CR-Varieties of the Type $(1, 2)$ as Varieties of "Super-High" Codimension

VALERII K. BELOSHAPKA

Dept. of Mechanics and Mathematics, Moscow State University, Vorob'evy Gory, Moscow 119899, Russia

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In the paper [1] the role of the tangent hyperquadric in the study of biholomorphic mappings and automorphisms of a nondegenerate real hypersurface was clarified. In [2] the automorphisms and mappings of a real surface of high codimension (i.e., exceeding 1) were studied. It turns out that an analog of the tangent hyperquadric, which is a tangent quadric of high codimension, still plays the role of a model surface for the entire class of nondegenerate surfaces of high codimension and possesses a number of extremal properties (the quadric is the most symmetric surface, and its local groups are the richest ones). Moreover, when passing from the "hyper" case to the situation of high codimension, a number of specific phenomena was discovered: the "rigidity" phenomenon of a quadric, the appearance of a rich system of CR-invariants, etc. [3, 4].

If (n, k) is the type of a CR-variety, then n is the complex dimension of the complex part of the tangent space and k is the dimension of the quotient space of the total tangent space by the complex part, i.e., the CR-codimension. For a variety realized as the generating real subvariety of a complex space, k is its real codimension.

The above theory is rich in content provided that the generic quadric is nondegenerate in the sense of [2]. The relation $1 \leq k \leq n^2$ is a necessary condition. For this reason we call CR-varieties of the type (n, k), with $k > n^2$, *varieties of super-high codimension* to distinguish them from the well-studied case of high codimension in which $2 \leq k \leq n^2$.

The situation of minimal dimension in which the varieties of super-high codimension occur is $n = 1$, $k = 2$, i.e., a real surface in $\mathbb{C}^3$ with one-dimensional complex tangent and of codimension two. The surfaces with one-dimensional complex tangent were studied in the paper by Abrosimov [5], however, his original approach first works at $k = 3$ and is valid for higher dimensions. Studying the situation for $n = 1$, $k = 2$, we show that in this case the role of a quadric is played by another object, namely, a $(1, 2)$-cubic surface defined by equations of degree three.

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Suppose that M is a smooth real surface in $\mathbb{C}^3$, $\xi \in M$, $\dim_C T_\xi^C M = 1$, $\operatorname{codim}_R T_\xi M = 2$, and the equation for M in a neighborhood of ξ, in some coordinates with center at the point ξ, is a relation of the form $(z,\ w^1 = u^1 + iv^1,\ w^2 = u^2 + iv^2)$, where as $v^1 = F^1(z, \bar{z}, u)$, $v^2 = F^2(z, \bar{z}, u)$, the functions F being smooth and real-valued in a neighborhood of the origin and F and dF being zero at the origin. To this end it suffices to assume that M is generating at the point ξ. Moreover, we have $T_0 M = \{v = 0\}$, $T_0^C M = \{w = 0\}$. The component of the Taylor expansion of F whose degree with respect to z is one, with respect to $\bar{z}$ is one, and with respect to u is zero, has the form $Az\bar{z}$, where $A \in \mathbb{R}^2$. If $A \neq 0$, then after a linear change of variables with respect to w, we may assume that $A = (1, 0)$. Furthermore, the term of degree two with respect to z, of degree zero with respect to $\bar{z}$, and of degree zero with respect to u, is Bz^2, and its complex conjugate term is $\overline{B}\bar{z}^2$. After the change of coordinates $z \mapsto z$, $w \mapsto w + 2iBz^2$, this parameter vanishes. Introduce a grading by setting the weights of the variables as follows: $[z] = 1$, $[w^1] = [u^1] = 2$, and $[w^2] = [u^2] = 3$; here the terms of weight at least p will be denoted by $O(p)$. Now the equations of M become

$$v^1 = |z|^2 + O(3), \quad v^2 = |z|^2 2\operatorname{Re} \mu z + u^1 2 \operatorname{Re} \nu z + O(4). \tag{1}$$

