^2$; namely, the following assertions hold ([11], [12]).

1. *Universality*: an arbitrary generating germ of type (n,k) in $\mathbb{C}^{n+k}$ is equivalent to a germ of the form (11).

2. *Finite dimension*: (a) the group of holomorphic automorphisms of a generic quadric is a finite-dimensional Lie group; (b) the group of holomorphic automorphisms of Q is finite-dimensional if and only if the form $\langle z,\overline{z}\rangle$ is non-degenerate (in the sense of Definition 1); (c) every surface specified by equations of degree less than two has an infinite-dimensional group.

3. *Homogeneity*: the quadric Q is homogeneous; that is, its holomorphic automorphisms act on Q transitively. The homogeneity is provided by affine automorphisms.

4. *Symmetry*: (a) the quadric is the most symmetric non-degenerate surface in that the dimension of the germ group of a non-degenerate surface does not exceed the dimension of the germ group of the tangent quadric; (b) the automorphism algebra of the quadric parametrizes the family of maps of one non-degenerate germ into another.

5. *Algebraic properties*: (a) the Lie algebra of holomorphic vector fields on a non-degenerate quadric is an algebra of polynomial vector fields of bounded degree, and the degrees of the coefficients do not exceed two; (b) the automorphism group of a non-degenerate quadric is a Lie subgroup of the group of birational transformations of $\mathbb{C}^{n+k}$ with uniformly bounded degrees [60]; more precisely, one can estimate the degrees of numerators and denominators in a non-cancellable representation by $4(n+k)$ [13]; (c) if two germs are equivalent, then so are their tangent quadrics; two quadrics are holomorphically equivalent if and only if they are linearly equivalent.

The algebra $\operatorname{aut} Q$ has the same structure $\operatorname{aut} Q = g_{-2}+g_{-1}+g_0+g_1+g_2$ (and the same gradation), and the components admit the following explicit description:

$$g_{-2}=\biggl\{2\operatorname{Re}\biggl(q\frac{\partial}{\partial w}\biggr)\biggr\}, \quad \text{where} \quad q\in\mathbb{R}^k;$$

$$g_{-1}=\biggl\{2\operatorname{Re}\biggl(p\frac{\partial}{\partial z}+2i\langle z,\overline{p}\rangle\frac{\partial}{\partial w}\biggr)\biggr\}, \quad \text{where} \quad p\in\mathbb{C}^n;$$

$$g_0=\biggl\{2\operatorname{Re}\biggl(\lambda z\frac{\partial}{\partial z}+\rho w\frac{\partial}{\partial w}\biggr)\biggr\},$$

where λ and ρ are an $n\times n$ and a $k\times k$ matrix, respectively, related by the formula $2\operatorname{Re}\langle\lambda z,\overline{z}\rangle=\rho\langle z,\overline{z}\rangle$;

$$g_1=\biggl\{2\operatorname{Re}\biggl((aw+A(z,z))\frac{\partial}{\partial z}+2i\langle z,\overline{a}w\rangle\frac{\partial}{\partial w}\biggr)\biggr\},$$

where A is a quadratic $\mathbb{C}^n$-valued form and a is a linear map from $\mathbb{C}^k$ into $\mathbb{C}^n$ related to A by the formula $\langle A(z,z),\overline{z}\rangle=2i\langle z,\overline{a}\langle z,\overline{z}\rangle\rangle$;

$$g_2=\biggl\{2\operatorname{Re}\biggl((B(z,w))\frac{\partial}{\partial z}+r(w,w)\frac{\partial}{\partial w}\biggr)\biggr\},$$

where B is a bilinear $\mathbb{C}^n$-valued form and r is a quadratic $\mathbb{C}^k$-valued form related to B by the formula $\operatorname{Re}\langle B(z,u),\overline{z}\rangle=r(\langle z,\overline{z}\rangle,u)$, $\operatorname{Im}\langle B(z,\langle z,\overline{z}\rangle),\overline{z}\rangle=0$.

The group corresponding to $g_-=g_{-2}+g_{-1}$ is completely similar to the group arising for $k=1$. This is the group $\operatorname{Aut}_- Q$ of affine transformations of the form

$$\bigl\{z\to z+p,\ w\to w+2i\langle z,\overline{p}\rangle+(q+i\langle p,\overline{p}\rangle)\bigr\},$$

which acts on Q transitively (an analogue of the Heisenberg group).

One can readily show that the subgroup $\operatorname{LAut} Q$ of linear automorphisms consists of transformations $z\to\Lambda z$, $w\to\rho w$, where $\Lambda\in GL(n,\mathbb{C})$, $\rho\in GL(k,\mathbb{R})$, and moreover, $\langle\Lambda z,\overline{\Lambda z}\rangle=\rho\langle z,\overline{z}\rangle$. The algebra g_0 is the Lie algebra of this group.

The subalgebra $g_+=g_1+g_2$ generates the subgroup $\operatorname{Aut}_+ Q$ of transformations Q of the form $z\to z+f_2+\cdots$, $w\to w+g_3+\cdots$. This is a subgroup of non-linear automorphisms preserving the origin. There is no clear explicit description of this subgroup similar to that in the case $k=1$. Results concerning the structure of this subgroup will be discussed later in this paper. Here we mention only that instead of linear-fractional transformations one has birational transformations of bounded degree (Tumanov's theorem); we have included this result in the list as property 5(b).

The key result in the list is the assertion that the algebra of a non-degenerate quadric consists of fields with quadratic coefficients. For $k=1$ this follows from the main lemma in the first part of the Chern–Moser paper [21]. For $k>1$ there is a very similar construction that permits one to reduce the solution of the main equation (the tangency condition) in formal series to the solution of a system of linear differential equations with constant coefficients. However, in contrast to the case of a hypersurface, these are partial rather than ordinary differential equations. The main role in the derivation of the criterion for the automorphism group to be finite-dimensional is played by the Ehrenpreis–Malgrange–Palamodov theorem on an exponential representation of solutions of a system of linear partial differential equations with constant coefficients. By this theorem, the solution space of the system is finite-dimensional if and only if the characteristic set is finite. The non-degeneracy of the Hermitian form for the system of equations to which the tangency condition can be reduced implies that the characteristic set contains only the origin. This simultaneously proves that the solutions are polynomial. An additional argument shows that the degrees of the solutions actually do not exceed two [11], [12].

Item 5(c) of the list follows from the fact that the action of holomorphic maps on surface germs induces a linear action on the tangent quadrics. Indeed, suppose that there are two equivalent germs

$$M_1 = \{\operatorname{Im} w = \langle z, \overline{z}\rangle_1 + \cdots\} \quad \text{and} \tag{12}$$
$$M_2 = \{\operatorname{Im} w = \langle z, \overline{z}\rangle_2 + \cdots\} \tag{13}$$

and $z \to f(z,w)$, $w \to g(z,w)$ is a map of the first germ into the second. By analyzing lower-order components of the corresponding relation, one can readily see that $f = \Lambda z + f_2 + \cdots$, $g = \rho w_2 + g_3 + \cdots$, and moreover, $\langle \Lambda z, \overline{\Lambda z}\rangle_2 = \rho\langle z, \overline{z}\rangle_1$. Two conclusions are now in order. First, if M_1 and M_2 are quadrics, then the linear map $z \to \Lambda z$, $w \to \rho w$ takes M_1 to M_2. Second, the action of a holomorphic map on a surface germ induces the following action of the group $GL(n,\mathbb{C}) \oplus GL(k,\mathbb{R})$ on the form $\langle z, \overline{z}\rangle$: $\langle z, \overline{z}\rangle \to \rho\langle \Lambda^{-1}z, \overline{\Lambda^{-1}z}\rangle$. In turn, this means that all invariants of the form with respect to this action are invariants of the germ with respect to biholomorphic transformations. We shall return to the construction of these invariants in what follows.

The Levi form has an intrinsic definition (for example, see [22]) via the commutator of vector fields. Let us choose a complement of the complex tangent space in the entire tangent space. Then the Levi form is a Hermitian form on the complex tangent space with values in this complement and coinciding with the Levi form $\langle z, \overline{z}\rangle$ of the germ at the marked point. This definition can be extended to abstract CR-manifolds. If two CR-manifold germs are CR-equivalent, then their Levi forms are related by the action of $GL(n,\mathbb{C}) \oplus GL(k,\mathbb{R})$ mentioned above. Hence arbitrary invariants of a vector Hermitian form are automatically CR-invariants of a CR-manifold.

The non-degeneracy condition is the weakest condition that guarantees finite dimensionality. There are other natural conditions that imply the non-degeneracy condition.

For example, a quadric is said to be strongly non-degenerate if the coordinate forms $\langle z, \overline{z}\rangle = (\langle z, \overline{z}\rangle_1, \ldots, \langle z, \overline{z}\rangle_k)$ are linearly independent and some linear combination of these forms is non-degenerate in the ordinary (scalar) sense. The two definitions are equivalent for quadrics of type $(2,2)$. The case of type $(3,2)$ is the lowest-dimensional type in which they differ. The simplest example of a non-degenerate quadric that is not strongly non-degenerate is as follows:

$$H_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

A strongly non-degenerate quadric is said to be positive definite if there is a positive-definite linear combination of coordinate forms. Positive-definite quadrics are the skeletons (Shilov boundaries) of Siegel domains of second kind. Siegel domains play an important role in the theory of homogeneous bounded domains and automorphic functions [34], [48], [42].

To each vector-valued Hermitian form $\langle z, \overline{z} \rangle$ one can assign the cone $C = \operatorname{int}(\operatorname{conv}(\{\langle z, \overline{z} \rangle : z \in \mathbb{C}^n\}))$, that is, the set of interior points of the convex hull of the image of $\mathbb{C}^n$ under the map specified by the form. If the cone C is non-empty and acute, then the corresponding Siegel domain

$$D = \big\{(z, w) \in \mathbb{C}^n \oplus \mathbb{C}^k : \operatorname{Im} w - \langle z, \overline{z} \rangle \in C\big\}$$

is a homogeneous domain biholomorphically equivalent to a bounded domain.

Moreover, D is the hull of holomorphy of the quadric $Q = \{(z, w) \in \mathbb{C}^n \oplus \mathbb{C}^k : \operatorname{Im} w = \langle z, \overline{z} \rangle\}$ [42]. Thus, positive-definite quadrics are shared by the theory of homogeneous bounded domains and our theory of good models. A number of results discussed here (the finite dimensionality of the automorphism group, the birationality of automorphisms, and the description of the structure of the infinitesimal automorphism algebra [50]) were obtained in the special case of positive-definite quadrics as results about Siegel domains and their skeletons.

4.2. The classification of quadrics. We have already noted that the action of holomorphic maps on germs of type (n, k) induces the following action of the group $GL(n, \mathbb{C}) \oplus GL(k, \mathbb{R})$ on the form $\langle z, \overline{z} \rangle$: $\langle z, \overline{z} \rangle \to \rho \langle \Lambda^{-1} z, \overline{\Lambda^{-1} z} \rangle$. If we represent the form by a finite sequence $(H_1, \ldots, H_k)$ of Hermitian matrices, then the action is given by $(\Lambda, \rho)(H_1, \ldots, H_k) = \rho((\Lambda^*)^{-1} H_1 \Lambda^{-1}, \ldots, (\Lambda^*)^{-1} H_k \Lambda^{-1})$. There is a simple way to reduce this action to an action of $GL(n, \mathbb{C})$. One proceeds from the representation of a quadric by a set of k linearly independent vectors in the real linear space of Hermitian matrices to the representation by a k-dimensional plane in this space. Thus, if we do not distinguish quadrics that differ by a linear change of the variable w, then the set of quadrics of type (n, k) can be identified with the real Grassmannian manifold $Gr_k(\mathbb{R}^{n^2})$. In this representation, the action is reduced to the standard action of $GL(n, \mathbb{C})$ on this Grassmannian manifold. Classes of equivalent quadrics are orbits of this action. The linear subgroup of the automorphism group of a quadric corresponds to the subgroup of $GL(n, \mathbb{C})$ preserving the corresponding k-dimensional plane.

