Automorphisms of Degenerate Hypersurfaces in $\mathbf{C}^2$ and a Dimension Conjecture

VALERIĬ K. BELOSHAPKA

Dept. of Mechanics and Mathematics, Moscow State University, Vorob'evy Gory, Moscow 119899, Russia

Received March 1, 1996

The automorphisms of four hypersurfaces in $\mathbb{C}^2$ with characteristic degenerations are calculated and the problem of estimating the dimension of the local group of holomorphic automorphisms of an arbitrary surface is posed.

In the papers [1, 2] and others, real hypersurfaces in a complex space and their holomorphic transformations and automorphisms are studied. In these papers it is assumed that the Levi form is nondegenerate. In the present paper we study automorphisms of degenerate germs of real analytic hypersurfaces.

Let z, $w = u + iv$ be the coordinates in $\mathbb{C}^2$. Then the equation of a Levi nondegenerate germ of a real analytic hypersurface can be written in the form

$$v = |z|^2 + \dots,$$

where the dots stand for terms of degree at least six [1]. This means that the quadric $Q = \{v = |z|^2\}$ is a model hypersurface and its properties govern in many respects the properties of the entire class of nondegenerate hypersurfaces. In the present paper we consider some surfaces with degeneration that can play the same role for classes of hypersurfaces with a degeneration of some kind as the quadric plays for nondegenerate classes. For all the examples considered, we calculate the algebra of infinitesimal holomorphic transformations that preserve the germ and describe the corresponding local group.

Let M_ξ be the germ of a real hypersurface at the point ξ. The algebra $\mathrm{aut}M_\xi$ is the algebra of real vector fields of the form

$$X = 2\Re\Big(f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}\Big)$$

1061-9208/96/0393–4

such that the restriction of X to the hypersurface is tangent to this hypersurface and f and g are holomorphic in a neighborhood of ξ. By integrating any such field, we obtain a one-parameter local group of biholomorphic transformations of a neighborhood of ξ that preserve this germ, and these groups generate a local group $\operatorname{Aut} M_\xi$ for which $\operatorname{aut} M_\xi$ is a corresponding algebra. In $\operatorname{Aut} M_\xi$ we can select the subgroup $Aut_\xi M_\xi$ formed by transformations that preserve the point ξ. The corresponding algebra $aut_\xi M_\xi$ consists of the fields that vanish at the point ξ.

After an affine change of coordinates, we may assume that the equation of a representative of the germ M_ξ is written in the form $v = F(z, \overline{z}, u)$, where $\xi = (0,0) = 0$ and F and dF vanish at the origin. By simple additional changes, we can obtain the following form of the equation:

$$v = c_{110}|z|^2 + c_{210}z^2\overline{z} + c_{120}z\overline{z}^2 + c_{111}|z|^2u + \dots, \tag{1}$$

where the dots denote terms of degree at least four. The term $c_{110}|z|^2$ is the Levi form of the germ, and if $c_{110} \neq 0$, then the germ is nondegenerate; therefore, we assume that $c_{110} = 0$.

Example A. Consider the hypersurface A given by the equation

$$v = 2\Re(z^2\overline{z})$$

and its germ A_0. Imposing the condition that the restriction of the vector field X (defined in a neighborhood of the origin) to the hypersurface A is tangent to A, we see that for $w = u + 2i\Re(z^2\overline{z})$ we have

$$\Re(ig + 2\overline{z}^2 f + 4|z|^2 f) = 0.$$

Applying a technique of power series similar to that used in the paper [1], we can obtain the solutions of this functional equation:

$$f = i\alpha + \beta z, \quad g = \gamma + 3\beta w + 2\alpha z^2; \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

Thus, $\operatorname{aut} A_0$ is generated by three fields

$$X_1 = 2\Re\Big(\frac{\partial}{\partial w}\Big), \qquad X_2 = 2\Re\Big(z\frac{\partial}{\partial z} + 3w\frac{\partial}{\partial w}\Big), \qquad X_3 = 2\Re\Big(i\frac{\partial}{\partial z} + 2z^2\frac{\partial}{\partial w}\Big),$$

to each of which correspond one-parameter subgroups, which generate $\operatorname{Aut} A_0$. The subgroups corresponding to the first two fields clearly exist. These subgroups are formed by shifts and dilations:

$$z \mapsto z, \quad w \mapsto w + c, \quad \Im c = 0; \quad z \mapsto bz, \quad w \mapsto b^3 w, \quad b \neq 0, \quad \Im b = 0.$$

To the third field corresponds a group of quadratic transformations of the form

$$z \mapsto z + ia, \quad w \mapsto w - \frac{2}{3}a^3 + 2ia^2 z + 2az^2, \quad \Im a = 0.$$

Let us list some properties of the hypersurface A.

(A1) $\dim \operatorname{Aut} A_0 = 3$, $\quad \dim \operatorname{Aut}_0 A_0 = 1$.

(A2) Fields and automorphisms, defined *a priori* only locally, turn out to be globally defined, i.e., $\operatorname{aut} A_0 = \operatorname{aut} A$ and $\operatorname{Aut}_0 A_0 = \operatorname{Aut}_0 A$.

(A3) The hypersurface A consists of the three parts $A^+ \cup A^0 \cup A^-$, where $A^+ = A \cap \{\Re z > 0\}$, $A^0 = A \cap \{\Re z = 0\}$, and $A^+ = A \cap \{\Re z < 0\}$. The group $\operatorname{Aut} A$ acts transitively on A^+ and on A^-, and A^0 is formed by the points of A at which the Levi form is degenerate.

(A4) The points of A_+ and A_- are points of Kohn type 2 [3], and the points of A^0 are the points of type 3.