If $\mu \neq 0$, then after the linear transformation $z \mapsto \bar{\mu} z$, $w^1 \mapsto |\mu|^2 w^1$, $w^2 \mapsto |\mu|^4 w^2$ we obtain

$$v^1 = |z|^2 + O(3), \quad v^2 = |z|^2 2\operatorname{Re} z + u^1 2 \operatorname{Re} \nu z + O(4). \tag{2}$$

Let us apply to this surface the invertible holomorphic transformation of the form

$$z \mapsto f(z, w^1, w^2) = \sum_{p=0}^{\infty} f_p, \quad w^1 \mapsto g(z, w^1, w^2) = \sum_{p=0}^{\infty} g_p, \quad w^2 \mapsto h(z, w^1, w^2) = \sum_{p=0}^{\infty} h_p,$$

where f_p, g_p, h_p are the components of the weight p. Suppose that under this transformation the surface becomes another surface of the same form with parameter $\tilde{\nu}$, and the origin is fixed. Then $f_0 = g_0 = h_0 = 0$, and the transforming functions are related as follows:

$$\begin{aligned} \operatorname{Im}(g_1 + g_2) &= |f_1|^2 + O(3), \\ \operatorname{Im}(h_1 + h_2 + h_3) &= |f_1|^2 2 \operatorname{Re} f_1 + \operatorname{Re}(g_1 + g_2) 2 \operatorname{Re} \tilde{\nu}(f_1 + f_2) + O(4). \end{aligned} \tag{3}$$

Lemma 1. *If $\nu \neq -i$, then the surface (1) can be transformed by a quadratic change of variables to the form*

$$v^1 = |z|^2 + O(3), \qquad v^2 = |z|^2 2 \operatorname{Re} z + O(4). \tag{4}$$

Proof. Let us calculate the lowest weight components in relation (3). For weight one, we obtain $\operatorname{Im} g_1 = 0$ and $\operatorname{Im} h_1 = 0$, but these expressions are linear with respect to z, and therefore $g_1 = 0$ and $h_1 = 0$. For weight two, we obtain $\operatorname{Im} g_2 = |f_1|^2$, $\operatorname{Im} h_2 = 0$, and hence $f_1 = \lambda z$, $g_2 = |\lambda|^2 w^1$, and $h_2 = 0$. For weight three we have $\operatorname{Im} h_3 = |f_1|^2 2 \operatorname{Re} f_1 + \operatorname{Re} g_2 2 \operatorname{Re} \tilde{\nu} f_1$. If $h_3 = az^3 + bzw^1 + cw^2$, then the relation becomes

$$\operatorname{Re}(iaz^3 + ibz(u^1 + i|z|^2) + ic(u^2 + i|z|^2 2 \operatorname{Re} z + u^1 2 \operatorname{Re} \nu z) + 2\lambda^2 \bar{\lambda} z^2 \bar{z} + 2\tilde{\nu}\lambda^2\bar{\lambda} u^1 z) = 0$$

Hence, on equating the coefficients, we obtain

$$\operatorname{Im} c = 0, \quad b = 2\lambda^2\bar{\lambda} - 2c, \quad \tilde{\nu} = c\lambda^{-2}\bar{\lambda}^{-1}(\nu + i) - i,$$

and therefore for $\nu \neq i$ we can set $\tilde{\nu} = 0$, and hence $-ic(\nu + i) = \lambda^2 \bar{\lambda}$. This equation, for any $c \neq 0$, has a unique solution with respect to λ, which is nonzero. Thus, the mapping is invertible. This proves the lemma.