The classification of quadrics of arbitrary type with respect to this action seems to be intractable. However, the problem has a fairly good answer for some types of quadrics, namely, for $k = 1, 2, n^2 - 2, n^2 - 1, n^2$. The number of equivalence classes of non-degenerate quadrics of type $(n, 1)$ is finite and is equal to the integer part of $\frac{n+2}{2}$. All non-degenerate quadrics of type (n, n^2) are equivalent, since each of them can be represented by a basis consisting of n^2 matrices and any two bases are related by a linear transformation. Thus, there is only one class of such quadrics. The classification problem for quadrics of codimension two was completely solved by Shevchenko [52], [53].

4.3. Quadrics of codimension two. Quadrics of codimension two are quadrics of type $(n, 2)$, the type that follows hyperquadrics. Each quadric of type $(n, 2)$ is specified by a pair of Hermitian matrices H_1 and H_2 by the formula $Q = \{\operatorname{Im} w_1 = H_1 z \cdot \overline{z},\ \operatorname{Im} w_2 = H_2 z \cdot \overline{z}\}$. The classification of a pair of Hermitian forms modulo linear changes of the variable z is a classical problem of linear algebra. It is well known that two forms can be reduced to a diagonal form simultaneously provided that one of them is positive definite. The theorem [35] on the canonical form of a pair of forms one of which is non-degenerate is also fairly well known. However, it is not sufficient even for the classification of non-degenerate quadrics, since there are non-degenerate quadrics of codimension two such that all linear combinations of coordinate forms are degenerate. Thus, one needs the most general classification theorem [62]. Using this classification together with the additional possibility of passing to linear combinations of the forms, Shevchenko constructed a complete classification of quadrics of codimension two.

To each pair (H_1, H_2) of Hermitian matrices one assigns the matrix pencil $(t_1 H_1 + t_1 H_2)$, its characteristic polynomial $\det(t_1 H_1 + t_2 H_2)$, where $(t_1 : t_2) \in \mathbb{CP}^1$, the set of roots of this polynomial, and also a set of integer parameters including the minimal indices and factors of inertia (see [62]). Two pairs of Hermitian forms are equivalent with respect to changes of the variable z if and only if they have the same roots, minimal indices of inertia, and factors of inertia. By studying the action of $GL(2, \mathbb{R})$ (changes of the variable w) on this set of data, one can obtain a criterion for the equivalence of two quadrics. Namely, the first set of roots must be taken to the second by a linear-fractional transformation with real coefficients, and some relations between the minimal indices and factors of inertia of the first and the second pair must hold.

The canonical form of a pair of matrices is block diagonal (like the Jordan normal form), where the diagonals contain pairs of cells of three types. The first type corresponds to a real root, and the parameters of the corresponding cells include the root itself, the cell size, and the inertia factor. The second type corresponds to a pair of complex-conjugate roots, and the parameters include one of the roots and the cell size (which is even). The third type corresponds to the root at infinity and is solely determined by the cell size (which is odd). The cell sizes are determined by the elementary divisors of the pencil for the first two types and by the minimal indices for the third type. The canonical form of a strongly non-degenerate quadric contains only cells of the first two types.

In general position, the characteristic polynomial has n distinct roots. For $n \geqslant 4$, a system of n distinct ordered points on the complex plane has $n-3$ numerical invariants (double ratios) with respect to the group of linear-fractional transformations. This set of parameters, together with the discrete invariants, specifies an equivalence class of quadrics of codimension two. It follows that for $n \leqslant 3$ the number of equivalence classes of quadrics of type $(n, 2)$ is finite: there are three classes for $n = 2$ and ten classes for $n = 3$.

The above description of the infinitesimal automorphism algebra of a quadric is explicit modulo finding the values of the parameters $(p, q, \Lambda, \rho, a, A, B, r)$ from the defining relations (see 4.1). For a given quadric, these relations form a system of linear algebraic equations for the unknown parameter values. The coefficients of these equations depend on the entries of the Hermitian matrices specifying the form $\langle z, \overline{z}\rangle$. This dependence is linear in all relations except for one, where the dependence is quadratic. The classification obtained for the case of codimension two permits one to describe the solution space of this system explicitly and to answer a number of qualitative questions, mainly about the dimensions of the graded components of the algebra $\operatorname{aut} Q = g_{-2} + g_{-1} + g_0 + g_1 + g_2$. The parameters p and q specifying the components g_{-2} and g_{-1} are not subjected to any relations, and accordingly, $\dim g_{-2} = 2$ and $\dim g_{-1} = 2n$ for an arbitrary quadric of codimension two. A description of g_0, g_1, and g_2, as well as a computation of their dimensions in terms of invariants of the pencil $t_1 H_1 + t_2 H_2$, can be found in [52]. Let us state some results of that paper.

1. Suppose that the quadric is strongly non-degenerate. We can assume that $\det H_1 \neq 0$ and set $H = (H_1)^{-1} H_2$. There are three possible cases.

1.1. If the matrix H is diagonalizable (by changes of the variable z) and has two distinct eigenvalues, then $\dim g_1 = 2n$ and $\dim g_2 = 2$.

1.2. If H has a single eigenvalue λ and $(H - \lambda E)^2 = 0$, then also $\dim g_1 = 2n$ and $\dim g_2 = 2$.

1.3. In all other cases, $\dim g_1 = 0$ and $\dim g_2 = 0$.

2. If the quadric is non-degenerate but not strongly non-degenerate, then $\dim g_1 < 2n$ and $\dim g_2 = 2$.

3.1. If $n = 2$, then $4 \leqslant \dim g_0 \leqslant 5$.

3.2. If $n \geqslant 3$, then $n + 1 \leqslant \dim g_0 \leqslant (n-1)^2 + 4$.

Thus,

$$10 \leqslant \dim g_0 + \dim g_1 + \dim g_2 \leqslant 11 \text{ for } n = 2;$$

$$n + 1 \leqslant \dim g_0 + \dim g_1 + \dim g_2 \leqslant n^2 + 7 \text{ for } n \geqslant 3.$$

Every non-degenerate quadric of type $(2, 2)$ is equivalent to one of three quadrics Q_1, Q_{-1}, and Q_0. Let us describe these quadrics. First,

$$Q_1 = \left\{\operatorname{Im} w_1 = |z_1|^2 + |z_2|^2,\ \operatorname{Im} w_2 = z_1 \overline{z}_2 + z_2 \overline{z}_1\right\}.$$

The characteristic polynomial of Q_1 has two real roots. This quadric is the direct product of two quadrics of type $(1, 1)$ and can be represented in some other coordinates in the form $Q_1 = \{\operatorname{Im} w_1 = |z_1|^2, \operatorname{Im} w_2 = |z_2|^2\}$. The dimensions of components of the corresponding algebra are $\dim g_0 = 4$, $\dim g_1 = 4$, and $\dim g_2 = 2$. We note that this quadric is positive definite, that is, it is the canonical form of any positive-definite quadric of type $(2, 2)$. Second,

$$Q_{-1} = \left\{\operatorname{Im} w_1 = |z_1|^2 - |z_2|^2,\ \operatorname{Im} w_2 = z_1 \overline{z}_2 + z_2 \overline{z}_1\right\}.$$

The characteristic polynomial of Q_{-1} has two complex-conjugate roots. The dimensions of the components of the corresponding algebra are the same as for Q_1. Finally,

$$Q_0 = \left\{\operatorname{Im} w_1 = |z_1|^2,\ \operatorname{Im} w_2 = z_1 \overline{z}_2 + z_2 \overline{z}_1\right\}.$$

The characteristic polynomial of Q_0 has a single multiple root, $\dim g_0 = 5$, $\dim g_1 = 4$, and $\dim g_2 = 2$.

Ezhov and Schmalz suggested the following terminology: Q_1 is a hyperbolic quadric, Q_{-1} is an elliptic quadric, and Q_0 is a parabolic equation. In the eight-dimensional real linear space of quadrics of type $(2,2)$, quadrics equivalent to Q_1 and Q_{-1} lie on different sides of the conical second-order hypersurface of quadrics equivalent to Q_0. All three quadrics are strongly non-degenerate, and so each non-degenerate quadric of type $(2,2)$ is strongly non-degenerate [39], [12], [26].

Each quadric of type $(3,2)$ is equivalent to a single quadric from the following list, which contains ten entries. We arrange the data about the dimensions of the components g_0, g_1, and g_2 for each quadric in the form $(d_0+d_1+d_2=d)$.

The quadrics Q_1 and Q_2 correspond to the case of three distinct real eigenvalues, which implies that the quadric is diagonalizable (simultaneous reduction to sums of squares). Furthermore, Q_1 is positive definite.

$$Q_1=\bigl\{\operatorname{Im} w_1=|z_1|^2+|z_2|^2+|z_3|^2,\ \operatorname{Im} w_2=|z_2|^2-|z_3|^2\bigr\} \qquad (4+0+0=4),$$
$$Q_2=\bigl\{\operatorname{Im} w_1=-|z_1|^2+|z_2|^2+|z_3|^2,\ \operatorname{Im} w_2=|z_2|^2-|z_3|^2\bigr\} \qquad (4+0+0=4).$$

The following four quadrics correspond to the case of two real eigenvalues. All cells in the canonical form of Q_3 and Q_4 are of first order, and both quadrics are reducible (to a direct product of two hyperquadrics); moreover, Q_3 is positive definite and Q_4 is not. The canonical forms of Q_5 and Q_6 contain a second-order cell.

$$Q_3=\bigl\{\operatorname{Im} w_1=|z_1|^2+|z_2|^2,\ \operatorname{Im} w_2=|z_3|^2\bigr\} \qquad (7+6+2=15),$$
$$Q_4=\bigl\{\operatorname{Im} w_1=-|z_1|^2+|z_2|^2,\ \operatorname{Im} w_2=|z_3|^2\bigr\} \qquad (7+6+2=15),$$
$$Q_5=\bigl\{\operatorname{Im} w_1=|z_1|^2+2\operatorname{Re}(z_2\overline{z}_3),\ \operatorname{Im} w_2=|z_1|^2+|z_2|^2\bigr\} \qquad (6+0+0=6),$$
$$Q_6=\bigl\{\operatorname{Im} w_1=-|z_1|^2+2\operatorname{Re}(z_2\overline{z}_3),\ \operatorname{Im} w_2=-|z_1|^2+|z_2|^2\bigr\} \qquad (6+0+0=6).$$

The following two quadrics correspond to the case of a single eigenvalue; Q_7 has first- and second-order cells, while Q_8 has a third-order cell.

$$Q_7=\bigl\{\operatorname{Im} w_1=|z_1|^2+2\operatorname{Re}(z_2\overline{z}_3),\ \operatorname{Im} w_2=|z_2|^2\bigr\} \qquad (8+6+2=16),$$
$$Q_8=\bigl\{\operatorname{Im} w_1=|z_2|^2+2\operatorname{Re}(z_1\overline{z}_3),\ \operatorname{Im} w_2=2\operatorname{Re}(z_1\overline{z}_2)\bigr\} \qquad (6+0+0=6).$$

The following quadric corresponds to the case of one real and two complex-conjugate eigenvalues:

$$Q_9=\bigl\{\operatorname{Im} w_1=|z_1|^2+2\operatorname{Re}(z_2\overline{z}_3),\ \operatorname{Im} w_2=2\operatorname{Im}(z_2\overline{z}_3)\bigr\} \qquad (4+0+0=4).$$

The tenth and last quadric is non-degenerate but not strongly non-degenerate (a zero-quadric). Its canonical form is given by a cell of the third type.

$$Q_{10}=\bigl\{\operatorname{Im} w_1=2\operatorname{Re}(z_1\overline{z}_3),\ \operatorname{Im} w_2=2\operatorname{Re}(z_2\overline{z}_3)\bigr\} \qquad (8+2+0=10).$$

The dimension of the solution space of a system of linear equations is determined by the rank of the system. Since the coefficients of our system depend on the coefficients of the Hermitian forms at most quadratically, it follows that the stratification of quadrics with respect to the ranks of graded components is given by polynomial relations. (Certain minors are equated to zero.) Hence, all dimensions attain their minimum possible values outside a proper algebraic subset of the space of quadrics (the direct sum of k copies of the space of Hermitian matrices). Thus, a generic quadric of type $(3,2)$ is either Q_1, or Q_2, or Q_9. In each of these three cases, there are three distinct eigenvalues. They are real in the first two cases, and there is a pair of complex-conjugate eigenvalues in the third case. The first case, in contrast with the second, is positive definite.

A generic quadric of type $(n,2)$ has the following dimensions of components for $n \geqslant 3$: $\dim g_0 = n+1$ and $\dim g_1 = \dim g_2 = 0$. The fact that this lower bound is attained and hence indeed is realized for a generic quadric can be proved by considering an arbitrary diagonalizable quadric with distinct roots of the characteristic polynomial, say,

$$\bigl\{\operatorname{Im} w_1 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2,\ \operatorname{Im} w_2 = 1|z_1|^2 + 2|z_2|^2 + \cdots + n|z_n|^2\bigr\}.$$

The upper bounds for the dimensions of components for a quadric of type $(n,2)$, $n \geqslant 2$, are attained at the quadric

$$\bigl\{\operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2) + |z_3|^2 + \cdots + |z_n|^2,\ \operatorname{Im} w_2 = |z_1|^2\bigr\}.$$

The dimensions of components are $\dim g_{-2} = \dim g_2 = 2$, $\dim g_{-1} = \dim g_1 = 2n$, and $\dim g_0 = (n-1)^2 + 4$. We note that here, owing to g_0, the total dimension is greater by one than the dimension of the algebra corresponding to the reducible quadric

$$\bigl\{\operatorname{Im} w_1 = |z_2|^2 + |z_3|^2 + \cdots + |z_n|^2,\ \operatorname{Im} w_2 = |z_1|^2\bigr\}.$$

This description of the automorphism algebra enabled Ezhov and Schmalz to prove that an arbitrary automorphism of a non-degenerate $(n,2)$-quadric is realized by a rational map of degree ≤ 2 [26]. We recall that automorphisms of hyperquadrics are linear-fractional transformations. For strongly non-degenerate quadrics of codimension two this result was obtained earlier by Abrosimov [1], whose work is based on a different technique and is independent of Shevchenko's classification.

4.4. The rigidity phenomenon and exceptional quadrics. The one-codimensional case has a peculiarity rarely observed in higher codimensions: a small deformation of a non-degenerate hyperquadric in the class of hyperquadrics gives an equivalent hyperquadric. This permits one to apply the technique due to E. Cartan and methods of the theory of G-structures ([55], [58], [21], Part II, [61]). There are two other cases with the same property, namely, the last two possible codimensions $k = n^2 - 1$ and $k = n^2$ of non-degenerate quadrics. Moreover, these cases are also distinguished in another respect: every such quadric has a non-trivial non-linear subgroup $\operatorname{Aut}_+ Q$. It turns out that for $2 \leqslant k \leqslant n^2 - 2$ a quadric in general position (that is, outside some proper algebraic set) has no non-linear automorphisms.

Moreover, for $3 \leqslant k \leqslant n^2 - 3$ the linear subgroup $\operatorname{LAut} Q$ of the automorphism group of a generic quadric is trivial in the sense that it contains only scalar dilations of the form $\{z \to \lambda z,\ w \to |\lambda|^2 w,\ \lambda \in \mathbb{C}^*\}$ [12], [45], [30]. Such quadrics are said to be rigid. The absence of non-linear automorphisms is referred to as N-rigidity, and the absence of non-trivial linear automorphisms is called L-rigidity; thus, one speaks of rigidity if both conditions hold simultaneously. One can also speak of g_0-rigidity ($\dim g_0 = 2$), g_1-rigidity ($\dim g_1 = 0$), and g_2-rigidity ($\dim g_2 = 0$). These conditions are related to each other; for example, g_2-rigidity follows from g_0-rigidity [12] as well as from g_1-rigidity [44]. The automorphism group of a rigid quadric consists of affine transformations, forming the same group $\operatorname{Aut}_- Q$ for all quadrics of given type, and scalar dilations. The cases $k = 2$ and $k = n^2 - 2$ require special treatment, since the subgroup of linear automorphisms of a generic quadric is $(n+1)$-dimensional and hence is not exhausted by scalar dilations in these cases.

There is a deep cause behind the symmetry between quadrics of types (n, k) and $(n, n^2 - k)$. A quadric of codimension k is a set of k linearly independent Hermitian matrices. If we do not distinguish quadrics related by a linear change of the variable w, then this is a k-dimensional plane in the n^2-dimensional real linear space $\mathcal{H}$ of Hermitian matrices. This plane admits a dual description as the set of common zeros of $n^2 - k$ linearly independent linear forms on $\mathcal{H}$. By choosing an isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$, we obtain a correspondence between quadrics of types (n, k) and $(n, n^2 - k)$. Moreover, the linear subgroups $\operatorname{LAut} Q$ corresponding to dual quadrics prove to be isomorphic.

Although a generic quadric of codimension between 3 and $n^2 - 3$ is rigid, this does not prohibit the existence of exceptional quadrics that have non-linear automorphisms and a non-trivial linear subgroup. There is a simple method for constructing such quadrics: one takes direct products of quadrics. If the factors are non-degenerate, then the algebra corresponding to the product is the direct sum of the algebras corresponding to the factors, and accordingly, the automorphism group is the direct product of the automorphism groups of the factors [12]. If the factors have non-trivial groups, then so does the product. The list of quadrics of type $(3, 2)$ contains the reducible quadrics Q_3 and Q_4. The dimension of the subgroup preserving the origin is equal to 15 for both quadrics. It is clear why these quadrics have large automorphism groups. The dimension is equal to the sum of the dimensions of the corresponding groups for the factors: 10 for a hyperquadric of type $(2, 1)$ and 5 for a hyperquadric of type $(1, 1)$. However, the list also contains the irreducible quadric Q_7 with the corresponding stabilizer subgroup of dimension 16. Thus, the classification problem for quadrics with non-linear automorphisms and the problem of finding a maximally symmetric non-degenerate quadric, that is, a quadric whose group is of maximal dimension for a given type, is rather interesting. For quadrics of codimension two this problem can be solved on the basis of Shevchenko's classification (see above).

For quadrics of type $(3, 3)$, that is, quadrics of codimension three in $\mathbb{C}^6$, this problem was solved by Palinchak [43], [44]. Each $(3, 3)$-quadric possessing non-linear automorphisms is equivalent to one of the following eight quadrics. We give the data on the dimensions of components in the same format: $\dim g_0 + \dim g_1 + \dim g_2$. We recall that $\dim g_{-2} + \dim g_{-1} = 3 + 6 = 9$ for all quadrics of this type and $\dim Q = 2n + k = 9$.

$$
\begin{aligned}
&Q_1\colon \quad \operatorname{Im} w_1 = |z_1|^2, \quad \operatorname{Im} w_2 = |z_2|^2, \quad \operatorname{Im} w_3 = |z_3|^2;\\
&Q_2\colon \quad \operatorname{Im} w_1 = |z_1|^2 - |z_2|^2, \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_3 = |z_3|^2;\\
&Q_3\colon \quad \operatorname{Im} w_1 = |z_1|^2, \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_3 = |z_3|^2.
\end{aligned}
$$

Each of the first three quadrics is the direct product of a $(2,2)$-quadric (see the list above) by the unique $(1,1)$-quadric. Since the first $(2,2)$-quadric in the list is the direct product of two $(1,1)$-quadrics, it follows that Q_1 is the direct product of three $(1,1)$-quadrics. The dimensions of the groups are as follows: $Q_1 : (6+6+3 = 15)$, $Q_2 : (6+6+3 = 15)$, and $Q_3 : (7+6+3 = 16)$.

$$
\begin{aligned}
&Q_4 : \quad \operatorname{Im} w_1 = |z_1|^2, \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_3 = 2\operatorname{Re}(z_1\overline{z}_3) + |z_2|^2\\
&\qquad (8+6+3 = 17);\\
&Q_5\colon \quad \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_2\overline{z}_3), \quad \operatorname{Im} w_3 = 2\operatorname{Im}(z_1\overline{z}_2)\\
&\qquad (10+6+3 = 19);\\
&Q_6\colon \quad \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_2\overline{z}_3), \quad \operatorname{Im} w_3 = |z_3|^2\\
&\qquad (10+6+3 = 19);\\
&Q_7\colon \quad \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_2\overline{z}_3), \quad \operatorname{Im} w_3 = 2\operatorname{Im}(z_2\overline{z}_3)\\
&\qquad (8+6+3 = 17);\\
&Q_8\colon \quad \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = 2\operatorname{Im}(z_1\overline{z}_2), \quad \operatorname{Im} w_3 = 2\operatorname{Re}(z_1\overline{z}_3) + |z_2|^2\\
&\qquad (7+2+1 = 10).
\end{aligned}
$$

We recall that a generic $(3,3)$-quadric is rigid: the dimensions are $(2+0+0=2)$. The above list does not contain quadrics whose automorphism group is linear but non-trivial ($\dim g_0 > 2$). However, very coarse estimates show that $\dim g_0 \leqslant 18$. Hence, one can claim that the dimension of the automorphism group of an arbitrary $(3,3)$-quadric does not exceed 19, and if it is equal to 19, then the quadric itself is equivalent to Q_5 or Q_6. Arbatskii [4] computed the automorphism group for each of the quadrics in the list. All automorphisms are realized by birational transformations of $\mathbb{C}^6$ of degree ≤ 3.