Thus, each surface of type $(1,2)$ satisfying certain nondegeneracy conditions, which we indicated in the course of our discussion (namely, $A \neq 0$, $\mu \neq 0$, $\nu \neq -i$), has an equation of the form (4). A surface satisfying these conditions is said to be *nondegenerate* and $Q = \{v^1 = |z|^2,\ v^2 = |z|^2 2\operatorname{Re} z\}$ is called the *tangent cubic.*

Under the traditional approach (see [3]), we would arrive at a tangent quadric, which has the form $\{v^1 = |z|^2,\ v^2 = 0\}$ for the case under consideration. However, this quadric is degenerate. In particular, this means that the group of biholomorphic automorphisms of such a quadric is infinite-dimensional. However, a nondegenerate quadric possesses the following series of properties:

1. All surfaces given by linear equations have infinite-dimensional group. A quadric is given by quadratic equations and its group is finite-dimensional.
2. Each germ is equivalent to a quadric modulo cubics.
3. A quadric is homogeneous, i.e., its automorphism group acts transitively.
4. The algebra of infinitesimal automorphisms of a quadric consists of fields with polynomial coefficients whose degree is at most two. The automorphism group consists of birational mappings of bounded degree.
5. A quadric has the following extremal property: the dimension of the group of an arbitrary surface is majorized by the dimension of the group of its tangent quadric.

In accordance with our objective, we wish to clarify to what extent the analogy between the tangent $(1,2)$-cubic and the nondegenerate quadric holds. To this end we calculate the algebra and the group of the automorphisms of the cubic. Let us make more precise what we mean. Let M_ξ be a germ of a real surface at some point $\xi \in \mathbb{C}^N$. Consider the algebra $\operatorname{aut}_\xi M_\xi$ of germs at the point ξ of real vector fields with holomorphic coefficients that vanish at ξ and whose restriction to the surface is tangent to this surface. On integrating each field representing such a germ, we obtain a one-parameter local group of biholomorphic transformations of a neighborhood of ξ that keep ξ fixed and map M_ξ into itself. The set of these transformations generates a group. Denote this group by $\operatorname{Aut}_\xi M_\xi$. If we omit the condition that the fields vanish at ξ, then the algebra $\operatorname{aut}(M_\xi)$ arises that contains $\operatorname{aut}_\xi M_\xi$ as a subalgebra. On integrating the fields from $\operatorname{aut}(M_\xi)$, we obtain one-parameter local subgroups of biholomorphic transformations of neighborhoods of ξ, which need not leave the point ξ fixed. However, if M is a representative of M_ξ, then each of these transformations maps a neighborhood of ξ on M into M. The collection of these transformations generates a local group, which we denote by $\operatorname{Aut}(M_\xi)$. Here the dimensions of the algebras and of the corresponding groups are equal.

Let us pass to the calculation of the algebra $\operatorname{aut}(Q_0)$. Let a real vector field

$$X = 2\operatorname{Re}\left(f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w^1} + h(z,w)\frac{\partial}{\partial w^2}\right)$$

belong to this algebra. This means that f, g, h are holomorphic in a neighborhood of zero and satisfy on Q the relations that determine the tangent plane

$$\operatorname{Re}(ig + 2\bar{z}f) = 0, \quad \operatorname{Re}(ih + 2\bar{z}^2 f + 4|z|^2 f) = 0, \tag{5}$$

where $w^1 = u^1 + i|z|^2$, $w^2 = u^2 + i|z|^2 2\operatorname{Re} z$. On applying the relations of the form

$$\phi(z, u^1 + i|z|^2, u^2 + i|z|^2 2\operatorname{Re} z) = \phi(z,u^1,u^2) + \frac{\partial\phi}{\partial u^1}(z,u^1,u^2)i|z|^2 + \frac{\partial\phi}{\partial u^2}(z,u^1,u^2)i|z|^2 2\operatorname{Re} z + \ldots$$

$$\phi(z,u) = \sum \phi_q(u) z^q$$

(do not confuse this with the decomposition with respect to the weight components) we select in (5) the coefficients of the monomials $z^m\bar{z}^n$. For $n = 0$ we obtain

$$(0,0)\quad \operatorname{Im} g_0 = 0, \qquad \operatorname{Im} h_0 = 0; \qquad\qquad (1,0)\quad ig_1 + 2\bar{f}_0 = 0, \quad ih_1 = 0;$$
$$(2,0)\quad ig_2 = 0, \quad ih_2 + 2\bar{f}_0 = 0; \qquad (m,0) \text{ for } \quad m > 2 \quad ig_m = 0, \quad ih_m = 0.$$

Denote f_0 by a, g_0 by b, and h_0 by c, and denote the derivatives with respect to u^1 and u^2 by superscripts. We have

$$(1,1)\quad 2\operatorname{Re} f_1 = b^{(10)}, \quad 4\operatorname{Re} a = c^{(10)},$$
$$(2,1)\quad f_2 = 2i\bar{a}^{(10)} + b^{(01)}, \quad 2f_1 + \bar{f}_1 = c^{(01)},$$
$$(3,1)\quad f_3 = 2i\bar{a}^{(01)}, \quad f_2 = i\bar{a}^{(10)}, \qquad (4,1)\quad f_4 = 0, \quad f_3 = i\bar{a}^{(01)},$$
$$(m,1) \quad \text{for} \quad m > 4 \quad f_m = 0, \qquad (2,2) \quad \operatorname{Im} a^{(01)} = 0, \quad \operatorname{Im} a^{(10)} = 0.$$

Hence, $f_1 = c^{(01)}/3$ and $f_2 = f_3 = 0$, and the functions a, b, c satisfy the following relations: $a^{(01)} = a^{(10)} = 0$, $2c^{(01)} = 3b^{(10)}$, $c^{(10)} = 4\operatorname{Re} a$, $b^{(20)} = b^{(11)} = b^{(02)}$, $c^{(20)} = c^{(11)} = c^{(02)}$,

Thus,

$$a(u) = A + iE, \quad b(u) = B + 2Du^1, \quad c(u) = C + 4Au^1 + 3Du^2.$$

Here A, B, C, D, and E are real constants. Hence,

$$f = (A + iE) + Dz, \quad g = B + 2Dw^1 + 2i(F + iA)z,$$
$$h = C + 4Aw^1 + 3Dw^2 + 2i(F + iA)z^2.$$

By substituting the functions f, g, and h into (5), we see that the relations identically hold. We finally obtain the following assertion.

Proposition 2. *The algebra* $\operatorname{aut}(Q_0)$ *has dimension five and is the set of linear combinations of the following vector fields:*

$$X_1 = \frac{\partial}{\partial u^1}, \quad X_2 = \frac{\partial}{\partial u^2}, \quad X_3 = 2\operatorname{Re}\left(z\frac{\partial}{\partial z} + 2w^1\frac{\partial}{\partial w^1} + 3w^2\frac{\partial}{\partial w^2}\right),$$
$$X_4 = 2\operatorname{Re}\left(\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial w^1} + (4w^1 - 2z^2)\frac{\partial}{\partial w^2}\right),$$
$$X_5 = 2\operatorname{Re}\left(i\frac{\partial}{\partial z} + 2iz\frac{\partial}{\partial w^1} + 2iz^2\frac{\partial}{\partial w^2}\right),$$

the subalgebra $\operatorname{aut}_0(Q_0)$ *is one-dimensional and is generated by the field* X_3, *and here the first two fields generate the translations along* u^1 *and* u^2, *the third field generates the dilations*

$$\{z \mapsto tz,\ w^1 \mapsto t^2w^1,\ w^2 \mapsto t^3w^2\},$$

and the fourth and the fifth fields generate the one-parameter subgroups of the form

$$\{z \mapsto z + t,\ w^1 \mapsto w^1 - 2zt - t^2,\ w^2 \mapsto w^2 + (4w^1 - 2z^2)t - 6zt^2 - 2t^3\},$$
$$\{z \mapsto z + it,\ w^1 \mapsto w^1 + 2izt - 2t^2,\ w^2 \mapsto w^2 + 2iz^2t - 2zt^2 - (2i/3)t^3\},$$

respectively.