A generic $(3,4)$-quadric is rigid, that is, has the structure $(2+0+0)$. The classification of quadrics of type $(3,4)$ (quadrics of codimension four in $\mathbb{C}^7$) with non-linear automorphisms was constructed by Anisova [2], [3]. Every such quadric is equivalent to one of the following nine pairwise non-equivalent quadrics:

$$
\begin{aligned}
Q_1\colon \quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_2\overline{z}_3),\\
& \operatorname{Im} w_3 = 2\operatorname{Im}(z_1\overline{z}_3), \quad \operatorname{Im} w_4 = 2\operatorname{Im}(z_2\overline{z}_3) \quad (10+6+4 = 20);\\
Q_2\colon \quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_2\overline{z}_3),\\
& \operatorname{Im} w_3 = |z_3|^2, \quad \operatorname{Im} w_4 = 2\operatorname{Im}(z_2\overline{z}_3) \quad (11+6+4 = 21);\\
Q_3\colon \quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_2 = 2\operatorname{Re}(z_1\overline{z}_2),\\
& \operatorname{Im} w_3 = |z_2|^2, \quad \operatorname{Im} w_4 = |z_1|^2 \quad (8+2+1 = 11);
\end{aligned}
$$

$$\begin{aligned}
Q_4\colon\quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = -2\operatorname{Im}(z_1\overline{z}_2),\\
& \operatorname{Im} w_3 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_4 = |z_2|^2 \quad (8+2+1=11);\\
Q_5\colon\quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = -2\operatorname{Im}(z_1\overline{z}_2),\\
& \operatorname{Im} w_3 = 2\operatorname{Re}(z_1\overline{z}_3), \quad \operatorname{Im} w_4 = |z_1|^2 + |z_2|^2 \quad (7+2+1=10);\\
Q_6\colon\quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = -2\operatorname{Im}(z_1\overline{z}_2),\\
& \operatorname{Im} w_3 = |z_1|^2, \quad \operatorname{Im} w_4 = |z_3|^2 \quad (10+4+4=18);\\
Q_7\colon\quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = -2\operatorname{Im}(z_1\overline{z}_2),\\
& \operatorname{Im} w_3 = |z_1|^2, \quad \operatorname{Im} w_4 = 2\operatorname{Re}(z_1\overline{z}_3) + |z_2|^2 \quad (8+4+2=14);\\
Q_8\colon\quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = |z_1|^2,\\
& \operatorname{Im} w_3 = |z_2|^2, \quad \operatorname{Im} w_4 = |z_3|^2 \quad (7+2+2=11);\\
Q_9\colon\quad & \operatorname{Im} w_1 = 2\operatorname{Re}(z_1\overline{z}_2), \quad \operatorname{Im} w_2 = 2\operatorname{Im}(z_1\overline{z}_2),\\
& \operatorname{Im} w_3 = |z_1|^2 - |z_2|^2, \quad \operatorname{Im} w_4 = |z_3|^2 \quad (7+2+1=10).
\end{aligned}$$

We point out that the first two quadrics are not strongly non-degenerate, and moreover, the second quadric Q_2 is the most symmetric quadric of type $(3,4)$. The list includes the reducible quadrics Q_6, Q_8, and Q_9, which are the direct products of various $(2,3)$-quadrics by the unique hyperquadric of type $(1,1)$.

Palinchak's and Anisova's classifications are based on Shevchenko's classification of pairs of third-order Hermitian matrices. The proof of pairwise non-equivalence of the canonical quadrics in both classifications uses the dimensions of the components of the corresponding algebras as well as the quadratic invariants constructed in [13], [14]. These invariants will be discussed below.

One can observe from these classifications that in all cases there is a curious symmetry: $\dim g_j \leqslant \dim g_{-j}$, $j=1,2$, and the upper bound is attainable. In this connection, there was a conjecture that this is true for all types of quadrics. However, Utkin [63] has recently shown that the symmetry can be violated for $(5,3)$-quadrics. His example can be generalized to $n \geqslant 5$ and $k \geqslant 3$.

It would be of interest to estimate the degree of non-linear automorphisms for the exceptional quadrics. By Tumanov's theorem [60], the automorphism group is a Lie group that acts on the quadric by birational transformations of $\mathbb{C}^{n+k}$ of bounded degree. Since the automorphism algebra consists of fields of degree ≤ 2, one can prove that the degrees of the automorphisms do not exceed $4(n+k)$. However, no one has been able to produce quadrics with automorphisms of degree greater than k, even though quite a few situations have been studied. There are fairly many quadrics whose degree is equal to the codimension: they occur in Shevchenko's, Palinchak's, and Anisova's lists, one can readily construct them in the form of a direct product of hyperquadrics, and automorphisms of homogeneous Siegel domains of the second kind have the same degree. But the general assertion is only a conjecture yet.

The degree conjecture. The automorphism group of a non-degenerate quadric of type (n,k) acts by birational transformations of $\mathbb{C}^{n+k}$ whose degrees (the degrees of numerators and denominators in the non-cancellable representation) do not exceed the codimension k.

The sort of groups that arise here as automorphism groups of non-degenerate quadrics is worth attention. These are subgroups of the Cremona group (the group of birational transformations of the complex linear space) consisting of transformations of bounded degree. In general, by substituting a rational transformation into itself, we obtain a transformation of larger degree. This assertion is valid only in general position, and our groups provide counterexamples. There is an example in which it is obvious why the degree remains bounded under the substitution. This example is given by projective transformations. Hence, one naturally attempts to explain the existence of such subgroups by reducing them to projective transformations. In this connection, the following theorem is of interest (Zaitsev [66]).

Let G be a real connected Lie group acting on $\mathbb{C}^n$ by birational transformations holomorphic in some domain (the same for the all elements of the group). Then there is a linear representation of G in some $\mathbb{C}^{N+1}$ and a birational G-equivariant map from $\mathbb{C}^n$ onto $\mathbb{CP}^N$.

A condition ensuring that automorphisms of a domain can be extended to birational transformations of the ambient space was given by Webster [65]. Zaitsev's theorem cannot be applied to automorphism groups of quadrics, but this does not necessarily imply that its conclusion fails. We note, looking slightly ahead, that the automorphism groups of model surfaces of degree greater than two have the same property. They are subgroups of bounded degree in the Cremona group.

4.5. Real associative quadrics. Ezhov and Schmalz [28] described a very curious class of quadrics of type (k,k). Let $\mathbf{A}$ be a real k-dimensional commutative associative algebra, and let $\mathbf{A}_{\mathbb{C}} = \mathbf{A} \otimes \mathbb{C}$ be its complexification. To this algebra we can assign the quadric

$$Q = \left\{ (z,w) \in \mathbf{A}_{\mathbb{C}} \otimes \mathbf{A}_{\mathbb{C}} : \frac{1}{2i}(w - \overline{w}) = z\overline{z} \right\};$$

here the $\mathbb{R}^k$-valued Hermitian form is the product $z\overline{z}$ in the algebra $\mathbf{A}_{\mathbb{C}}$. Conversely, a quadric of type (k,k) admits a representation of this kind if and only if in appropriate coordinates the form $\langle \cdot , \cdot \rangle$ determines a real associative product on $\mathbb{R}^k$. The commutativity of this product follows from the Hermitian property.

These quadrics are called RAQ-quadrics (Real Associative Quadrics).

The associativity condition is equivalent to the requirement that the map

$$\tau \colon \mathbf{A} \to \mathbf{gl}(\mathbb{R}, k), \qquad x \mapsto \langle \cdot , x \rangle$$

be an algebra homomorphism. An RAQ-quadric is non-degenerate if and only if and only if $\mathbf{A}$ is unital. The quadric splits into a direct product whenever the algebra has a direct sum decomposition.

The only non-degenerate $(1,1)$-quadric is an RAQ-quadric. The corresponding algebra is $\mathbb{R}$.

All three non-degenerate $(2,2)$-quadrics Q_1, Q_0, and Q_{-1} are also RAQ-quadrics. The corresponding algebras are $\mathbb{R} \otimes \mathbb{R}$, $\mathbb{C}$, and $\mathbb{R} \otimes \mathbf{n}\mathbb{R}$, respectively. (Here $\mathbf{n}$ is a nilpotent with $\mathbf{n}^2 = 0$.)

For the $(3,3)$-quadrics from Palinchak's list, the situation is as follows. The quadrics Q_1, Q_2, Q_3 are reducible and hence are RAQ-quadrics. Only two of the five irreducible quadrics, namely, Q_4 and Q_6, are RAQ-quadrics. The corresponding algebras are, respectively, $\mathbb{R} \otimes \mathbf{n}_1\mathbb{R} \otimes \mathbf{n}_2\mathbb{R}$, where $\mathbf{n}_1^2 = \mathbf{n}_2^2 = \mathbf{n}_1\mathbf{n}_2 = 0$, and $\mathbb{R} \otimes \mathbf{n}\mathbb{R} \otimes \mathbf{n}^2\mathbb{R}$, where $\mathbf{n}^3 = 0$.

The group $\operatorname{Aut}_+ Q$ of non-linear automorphisms of an RAQ-quadric Q is determined by a formula completely similar to the formula for the automorphisms of the $(1,1)$-quadric:

$$\begin{aligned} z &\mapsto (\mathrm{Id}_k - 2i\overline{a}z - (r + ia\overline{a})w)^{-1}(z + aw), \\ w &\mapsto (\mathrm{Id}_k - 2i\overline{a}z - (r + ia\overline{a})w)^{-1}w, \end{aligned}$$

where $a \in \mathbf{A}_{\mathbb{C}}$ and $r \in \mathbf{A}$. Thus, $\dim \operatorname{Aut}_+ Q = 3k$ for an RAQ-quadric, and moreover, $\dim g_1 = 2k$ and $\dim g_2 = k$. In particular, this means that Palinchak's list contains all $(3,3)$-RAQ-quadrics. It is not clear what causes the existence of non-linear automorphisms for the remaining three quadrics. Even one of the two quadrics Q_5 and Q_6 with the richest automorphism groups, namely, Q_5, is not an RAQ-quadric. Apparently, some other algebraic structures underlie these quadrics.

4.6. The Ezhov–Schmalz matrix arithmetic. The results of this subsection are also due to Ezhov and Schmalz [26]–[29], [31].

Shevchenko's classification shows that $(n,2)$-quadrics possessing non-linear automorphisms can be divided into hyperbolic, elliptic, parabolic, and null quadrics by analogy with the classification of $(2,2)$-quadrics. A hyperbolic quadric is the direct product of two hyperquadrics and can be specified by relations of the form

$$v_1 = \sum_1^r \varepsilon_i |z_i|^2, \qquad v_2 = \sum_{r+1}^n \varepsilon_i |z_i|^2.$$

Elliptic quadrics exist only in even dimensions and are given by the relations

$$v_1 = \sum_1^{n/2} \operatorname{Re}(z_{2i-1}\overline{z}_{2i}), \qquad v_2 = \sum_1^{n/2} \operatorname{Im}(z_{2i-1}\overline{z}_{2i}).$$

Parabolic quadrics are given by

$$v_1 = \sum_1^s \varepsilon_i |z_{2i}|^2, \qquad v_2 = \sum_1^s 2\operatorname{Re}(z_{2i-1}\overline{z}_{2i}) + \sum_{2s+1}^n \varepsilon_i |z_i|^2.$$

In these formulae, $\varepsilon_i = \pm 1$.