Proof. The fact that the algebra is generated by the given fields is shown above. The one-parameter subgroups are obtained by integration of the corresponding differential equations.

Corollary 3. *The cubic* Q *is a globally homogeneous surface.*

This can also be proved directly. If we have the translations along u, we only need "translations" along z. These can be obtained as follows. We must perform the change of variables $z \mapsto z+a$ and then, by simple transformations, reduce the equations to the initial form.

Let us show that the automorphism groups of the cubic have the maximum possible dimensions in the class of nondegenerate surfaces.

Proposition 4. *If M and $\tilde{M}$ are nondegenerate surfaces given by the equations*

$$v^1 = |z|^2 + F_3 + \dots, \qquad v^1 = |z|^2 + \tilde{F}_3 + \dots,$$
$$v^2 = |z|^2 2\operatorname{Re} z + G_4 + \dots, \quad v^2 = |z|^2 2\operatorname{Re} z + \tilde{G}_4 + \dots$$

and if $z \mapsto f(z,w^1,w^2)$, $w^1 \mapsto g(z,w^1,w^2)$, $w^2 \mapsto h(z,w^1,w^2)$ is a mapping biholomorphic in a neighborhood of the origin that keeps the origin fixed and maps a germ M into $\tilde{M}$, then, for chosen M, and $\tilde{M}$ and for a given value $\partial f/\partial z(0,0,0)$, this mapping is unique.

Proof. It suffices to prove the assertion for $\partial f/\partial z(0,0,0) = 1$. Let us expand the mapping in weight components

$$z \mapsto f(z,w^1,w^2) = \sum_{p=0}^{\infty} f_p, \quad w^1 \mapsto g(z,w^1,w^2) = \sum_{p=0}^{\infty} g_p, \quad w^2 \mapsto h(z,w^1,w^2) = \sum_{p=0}^{\infty} h_p.$$

The origin remains fixed, and hence $f_0 = g_0 = h_0 = 0$. By assumption, $f_1 = z$. Then, according to the calculations in the proof of Lemma 1, for $A = B = 0$ we obtain $g_1 = 0$, $g_2 = w^1$, $h_1 = h_2 = 0$, $h_3 = w^2$. Let us write out a relation of the form (3) and extract in it the component of weight m. Then we have

$$\operatorname{Re}\{ig_m + 2\bar{z}f_{m-1}\} = F_m - \tilde{F}_m + \dots,$$
$$\operatorname{Re}\{ih_m + 2\bar{z}^2 f_{m-1} + 4|z|^2 f_{m-2}\} = G_m - \tilde{G}_m + \dots,$$

where $w^1 = u^1 + i|z|^2$, $w^2 = u^2 + i|z|^2 2\operatorname{Re} z$, and the dots denote the terms that depend on $F_n, \tilde{F}_n, G_n, \tilde{G}_n, g_n, f_{n-1}$ (in the first equation) and f_{n-2} (in the second equation) for $n < m$. These equations may be regarded as recurrence relations that allow the successive calculation of the components of the transforming functions. These relations are linear with respect to the unknown functions, and to find the number of solutions it suffices to find the number of solutions of these relations with zero right-hand side; this is just relation (5) discussed above in the calculation of the algebra of the cubic Q. In particular, our discussion means that the condition that the origin is fixed, together with the relation $f_1 = z$, determines the solution uniquely. This completes the proof of the proposition.

We immediately obtain the following obvious corollary.

Corollary 5. $\dim(\operatorname{Aut}(M_\xi)) \le \dim(\operatorname{Aut}(Q))$, $\dim(\operatorname{Aut}_\xi(M_\xi)) \le \dim(\operatorname{Aut}_0(Q))$.

Proof. The second inequality follows from the fact that a unique parameter in $\operatorname{Aut}_\xi M_\xi$ is the real number $\partial f/\partial z(0,0,0)$. This, together with the fact that Q is homogeneous, implies the first assertion of the corollary.