We represent the variables z by a matrix Z and the variables w by a matrix W. The correspondence is defined separately for each type of quadric. In the hyperbolic case we set

$$Z = \begin{pmatrix} z_1 & 0 \\ \vdots & \vdots \\ z_r & 0 \\ 0 & z_{r+1} \\ \vdots & \vdots \\ 0 & z_n \end{pmatrix},$$

$$\overline{Z} = \begin{pmatrix} \varepsilon_1 \overline{z}_1 & \dots & \varepsilon_r \overline{z}_r & 0 & \dots & 0 \\ 0 & \dots & 0 & \varepsilon_{r+1} \overline{z}_{z+1} & \dots & \varepsilon_n \overline{z}_n \end{pmatrix},$$

$$W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \qquad \overline{W} = \begin{pmatrix} \overline{w}_1 & 0 \\ 0 & \overline{w}_2 \end{pmatrix}.$$

For elliptic quadrics we set

$$Z = \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \\ \vdots & \vdots \\ z_{n/2-1} & -z_{n/2} \\ z_{n/2} & z_{n/2-1} \end{pmatrix},$$

$$\overline{Z} = \begin{pmatrix} \overline{z}_1 & -\overline{z}_2 & \dots & \overline{z}_{n/2-1} & -\overline{z}_{n/2} \\ \overline{z}_2 & \overline{z}_1 & \dots & \overline{z}_{n/2} & \overline{z}_{n/2-1} \end{pmatrix},$$

$$W = \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix}, \qquad \overline{W} = \begin{pmatrix} \overline{w}_1 & -\overline{w}_2 \\ \overline{w}_2 & \overline{w}_1 \end{pmatrix}.$$

Finally, for parabolic quadrics we set

$$Z = \begin{pmatrix} z_1 & 0 \\ z_2 & z_1 \\ \vdots & \vdots \\ z_{2s-1} & 0 \\ z_{2s} & z_{2s-1} \\ z_{2s+1} & 0 \\ z_n & 0 \end{pmatrix},$$

$$\overline{Z} = \begin{pmatrix} \varepsilon_1 \overline{z}_1 & 0 & \dots & \varepsilon_s \overline{z}_{2s-1} & 0 & 0 & \dots & 0 \\ \overline{z}_2 & \overline{z}_1 & \dots & \overline{z}_{2s} & \overline{z}_{2s-1} & \varepsilon_{2s+1} \overline{z}_{2s+1} & \dots & \varepsilon_n \overline{z}_n \end{pmatrix},$$

$$W = \begin{pmatrix} w_1 & 0 \\ w_2 & w_1 \end{pmatrix}, \qquad \overline{W} = \begin{pmatrix} \overline{w}_1 & 0 \\ \overline{w}_2 & \overline{w}_1 \end{pmatrix}.$$

In all three cases the quadric Q is determined by the same matrix equation

$$\frac{1}{2i}(W - \overline{W}) = \overline{Z} Z,$$

and all automorphisms in $\operatorname{Aut}_+ Q$ are given by a unified formula, completely similar to the formula for linear-fractional automorphisms of the $(1,1)$-quadric:

$$\begin{aligned} Z &\mapsto (Z + AW)(\operatorname{Id} - 2i\overline{A}Z - (R + iA\overline{A})W)^{-1}, \\ W &\mapsto W(\operatorname{Id} - 2i\overline{A}Z - (R + iA\overline{A})W)^{-1}. \end{aligned}$$

The parameters $a \in \mathbb{C}^n$ and $r \in \mathbb{R}^2$ are represented by matrices A and R according to the same rules as z and w. Moreover, $\dim \operatorname{Aut}_+ Q = 2n + k$, $\dim g_1 = 2n$, and $\dim g_2 = 2$ for all three classes. The formula is a matrix linear-fractional expression, and so the degree of the automorphisms described by this formula is equal to two, which corroborates the degree conjecture (the degree does not exceed the codimension).

These three classes do not exhaust all quadrics of codimension two with non-linear automorphisms. The fourth and last class, which is absent in the $(2,2)$ case, is the class of so-called null quadrics. These are non-degenerate quadrics given by pairs of matrices H_1 and H_2 such that all their linear combinations are singular, that is, the characteristic polynomial $P(t) = \det(t_1 H_1 + t_2 H_2)$ is zero identically. This means that the canonical form contains cells of the third type. Automorphisms of null quadrics are projective transformations of $\mathbb{CP}^{n+k}$.

These constructions have been completed by a general formula for matrix linear-fractional automorphisms of an arbitrary quadric.

Let $\mathbf{A}$ be a matrix algebra of the form

$$\mathbf{A} = \bigl\{(D, d) \in gl(\mathbb{C}, n) \times gl(\mathbb{C}, k) : \langle Dz, \overline{z}\rangle = d\langle z, \overline{z}\rangle \text{ for all } z \in \mathbb{C}^n\bigr\}.$$

We consider multilinear maps

$$\begin{gathered} \widehat{A}\colon \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \text{ (not necessarily symmetric)}, \\ a\colon \mathbb{C}^k \to \mathbb{C}^n, \\ B\colon \mathbb{C}^n \otimes \mathbb{C}^k \to \mathbb{C}^n, \\ \widehat{r}\colon \mathbb{C}^k \otimes \mathbb{C}^k \to \mathbb{C}^k \text{ (Hermitian)} \end{gathered}$$

satisfying the relations

$$\begin{gathered} \langle \widehat{A}(z, \zeta), \xi\rangle = 2i\langle z, \overline{a}\langle \xi, \zeta\rangle\rangle, \\ \langle B(z, w), \zeta\rangle = \widehat{r}(w, \langle \zeta, z\rangle). \end{gathered}$$

Then the quadruple (A, a, B, r), where $A(z, z) = \widehat{A}(z, z)$ and $r(u, u) = \widehat{r}(u, u)$, satisfies the relations determining an element of $\operatorname{Aut}_+ Q$ (see 4.1). The converse is not true in general.

The relations determining the quadruple $(\widehat{A}, a, B, \widehat{r})$ are equivalent to the relations

$$\begin{gathered} \bigl(A(z, \,\cdot\,), 2i\langle z, \overline{a}\,\cdot\,\rangle\bigr) \in \mathbf{A}, \\ \bigl(B(\,\cdot\,, w), r(w, \,\cdot\,)\bigr) \in \mathbf{A} \end{gathered}$$

for all z and w. Using this, one can show that the map

$$z \mapsto \bigl(\mathrm{Id}_n - A(z,\,\cdot\,) - B(\,\cdot\,,w) - \tfrac{1}{2}A(aw,\,\cdot\,)\bigr)^{-1}(z+aw),$$
$$w \mapsto \bigl(\mathrm{Id}_k - 2i\langle z, \overline{a}\,\cdot\,\rangle - r(w,\,\cdot\,) - i\langle aw, \overline{a}\,\cdot\,\rangle\bigr)^{-1} w$$

is an automorphism of Q. In view of the analogy between this formula and automorphisms of the 3-sphere in $\mathbb{C}^2$, already known to Poincaré, this formula is called the Poincaré formula. Automorphisms of this type form a subgroup of the automorphism group of the quadric. The automorphisms of a null quadric of codimension two are linear-fractional and hence are given by a formula of this type. Thus, all automorphisms of non-degenerate quadrics of codimension two are described by the Poincaré formula.

The first non-linear automorphisms that are not described by this formula were discovered in the $(3,3)$ case. Of the eight quadrics with non-linear automorphisms found by Palinchak, five $(Q_1, Q_2, Q_3, Q_4, Q_6)$ are RAQ-quadrics, their automorphisms being thereby described by the Poincaré formula. The Poincaré formula also describes the automorphisms of Q_5 and Q_8. The quadric Q_7 has a nine-dimensional group $\mathrm{Aut}_+ Q_7$, but the subgroup of Poincaré automorphisms is trivial.

After this had been discovered, Ezhov and Schmalz constructed a more general formula, which describes the automorphisms of an arbitrary non-degenerate quadric. We do not give this rather complicated expression but only note that it is a matrix linear-fractional expression, that is, not only linear operations and taking the reciprocal are allowed but also products of two matrix expressions [31].

§ 5. Quadratic invariants of *CR*-manifolds: three flag constructions

The absence of non-trivial point-preserving automorphisms for a generic quadric and the non-equivalence of two quadrics chosen at random can be viewed as manifestations of the same phenomenon: quadrics have a rich system of $(GL(n,\mathbb{C}) \oplus GL(k,\mathbb{R}))$-invariants. The same invariants are holomorphic invariants of a C^2 surface in the complex space or even of an abstract, not necessarily integrable CR-structure. The first invariant construction (the characteristic polynomial) is due to Mizner [41], who studied CR-manifolds of codimension two. The topic of CR-invariants was further developed in [13]–[16]. Garrity and Mizner [33] described a system of generators of rational invariant functions. The contents of this section reflects the author's approach [13]–[16].

The first series of invariant sets $\Gamma(p)$ is a system of embedded subsets of $\mathbb{C}^n$ given by homogeneous polynomial conditions. Such sets can be viewed naturally as algebraic subvarieties in $\mathbb{CP}^{n-1}$. Let $(H_1,\dots,H_k)$ be a k-tuple of Hermitian matrices specifying an (n,k)-quadric. Then

$$\Gamma(p) = \bigl\{z \in \mathbb{CP}^{n-1} : \mathrm{rank}_{\mathbb{C}}(H_1 z,\dots,H_k z) \leqslant p\bigr\}.$$

In other words, the set $\Gamma(p)$ consists of non-zero vectors z such that the dimension of the system of k vectors $(H_1 z,\dots,H_k z)$ does not exceed p. Moreover,

$$\varnothing = \Gamma(0) \subseteq \Gamma(1) \subseteq \cdots \subseteq \Gamma(k-1) \subseteq \Gamma(k) = \mathbb{CP}^{n-1},$$

that is, Γ is a system of embedded algebraic varieties (a flag) numbered by the integers $0 \leqslant p \leqslant k$.

The second series of sets $\gamma(q)$, $0 \leqslant q \leqslant n$, is defined as follows:

$$\gamma(q) = \bigl\{w \in \mathbb{CP}^{k-1} : \operatorname{rank}(w_1 H_1 + \dots + w_k H_k) \leqslant q\bigr\}.$$

In other words, the set $\gamma(q)$ consists of non-zero vectors w such that the rank of the matrix $(w_1 H_1 + \dots + w_k H_k)$ does not exceed q. For example, $\gamma(n-1)$ is the zero set of the characteristic polynomial of the pencil $\{P(w) = \det(w_1 H_1 + \dots + w_k H_k) = 0\}$. Here we again have a flag of algebraic subvarieties in the projective space:

$$\varnothing = \gamma(0) \subseteq \gamma(1) \subseteq \dots \subseteq \gamma(n-1) \subseteq \gamma(n) = \mathbb{CP}^{k-1}.$$

If $n = 2$, then $\gamma(n-1) \in \mathbb{CP}^1$ is the set of eigenvalues of the pencil $w_1 H_1 + w_2 H_2$ and $\Gamma(2) \in \mathbb{CP}^{n-1}$ is the set of eigenvectors corresponding to eigenvalues in $\gamma(n-1)$. The flags obtained in the general case are a natural generalization of these problems: the series γ generalizes the eigenvalue problem, and Γ generalizes the eigenvector problem.

The third flag δ arises if one considers singularities of the projectivization of a quadric. Let us pass from the affine coordinates (z, w), $z \in \mathbb{C}^n$, $w \in \mathbb{C}^k$, to projective coordinates (T, Z, W), $T \in \mathbb{C}^1$, $Z \in \mathbb{C}^n$, $W \in \mathbb{C}^k$, by the change of variables $z = Z/T$, $w = W/T$; then the equation of the closure $\overline{Q}$ of the quadric Q acquires the form

$$\operatorname{Im} W\overline{T} = \langle Z, \overline{Z}\rangle.$$

The projectivization of a hyperquadric is projectively holomorphic, projective automorphisms act on it transitively, and the hyperquadric does not have singularities at infinity. However, this is not the case for $k \geqslant 2$. Hypersurfaces whose intersection forms the quadric need not be transversal at infinity ($\{T = 0\}$). The coarsest characteristic of the singularities arising in this case is given by the dimension of the tangent space $T(\overline{Q})$. This dimension determines a stratification of the set $Q^\infty = \overline{Q} \setminus Q$ and is determined by the rank of the system of gradients of the functions determining the quadric. Let $e_j(T, Z, W) = \operatorname{grad}(-\operatorname{Im} W_j\overline{T} + \langle Z, \overline{Z}\rangle)$; then

$$\delta(r) = \bigl\{(T, Z, W) \in \mathbb{CP}^{n+k} : \operatorname{rank}_{\mathbb{C}}(e_1(T, Z, W), \dots, e_k(T, Z, W)) \leqslant r\bigr\}.$$

The system δ is also a flag of algebraic subsets:

$$\varnothing = \delta(0) \subseteq \delta(1) \subseteq \dots \subseteq \delta(n-1) \subseteq \delta(k) = \mathbb{CP}^{n+k}.$$

All three series are determined by inequalities involving ranks in the complex space. One can obtain a finer stratification of Q^∞ by considering the real counterparts of these flags:

$$\Gamma R(p) = \bigl\{z \in \mathbb{RP}^{2n-1} : \operatorname{rank}_{\mathbb{R}}(H_1 z, \dots, H_k z) \leqslant p\bigr\},$$
$$\gamma R(q) = \bigl\{u \in \mathbb{RP}^{k-1} : \operatorname{rank}(u_1 H_1 + \dots + u_k H_k) \leqslant q\bigr\},$$
$$\delta R(r) = \bigl\{(T, Z, W) \in \mathbb{RP}^{2n+2k} : \operatorname{rank}_{\mathbb{R}}(e_1(T, Z, W), \dots, e_k(T, Z, W)) \leqslant r\bigr\}.$$

We also note two other invariant sets, namely, the Hermitian light cone $S = \{z \in \mathbb{RP}^{2n-1} : \langle z, \overline{z} \rangle = 0\}$ and the zero set $S' = \{(z, \zeta) \in \mathbb{CP}^{2n-1} : \langle z, \zeta \rangle = 0\}$ of a bilinear form.

If two germs are equivalent by a holomorphic map or a CR-map, then their tangent quadrics are also equivalent, the equivalence being given by a linear map of the form $z \to Cz$, $w \to \rho w$. That is why all above-mentioned subsets of the first quadric are linearly equivalent to the respective subsets of the second quadric. This equivalence is established:

for the flags Γ and ΓR, as well as for S_1, S_2, and S_3, by the induced action of C on $\mathbb{CP}^{n-1}$ and $\mathbb{RP}^{2n-1}$;

for the flags γ and γR, by the induced action of the matrix ρ' (the transpose of ρ) on $\mathbb{CP}^{k-1}$ and $\mathbb{RP}^{k-1}$;

for the flags δ and δR, by the induced action of the pair (C, ρ) on $\mathbb{CP}^{n+k}$ and $\mathbb{RP}^{2n+2k}$.

Since w is subjected to real changes of variables, it follows that the real subspace $\mathbb{R}^k = \{\operatorname{Im} w = 0\}$ is invariant and γR is none other than $\gamma \cap \mathbb{R}^k$.

The characteristic polynomial $P(w) = \det(w_1 H_1 + \cdots + w_k H_k)$ is transformed as follows: $P(w) \to |\det \Lambda|^{-2} P(\rho' w)$.

Thus, all linear invariants of all the above-mentioned sets constructed for some given form $\langle z, \overline{z} \rangle$ are biholomorphic invariants of a quadric and a surface germ as well as CR-invariants of a CR-manifold germ with a given Levi form. Of these invariants, we mention first of all the sets of dimensions and degrees. The list of integer invariants also includes the dimensions ($\dim g_0$, $\dim g_1$, $\dim g_2$) of graded components. A biholomorphic (or CR-) equivalence of two real manifolds implies the linear equivalence of all the sets constructed above and the equality of all of their linear invariants at each pair of the corresponding points.

For $(2,2)$-quadrics each of the flags $\Gamma, \Gamma R, \gamma, \gamma R, \delta, \delta R$ has only one informative component, namely, the first. For the three standard $(2,2)$-quadrics, the picture is as follows:

Q_1:	$\Gamma(1)$ consists of two points;	$\Gamma R(1)$ consists of two points;
	$\gamma(1)$ consists of two points;	$\gamma R(1)$ consists of two points;
	$\delta(1)$ consists of one point;	$\delta R(1)$ consists of one point;
Q_0:	$\Gamma(1)$ consists of one point;	$\Gamma R(1)$ consist of two straight lines;
	$\gamma(1)$ consists of one point;	$\gamma R(1)$ consists of one point;
	$\delta(1)$ consists of two points;	$\delta R(1)$ consists of one point and one straight line;
Q_{-1}:	$\Gamma(1)$ consists of two points;	$\Gamma R(1)$ is empty;
	$\gamma(1)$ consists of two points;	$\gamma R(1)$ is empty;
	$\delta(1)$ consists of three points;	$\delta R(1)$ consists of one point.

To distinguish the $(3,3)$- and $(3,4)$-quadrics in Palinchak's and Anisova's lists, it suffices to use the characteristic polynomial $P(w)$ and the light cone S.

We start from Palinchak's list:

Q_1: $P(w) = w_1 w_2 w_3$, $\gamma R(2)$ consists of three straight lines;

Q_2: $P(w) = -w_3(w_1 + iw_2)(w_1 - iw_2)$, $\gamma R(2)$ consists of one straight line;

Q_3: $P(w) = -(w_2)^2 w_3$, $\gamma R(2)$ consists of two straight lines;

Q_4: $P(w) = -(w_3)^3$, $\gamma R(2)$ consists of one straight line,
S consists of one straight line;

Q_5: $P(w) = 0$, $\gamma R(2)$ is the entire space,
S is a cone of degree 4;

Q_6: $P(w) = 0$, $\gamma R(2)$ is the entire space,
S is a three-dimensional subspace;

Q_7: $P(w) = 0$, $\gamma R(2)$ is the entire space,
S consists of a three-dimensional subspace and a quadratic cone;

Q_8: $P(w) = -(w_3)^3$, $\gamma R(2)$ consists of one straight line,
S is a quadratic cone.

This suffices for the verification of the pairwise non-equivalence of these quadrics. If we use the dimensions of components of the algebra in the proof of non-equivalence, then it remains only to distinguish Q_3 from Q_4 and Q_5 from Q_9 in Anisova's list. The first pair differs in the structure of the cone S, which is flat for Q_3 and is not flat for Q_4. The second pair differs in the characteristic polynomial. The decomposition of Q_3 into irreducible components contains a multiple linear factor ($P(w) = {w_1}^2(w_1 + w_2)$), while for Q_4 all three linear factors are distinct ($P(w) = w_4(w_1 + iw_2)(w_1 - iw_2)$).

For a generic $(3,3)$-quadric, both $\Gamma(2)$ and $\gamma(2)$ are non-singular cubic curves in $\mathbb{CP}^2$. Such a curve is known to have nine inflection points; moreover, the set of these nine points has no projective symmetries. (The double ratios of all quadruples are distinct.) Using the asymmetry, we can order the points in such a way that if there is a projective map of one cubic onto the other, then it takes the inflection points of the first cubic to the inflection points of the second cubic with the order preserved. (The first point is taken to the first, and so on.) For a self-map of the quadric, the action on the complex tangent is thus reduced to scalar dilations $\{z \to \lambda z\}$, where λ is a non-zero complex factor. This is the geometric cause of the rigidity of the quadric.

Classification issues for a manifold with rigid tangent quadric can be solved very easily [13], [16]. In particular, the reduction of a local equation of the manifold to a normal form enables one to write out a complete system of local invariants of the germ.

The invariant constructions described above enable one to find obstructions not only to the equivalence of germs but also to the existence of non-invertible holomorphic maps of one germ into another. Needless to say, one can always define a constant map into an arbitrary germ, and so these obstructions have the form of bounds on the degree of degeneration (the rank of the tangent map). This pertains not only to maps of quadrics but also to maps of smooth manifold germs of various types [15]. Obstructions for quadrics imply obstructions for germs with the given tangent quadrics.

Although the assertions given below are stated for quadrics in general position on the basis of information about the dimensions of invariant sets, one can study singular quadrics in the same way. Thus, we claim that for each of the pairs listed below the rank of the tangent map for a map of the first quadric into the second is less by at least two than the maximum possible value. The list is as follows:

1. a $(3,1)$-hyperquadric is mapped into a $(3,3)$-quadric;
2. a $(4,1)$-hyperquadric is mapped into a $(4,3)$-quadric;
3. $(3,2)$-quadric is mapped into a $(3,3)$-quadric;
4. $(3,3)$-quadric is mapped into a $(4,4)$-quadric;
5. $(3,3)$-quadric is mapped into a $(3,5)$-quadric.

§ 6. Higher-order models

In the preceding sections we have tried to show that a non-degenerate real quadric in the complex space is an object remarkable in many respects and that the description of automorphisms, the construction of a system of invariants, and the solution of the classification problem for surface germs (or CR-manifold germs) can be obtained on the basis of the solution of the respective problems for the tangent quadric. The existence of this object is fortunate but rather surprising. Let us explain this by discussing the choice of the degree d of the model (which is equal to two for the quadric). Taken separately, the universality and homogeneity requirements both promote the reduction of d. Indeed, the model $\{\operatorname{Im} w = 0\}$ of degree one of a real manifold is universal and homogeneous. The requirement that the automorphism group must be finite-dimensional tends to increase d: the germ of a surface of degree greater than two in general position (a non-degeneracy condition) has a finite-dimensional automorphism group. It is good luck that there is a trade-off between the two opposite tendencies at $d = 2$; the remaining requirements from the list of properties of a good model can thus be viewed as free extras.

However, one encounters a difficulty as the codimension increases. The point is that the dimension n^2 of the space of $n \times n$ Hermitian matrices is finite. Once the codimension exceeds this threshold value, the first part of the finite-dimensionality condition cannot be satisfied. Accordingly, all quadrics of this type are degenerate, and their automorphism groups are infinite-dimensional. Thus, for $k > n^2$ one cannot state a criterion for the finite dimensionality of the (n,k)-germ group in terms of 2-jets, and we have to construct a new good model.

This phenomenon can be viewed from a different angle. Let $T^c(M)$ be the bundle of real vector fields on M belonging to the complex part of the tangent plane at each point. The Levi–Tanaka graded algebra is defined inductively by the relations

$$D^1 = T^c(M), \quad D^{j+1} = [D^j, D^1] + D^j,$$

where the bracket stands for the commutator of vector fields. If this sequence stabilizes at the lth step ($D^{l-1} \neq D^l = D^{l+1} = \cdots = D^\infty$), then we say that the length of the algebra is equal to l. The Levi form is non-degenerate if and only if the algebra is of length two and $D^\infty = T(M)$. This is clearly impossible for $k > n^2$. Higher codimensions require the use of higher-order commutators, which in coordinate terms implies proceeding to jets of order higher than two.

There is yet another reason for considering higher degrees. All work on quadratic models was performed in the second millennium, and the advent of the third millennium stimulates one to consider models of degree three or higher.

6.1. A cubic model. Up to equivalence, there is a unique Hermitian form of type (n, n^2). Its components make up an arbitrary sequence of n^2 scalar linearly independent Hermitian forms. We denote this form by $\langle z, \overline{z}\rangle$. Let $k > 0$. We consider the surface Q_3 in the space $\mathbb{C}^n \oplus \mathbb{C}^{n^2} \oplus \mathbb{C}^k$ with coordinates $(z \in \mathbb{C}^n, w_2 \in \mathbb{C}^{n^2}, w_3 \in \mathbb{C}^k)$, $n > 0$, $k > 0$, given by the equations

$$\operatorname{Im} w_2 = \langle z, \overline{z}\rangle, \qquad \operatorname{Im} w_3 = 2 \operatorname{Re} \Phi(z, z, \overline{z}), \tag{14}$$

where $\Phi(z, z, \overline{z})$ is a homogeneous $\mathbb{C}^k$-valued form of degree two in z and degree one in $\overline{z}$. This is a surface of type $(n, K = n^2 + k)$, which is a complete analogue of a non-degenerate quadric, that is, satisfies the entire list of requirements imposed on a good model.