Remark 6. By a slight modification of the proof of Proposition 4, we can show that a given equation of a nondegenerate surface can be reduced, by a formal holomorphic change of coordinates, to the following normal form:

$$v^1 = |z|^2 + \sum c^1_{mn}(u) z^m \bar{z}^n, \qquad v^2 = |z|^2 2\operatorname{Re} z + \sum c^2_{mn}(u) z^m \bar{z}^n,$$
$$c_{m0} = c^1_{m1} = c^2_{11} = c^2_{21} = c^2_{31} = c^2_{41} = c^1_{22} = c^2_{22} = 0,$$

and by an additional argument we can prove the convergence of the normalizing formal change of variables.

Returning to the discussion of the analogy between the $(1,2)$-cubic and a nondegenerate quadric, we can ascertain, as the result of our efforts, that it is very complete: the cubic satisfies all the properties from list (1)–(5) and, in this sense, is a correct model of a nondegenerate real surface of codimension two in $\mathbb{C}^3$.

In this connection, a natural question arises: what surface is a correct model surface for an arbitrary generic CR-variety of "super-high" codimension with positive dimension of the complex tangent (CR-dimension)? If the CR-dimension is zero, then a germ of such a variety is equivalent to a real plane, and if the codimension is not "super-high," then the correct model is the tangent quadric. Another (weakened) form of this question is as follows: whether in each CR-class (with chosen values of the dimension and of the CR-dimension) there are homogeneous representatives with finite-dimensional group.

Furthermore, as noted above, the $(1,2)$-cubic is a homogeneous surface because it admits complex "translations." This observation allows us to write out a series of homogeneous surfaces of type $(1,n)$ in $\mathbb{C}^{n+1}$ for arbitrary n. Indeed, consider in the space $\mathbb{C}^{n+1}$ with the coordinates (z,w), $w=(w^1,\dots,w^n)=u+iv$, the surface given by an equation $v=F(z,\bar{z})$, where $F=(F^1,\dots,F^n)$ is a vector polynomial without constant and linear term. The complex tangent of this surface is one-dimensional and the full real dimension is $n+2$. Since the equation does not contain the coordinates u, it follows that the automorphism group contains the subgroup of parallel translations by real vectors: $z\mapsto z$, $w\mapsto w+c$, where $c\in\mathbb{R}^n$. To prove its homogeneity, we need only the shifts along the z axis. Denote by $D_a(F)$ the increment of the polynomial under the shift by a, i.e., $D_a(F)(z,\bar{z})=F(z+a,\bar{z}+a)-F(z,\bar{z})$. Let $\mathbf{L}$ be the subspace of polynomials of degree at most one and let $\mathbf{F}^j$, $j=1,\dots,n$, be the subspace generated by $F^1,\dots,F^j$. Let $\mathbf{F}^0=0$.

Proposition 7. *Let the polynomials $F^1,\dots,F^n$ satisfy the following condition: $D_a(F^j)\in\mathbf{F}^{j-1}+\mathbf{L}$, $j=1,\dots,n$, for each complex a. Then the corresponding surface is homogeneous.*

Proof. Let us perform the change of variables $z\mapsto z+a$. Since $D_a(F^1)\in\mathbf{L}$, it follows that by a linear change of the variable w^1, we can reduce the first equation to the initial form. Since $D_a(F^2)\in\mathbf{F}^1+\mathbf{L}$, we can reduce the second equation as well to the initial form by a linear change of the variable w^2, and so on up to the last equation. Finally we obtain the desired automorphism. This completes the proof of the proposition.

We can readily specify some families of homogeneous surfaces of this type. For instance,

$$v^1=|z|^2,\quad v^2=2\,\mathrm{Re}\,z^2\bar{z},\quad v^3=2\,\mathrm{Re}\,z^3\bar{z},\quad v^4=|z|^4,\quad v^5=2\,\mathrm{Re}(\mu z^4\bar{z}+\nu z^3\bar{z}^2).$$

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