We say that Q_3 is the tangent cubic for a germ of the same type if local equations of the germ can be rewritten in the form

$$\operatorname{Im} w_2 = \langle z, \overline{z}\rangle + F_3 + \cdots, \qquad \operatorname{Im} w_3 = 2 \operatorname{Re} \Phi(z, z, \overline{z}) + G_4 + \cdots. \tag{15}$$

(We use the weights $[z] = 1$, $[\operatorname{Re} w_2] = 2$, and $[\operatorname{Re} w_3] = 3$.)

Definition. The surface (15) is said to be *non-degenerate* if $\langle z, \overline{z}\rangle$ is a non-degenerate Hermitian form of type (n, n^2) and the coordinates of the form $\Phi(z, z, \overline{z})$ are linearly independent.

Just as the Levi form admits an invariant definition, so does the cubic form Φ. To construct an invariant definition of Φ, one must consider the repeated commutator of $(1, 0)$-fields [17]. The length of the Levi–Tanaka algebra of Q_3 is equal to 3.

The use of such surfaces is also restricted, since the dimension $n^2(n + 1)$ of the space of cubic forms is finite. The fact that a non-degenerate surface Q_3 of type $(n, K = n^2 + k)$, $k \leqslant n^2(n + 1)$, is a good model surface for non-degenerate germs of the same type can be expressed by the same list of properties [17], which acquires the following form for $d = 3$.

1. *Universality*: an arbitrary generating Levi non-degenerate manifold germ of type $(n, K = n^2 + k)$ in $\mathbb{C}^{n+K}$ is equivalent to a germ of the form (15).

2. *Finite dimension*: (a) the group of holomorphic automorphisms of a generic surface Q_3 is a finite-dimensional Lie group; (b) the group of holomorphic automorphisms of Q_3 is finite-dimensional if and only if Q_3 is non-degenerate, that is, all coordinate forms are linearly independent; (c) any surface of this type specified by equations of degree less than three has an infinite-dimensional automorphism group.

3. *Homogeneity*: Q_3 is homogeneous; that is, its holomorphic automorphisms act on Q_3 transitively. The homogeneity is provided by quadratic-triangular transformations.

4. *Symmetry*: (a) the cubic is the most symmetric non-degenerate surface in that the dimension of the germ group of a non-degenerate surface does not exceed the dimension of the germ group of the tangent cubic; (b) the automorphism algebra of the cubic parametrizes the family of maps of one non-degenerate germ into another.

5. *Algebraic properties*: (a) the Lie algebra of holomorphic vector fields on a non-degenerate cubic is an algebra of polynomial vector fields of bounded degree, and the degrees of the coefficients do not exceed five; (b) the automorphism group of a non-degenerate cubic is a Lie subgroup of the group of birational transformations of $\mathbb{C}^{n+k}$ with uniformly bounded degrees; one can estimate the degrees of numerators and denominators in a non-cancellable representation by $15(n+k)$; (c) if two germs are equivalent, then so are their tangent cubics, and two cubics are holomorphically equivalent if and only if they are linearly equivalent.

The structure of the automorphism algebra $\operatorname{Aut} Q_3$ of the cubic can be described as follows. Let us introduce a gradation in the space of vector fields with coefficients depending on (z, w_2, w_3) by setting

$$[z]=1, \quad [w_2]=2, \quad [w_3]=3\left[\frac{\partial}{\partial z}\right]=-1, \quad \left[\frac{\partial}{\partial w_2}\right]=-2, \quad \left[\frac{\partial}{\partial w_3}\right]=-3;$$

then $\operatorname{Aut} Q_3$ becomes a graded Lie algebra of the form

$$g_{-3}+g_{-2}+g_{-1}+g_0+g_1+g_2+g_3+g_4+g_5+g_6.$$

Furthermore, the subalgebra $g_- = g_{-3}+g_{-2}+g_{-1}$ is the Lie algebra of the Lie group $\operatorname{Aut}_- Q_3$ of triangular-quadratic transformations of the form

$$\begin{gathered}
z \to p+z, \\
w_2 \to (q+i\langle p,\overline{p}\rangle)+2i\langle z,\overline{p}\rangle+w_2, \\
w_3 \to (r+2i\operatorname{Re}\Phi(p,p,\overline{p}))+4i\Phi(z,p,\overline{p})+2i\Phi(z,z,\overline{p})+2i\overline{\Phi}(\overline{p},\overline{p},z)+sw_2+w_3;
\end{gathered}$$

here the point $(p,q,r)\in Q_3$ is arbitrary and

$$\operatorname{Re}\Phi(z,p,\overline{z})=s\langle z,\overline{z}\rangle.$$

Thus, $\operatorname{Aut}_- Q_3$ is a subgroup providing the homogeneity of Q_3, and accordingly, $\dim \operatorname{Aut}_- Q_3=\dim Q_3=2n+n^2+k$.

The subalgebra g_0 corresponds to linear automorphisms

$$z\to\Lambda z, \quad \zeta\to\rho\zeta, \quad w\to\nu w$$

of the cubic, where

$$\langle\Lambda z,\overline{\Lambda z}\rangle=\rho\langle z,\overline{z}\rangle, \qquad \Phi(\Lambda z,\Lambda z,\overline{\Lambda z})=\nu\Phi(z,z,\overline{z}).$$

The dimension of this subgroup, which will be denoted by $\operatorname{Aut}_0 Q$, may vary depending on Φ. However, we see from the defining relations that for a given non-singular matrix Λ the values of ρ and ν are uniquely determined. Hence the estimate $\dim \operatorname{Aut}_0 Q \leqslant 2n^2$ is valid.

The action of holomorphic changes of variables on germs induces the linear action

$$\Phi(z, z, \overline{z}) \to \nu\Phi(\Lambda^{-1}z, \Lambda^{-1}z, \overline{\Lambda^{-1}z})$$

of the group $GL(n, \mathbb{C}) \oplus GL(k, \mathbb{R})$ on the tangent cubics. The subalgebra $g_+ = g_1 + g_2 + g_3 + g_4 + g_5 + g_6$ is a nilpotent Lie algebra. The corresponding Lie group is the subgroup of non-linear automorphisms of the cubic preserving the origin and such that $\Lambda = E_n$, $\rho = E_{n^2}$, and $\nu = E_k$. This group will be denoted by $\operatorname{Aut}_+ Q$. In all known examples (namely, $(1,2)$ [19], $(1,3)$, $(1,4)$, $(1,5)$, and $(1,6)$ [51], and (n, n^2+1) [49]), this subalgebra is trivial and all automorphisms preserving a point are linear. Thus, the problem of the existence of a cubic with non-trivial subgroup $\operatorname{Aut} Q_3$ remains open. Apparently, just in the case of quadrics, there are no such automorphisms for a generic cubic (rigidity), and one must seek exceptional cubics.

6.2. A quasiperiodic system of model surfaces. The construction carried out here for $d = 2, 3$ has a generalization to manifolds with arbitrarily large length of the Levi–Tanaka algebra [18]. The generation of model surfaces of increasing degrees resembles the successive occupation of atomic energy levels by electrons [38]. The entire series resembles Mendeleev's periodic system.

Let $\mathcal{F}_{m,n}$ be the space of real polynomials of degree m in $(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n)$ whose expansion in bidegrees does not contain components of bidegrees $(m, 0)$ and $(0, m)$. Let $k_{m,n} = \dim \mathcal{F}_{n,m}$; then $k_{m,n} = \binom{2n+m-1}{m} - 2\binom{n+m-1}{m}$. Now we set $K_{m,n} = k_{2,n} + k_{3,n} + \dots + k_{m,n}$; it can be shown that $K_{m,n} = \binom{2n+m}{m} - 2\binom{n+m}{m} + 1$. In particular,

$$k_{2,n} = n^2, \qquad k_{3,n} = n^2(n+1), \qquad k_{4,n} = n^2(n+1)(7n+1)/12.$$

A model surface Q_d of type (n, K) and degree d, where $K_{d-1,n} < K \leqslant K_{d,n}$, is a surface in the space

$$\mathbb{C}^n \oplus \mathbb{C}^{k_{2,n}} \oplus \dots \oplus \mathbb{C}^{k_{d-1,n}} \oplus \mathbb{C}^k,$$

$k = K - K_{d-1,n}$, with coordinates $(z, w_2, \dots, w_d)$ determined by relations of the form

$$\operatorname{Im} w_m = \Phi_m(z, \overline{z}), \qquad m = 2, \dots, d, \tag{16}$$

where Φ_m, $m = 2, \dots, d-1$, is a vector consisting of basis elements of the space $\mathcal{F}_m$, and Φ_d is a vector consisting of linearly independent elements of $\mathcal{F}_d$. We obtain a non-degenerate quadric for $d = 2$, and a non-degenerate cubic for $d = 3$. We point out that the degree d, in contrast to n and K, is not an independent parameter but is determined by the condition that K must lie in a certain interval.

The surface Q_d will be called the tangent model surface for a germ of the form

$$\begin{aligned} \operatorname{Im} w_2 &= \Phi_2 + \text{terms of weight} \geq 3, \\ &\dots \\ \operatorname{Im} w_d &= \Phi_d + \text{terms of weight} \geq (d+1). \end{aligned} \tag{17}$$

In the computation of weights we adopt the obvious convention that $[\operatorname{Re} w_j] = j$. A real-algebraic surface Q_d of the form (15) will be called the tangent model surface to a germ M_0 of the form (16). The surface and the germ are said to be non-degenerate if all the coordinate forms Φ are linearly independent.

Surfaces of the series Q_d satisfy all requirements imposed on good models except for one important property. We speak of universality. However, the surfaces Q_d are universal for $d = 2$ and 3. Using simple changes of variables, one can readily show that an arbitrary germ of type (n, K), $K_3 < K \leqslant K_4$, with non-degenerate 3-jet can be represented in the form

$$\begin{aligned} \operatorname{Im} w_2 &= \Phi_2(z, \overline{z}\,) + \text{terms of weight} \geq 3, \\ \operatorname{Im} w_3 &= \Phi_3(z, \overline{z}\,) + \text{terms of weight} \geq 4, \\ \operatorname{Im} w_4 &= \Phi_4(z, \overline{z}\,) + \text{terms of weight} \geq 5, \end{aligned} \tag{18}$$

which proves that the model is universal for $d = 4$. For $d \geqslant 5$ the model is no longer universal. The cause is that the fraction of the variables z (the complex tangent) in the set of all coordinates of the space becomes too small. As a result, one cannot ensure a proper tangency of the germ and the model surface by choosing local coordinates. The remedy is obvious: one must enlarge the class of model surfaces by introducing some dependence on the variables $\operatorname{Re} w$. This can be done without loss of homogeneity. However, the key point of our program is to obtain a criterion for the finite dimensionality. At the algebraic level the finite dimensionality, together with the polynomiality and degree estimates, was derived from the analysis of a linear system of differential equations. Once we pass to models involving a dependence on $\operatorname{Re} w$, this system no longer has constant coefficients. It follows that the differential operators occurring in the system no longer commute. This results in technical obstacles, which we hope to overcome in the near future.

There is one more question related to higher-degree model surfaces. Positive-definite quadrics are the skeletons of homogeneous domains. To each model surface one can assign its hull of holomorphy. This is a domain in $\mathbb{C}^{n+K}$. If the model surface is convex in some sense ('positive definiteness'), then its hull of holomorphy does not coincide with the entire space; if the surface can be placed in an acute cone, then the hull of holomorphy proves to be biholomorphically equivalent to a bounded domain. Since the skeleton is homogeneous, one can hope that the hull of holomorphy itself has the same property. If this is the case, then model surfaces can be a source of new examples of homogeneous bounded domains (new Siegel domains).

§ 7. Surfaces with degeneration

Throughout the paper we have avoided 'degenerate' situations in the construction of model surfaces. In this section we touch on this issue, which permit us to state one more property of good model surfaces as a conjecture. All surfaces considered in this section are assumed to be real-analytic and connected.

We again consider the simplest situation, which has served as our starting point. Let M be a connected surface of type $(1, 1)$, that is, a hypersurface in $\mathbb{C}^2$, and let $\xi \in M$. If the Levi form of M at ξ is non-degenerate, then $\dim \operatorname{Aut} M_\xi \leqslant \dim \operatorname{Aut} Q = 8$, where Q is the unique model surface of this type.

If the Levi form is degenerate at ξ, then M contains the non-empty analytic set $M_0 = \{p \in M : M$ is degenerate at $p\}$. If M_0 contains open parts of M, then $M_0 = M$ by the uniqueness theorem. In this case it follows from the Frobenius theorem that in some holomorphic coordinates in a neighbourhood of an arbitrary point M is a hyperplane. In particular, $\dim \operatorname{Aut} M_\xi = \infty$, and the same assertion is true for an arbitrary point of this hypersurface.

It turns out that this is the only case in which a hypersurface germ can have an infinite-dimensional automorphism group. Moreover, we claim that if a hypersurface is not locally flat, then $\dim \operatorname{Aut} M_\xi \leqslant \dim \operatorname{Aut} Q = 8$ for all $\xi \in M$, regardless of whether ξ is degenerate or non-degenerate.

The proof is easy. We consider nine arbitrary vector fields $(X_1, \dots, X_9)$ from $\operatorname{aut} M_\xi$ defined in a neighbourhood U_ξ of ξ. Since M is not locally flat, it follows that $M \setminus M_0$ is a dense open subset of M. Let $\xi' \in (M \setminus M_0) \cap U_\xi$ be an arbitrary point. Since M is non-degenerate at ξ', it follows that there is a non-trivial linear combination of these nine fields with real coefficients that vanishes identically in a neighbourhood of ξ'. These fields have holomorphic coefficients in a full-dimensional neighbourhood of ξ, and their restrictions to M have real-analytic coefficients. By the uniqueness theorem, the vanishing of the linear combination in a neighbourhood of ξ' can be continued to the whole of U_ξ, which completes the proof.

Thus if M is a connected real-analytic hypersurface in $\mathbb{C}^2$ and $\xi \in M$, then the following alternative holds.

Alternative (*for a hypersurface in* $\mathbb{C}^2$):

either $\dim \operatorname{Aut} M_\xi = \infty$ and M is equivalent to a hyperplane in a neighbourhood of an arbitrary point,

or $\dim \operatorname{Aut} M_\xi \leqslant \dim \operatorname{Aut} Q(1,1) = 8$.

Thus, the hyperquadric is the most symmetric hypersurface not only in the class of non-degenerate germs but also in the class of all germs with finite-dimensional automorphism group.

The subgroup $\operatorname{Aut}_\xi M_\xi$ of automorphisms preserving the point ξ (the stabilizer subgroup of ξ) is five-dimensional for the case of a hyperquadric in $\mathbb{C}^2$. The author does not know any examples of surfaces with finite-dimensional automorphism group for which the stabilizer subgroup is larger. However, a proof is lacking, and the problem remains open.

The possibility of local rectification of a surface has been studied by Freeman [32]. A surface germ M_ξ in $\mathbb{C}^N$ is said to be q-rectifiable if it is biholomorphically equivalent to a germ of the form $\widetilde{M}_0 \times \mathbb{C}^q$, where $\widetilde{M}_0$ is a surface germ in $\mathbb{C}^{N-q}$. A germ is said to be rectifiable if it is q-rectifiable for some $q > 0$. Freeman's argument is based on the Frobenius theorem.

A criterion for the finite dimensionality of the automorphism group of a hypersurface was suggested by Stanton [54]. It is more convenient to state it as an infinite-dimensionality criterion: $\dim \operatorname{aut} M_\xi = \infty$ if and only if there are no non-zero tangent $(1,0)$-fields with holomorphic coefficients. This criterion is not a trivial restatement, but its verification is a computational problem of the same type as the computation of the automorphism algebra. Stanton refers to germs with finite-dimensional group as holomorphically non-degenerate. A connected real-analytic surface is (or is not) holomorphically non-degenerate at all points simultaneously ([5]; see also [37]).

The above proof of the alternative cannot be extended even to hypersurfaces in $\mathbb{C}^3$. The Levi form of such a hypersurface is a Hermitian form on $\mathbb{C}^2$, and its rank may be equal to 0, 1, or 2. If the rank is zero on an open part of the hypersurface, then the hypersurface is rectifiable and the group is infinite-dimensional (holomorphic degeneracy). If the rank is equal to two, then the form is non-degenerate. If the rank is equal to two at least at one point, then it is equal to two on a dense open subset of the hypersurface. This permits us to reproduce our argument and obtain the estimate $\dim \operatorname{Aut} M_\xi \leqslant \dim \operatorname{Aut} Q(2,1) = 15$. However, the rank can be equal to unity everywhere. An example is given by the light cone

$$\bigl\{(z_1, z_2, w) \in \mathbb{C}^3 : (\operatorname{Im} w)^2 = (\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2\bigr\}.$$

The alternative remains valid, but the proof is much harder.

Baouendi, Ebenfelt, Hang, Rothschild, and Zaitsev [5]–[9] considered some characteristics of degenerate points of a real manifold, in particular, l-non-degeneracy. A manifold M is said to be finitely non-degenerate at a point ξ if there is an $l \geqslant 0$ such that for an arbitrary $(0,1)$-vector field L on M with $L(\xi) \neq 0$ there are $(0,1)$-fields $L_1, \dots, L_m$, $0 \leqslant m \leqslant l$, such that $[L_1, \dots, [L_m, \overline{L}], \dots](\xi) \notin T^c_\xi M \otimes \mathbb{C}$. If l is the minimum number with this property, then M is said to be l-non-degenerate at ξ. Ebenfelt [23] wrote out normal forms of the equation of a 2-non-degenerate real-analytic hypersurface in $\mathbb{C}^3$. In particular, he showed that the lower-order terms in the equation of the hypersurface can be represented in one of the following eight forms. If the Levi form of M at ξ is zero, that is, both eigenvalues of the Levi form are zero, then:

(A.1) $\operatorname{Im} w = |z_1|^2(z_2 + \overline{z}_2) + r(z_1^2\overline{z}_2 + \overline{z}_1^2 z_2) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$, where $r > 0$;
(A.2) $\operatorname{Im} w = |z_1|^2(z_2 + \overline{z}_2) + (z_1^2\overline{z}_2 + \overline{z}_1^2 z_2) + i|z_1|^2(z_1 - \overline{z}_1) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$;
(A.3) $\operatorname{Im} w = |z_1|^2(z_2 + \overline{z}_2) + (z_1\overline{z}_2^2 + \overline{z}_1 z_2^2) + |z_2|^2(\lambda z_2 + \overline{\lambda}\overline{z}_2) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$, where $\lambda \in \mathbb{C}$, $\lambda \neq 0$;
(A.4) $\operatorname{Im} w = |z_1|^2(z_1 + \overline{z}_1) + |z_2|^2(z_2 + \overline{z}_2) + (\mu z_1^2\overline{z}_2 + \overline{\mu}\overline{z}_1^2 z_2) + (\nu z_1\overline{z}_2^2 + \overline{\nu}\overline{z}_1 z_2^2) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$, where $\mu, \nu \in \mathbb{C}$, $\mu\nu \neq 1$;
(A.5) $\operatorname{Im} w = |z_1|^2(\eta z_1 + \overline{\eta}\overline{z}_1) + (z_1^2\overline{z}_2 + \overline{z}_1^2 z_2) + (z_1\overline{z}_2^2 + \overline{z}_1 z_2^2) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$, where $\eta \in \mathbb{C}$.

If the Levi form of M at ξ has exactly one non-zero eigenvalue, then

(B.1) $\operatorname{Im} w = |z_1|^2 + |z_2|^2(z_2 + \overline{z}_2) + \gamma(z_1^2\overline{z}_2 + \overline{z}_1^2 z_2) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$, where $\gamma = 0, 1$;
(B.2) $\operatorname{Im} w = |z_1|^2 + (z_1^2\overline{z}_2 + \overline{z}_1^2 z_2) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$;
(B.3) $\operatorname{Im} w = |z_1|^2 + |z_2|^2(z_1 + \overline{z}_1) + O(|z|^4 + |\operatorname{Re} w|\,|z|^2)$.

Ershova [24] applied the model-surface method to each of these eight types. In particular, a sharp bound for the dimension of the automorphism group was obtained for each of the types. The maximum dimension is attained for the hypersurface $\operatorname{Im} w = |z_1|^2 + |z_2|^2(z_2 + \overline{z}_2)$. (This is the type (B.1) for $\gamma = 0$.) The entire group is seven-dimensional, and the stabilizer subgroup of the origin is three-dimensional.

This computation (more precisely, the part pertaining to the types (B.1), (B.2), and (B.3)) permits us to complete the proof of the bound for the dimension of the automorphism group of an arbitrary non-flat real-analytic hypersurface in $\mathbb{C}^3$. Indeed, if M is a real-analytic hypersurface in $\mathbb{C}^3$ that is not locally flat, then the points of l-non-degeneracy, where $l \leqslant 2$, from a dense open subset of M [7]. Repeating our argument for sixteen arbitrary fields, we see that they are linearly dependent. Consequently, $\dim \operatorname{Aut} M_\xi \leqslant 15$.

Thus, the alternative stated above remains valid for $(2,1)$-surfaces.

Alternative (*for a hypersurface in* $\mathbb{C}^3$):

either $\dim \operatorname{Aut} M_\xi = \infty$ and M is equivalent to a hyperplane in a neighbourhood of an arbitrary point,

or $\dim \operatorname{Aut} M_\xi \leqslant \dim \operatorname{Aut} Q(2,1) = 15$.

Thus, hyperquadrics are the most symmetric surfaces in $\mathbb{C}^3$.

In $\mathbb{C}^3$ there are two non-equivalent hyperquadrics, but the dimensions of their automorphism groups are the same and are equal to 15. This is specific to hypersurfaces. Model surfaces of higher codimension (say, quadrics) may have groups of different dimensions, but there is a bound for each type. To state the general conjecture, we denote by $D(n,k)$ and $d(n,k)$ the maximum dimensions of the automorphism groups and stabilizer subgroups, respectively, over all model surfaces of type (n,k).

Dimension conjecture. The following alternative holds for the germ M_ξ of an arbitrary real-analytic manifold:

either $\dim \operatorname{Aut} M_\xi = \infty$,

or $\dim \operatorname{Aut} M_\xi \leqslant D(n,k)$ and $\dim \operatorname{Aut}_\xi M_\xi \leqslant d(n,k)$.

In closing, we note that the problem of constructing a system of model surfaces including all cases of degeneracy leads naturally to a tree-like classification of germs. Moreover, the classification tree is not finite even for hypersurfaces in $\mathbb{C}^2$.

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