For the germs of hypersurfaces of the form (1) with $c_{110} = 0$ and $c_{120} \neq 0$, a theory similar to that developed in [1] can be constructed by using similar tools. An appropriate normal form is constructed and a theorem on the reduction to this normal form by holomorphic changes of coordinates is proved. Since $\dim \mathrm{Aut}_0 A_0 = 1$, the reduction to the normal form can be performed uniquely up to the choice of a single real parameter. As a consequence, we obtain the following statement.

Assertion. *If Γ_0 is the germ of a hypersurface of the form* (1), *where $c_{110} = 0$ and $c_{120} \neq 0$, then $\dim Aut_0 \Gamma_0 \leq 1$.*

Example B. Consider the hypersurface B given by the equation $v = u|z|^2$. The relation for coefficients of the vector fields has the form

$$\Re(ig + |z|^2 g + 2u\bar{z}f) = 0,$$

where $w = u + iu|z|^2$. By solving this equation we obtain $f = i\theta$, $\theta \in \mathbf{R}$, $g = 0$. The corresponding group is the rotation group $z \mapsto e^{it}z$, $w \mapsto w$. Thus, $\dim \mathrm{Aut}\, B_0 = 1$ and $\dim \mathrm{Aut}_0\, B_0 = 1$. Note that the hypersurface B contains the complex line $w = 0$, and all points of this line are points of Kohn infinite type; finally, this line consists of the points at which the hypersurface B is Levi degenerate.

Example C. Consider the hypersurface C given by the equation $v = |z|^4$. The relation for coefficients of the fields has the form

$$2\Re(ig + 4z\bar{z}^2 f) = 0, \quad \text{where} \quad w = u + i|z|^4.$$

By solving this equation, we see that the algebra $\mathrm{aut} C_0$ consists of fields of the form

$$X = 2\Re\left((\beta + \alpha w)z\frac{\partial}{\partial z} + (\gamma + 4(\Re\beta)w + 2\alpha w^2)\frac{\partial}{\partial w}\right), \quad \text{where} \quad \alpha, \gamma \in \mathbb{R}, \ \beta \in \mathbb{C}.$$

Here, to the fields $2\Re(\gamma\partial/\partial w)$ correspond the shifts

$$z \mapsto z, \qquad w \mapsto w + c; \qquad \Im c = 0,$$

to the fields $2\Re(\beta z\partial/\partial z + 4(\Re\beta)w\partial/\partial w)$ there correspond dilations with complex factor

$$z \mapsto bz, \quad w \mapsto |b|^4 w; \quad b \neq 0,$$

and, finally, to the fields $X = 2\Re(\alpha wz(\partial/\partial z) + 2\alpha w^2(\partial/\partial w))$ corresponds the group of transformations of the form

$$z \mapsto \frac{z}{\sqrt{1 - aw}}, \qquad w \mapsto \frac{w}{\sqrt{1 - aw}}.$$

Thus, $\dim \mathrm{Aut}\, C_0 = 4$ and $\dim \mathrm{Aut}_0\, C_0 = 3$. This hypersurface is the image of the hyperquadric $v = |z|^2$ under the mapping $z \mapsto z^2$ and $w \mapsto w$. Therefore, the points with Levi degeneration belong to the real line $z = 0$, $\Im w = 0$. At every point outside of this line, the germ of the hypersurface is nondegenerate and spherical; in particular, the dimension of the local group at this point is equal to eight.

Example D. Let the hypersurface D be given by the equation $v = \exp(-1/|z|^2)$, where the exponent is extended to the origin by zero. The hypersurface D is not analytic, however, it is of class $\mathbb{C}^\infty$. The analyticity is violated at the real line $z = 0$, $\Im w = 0$. The points of this line are points of infinite type, and the formal power series of the defining function at these points is identically zero. By performing algebraic manipulations somewhat more complicated than in the above examples, we can show that the algebra contains only trivial fields corresponding to shifts and rotations:

$$z \mapsto z, \quad w \mapsto w + c; \quad \Im c = 0, \quad z \mapsto \exp(it)z, \quad w \mapsto w; \quad \Im t = 0.$$

That is, $\dim \mathrm{Aut} D_0 = 2$ and $\dim \mathrm{Aut}_0 D_0 = 1$.

All these examples corroborate the following conjecture.

Conjecture 1. The dimension of the local group of holomorphic transformations of the germ of an arbitrary real surface does not exceed the dimension of the group of a nondegenerate quadric of the same type (i.e., the same dimension and codimension).

In other words, if k is the real codimension of the germ M_ξ, n is the complex dimension of the complex tangent of this germ, and $D(k,n)$ and $d(k,n)$ are the maximum dimensions of $\mathrm{Aut} Q_\xi$ and $\mathrm{Aut}_\xi M_\xi$ over all nondegenerate quadrics Q of this type, then we have

$$\dim \mathrm{Aut} M_\xi \leq D(k,n), \qquad \dim \mathrm{Aut}_\xi M_\xi \leq d(k,n).$$

For the germ of a smooth nondegenerate hypersurface, this fact was proved by Poincaré for the space $\mathbb{C}^2$ and by Tanaka for $\mathbb{C}^n$ (see also [1]). For the germ of a nondegenerate surface of an arbitrary codimension, this statement was proved by the author [4]. Therefore, the conjecture is that the above bound holds without the nondegeneracy assumption.

Note that since the quadrics are homogeneous, we have $D(k,n) = d(k,n) + (2n+k)$. Furthermore, $d(1,n) = (n+1)^2 + 1$, and all nondegenerate hyperquadrics have groups of the same dimension [1], namely, $d(2,n) = n^2 + 7$; moreover, this is a real maximum, because for a generic quadric, the dimension of the group is much smaller [5]. For an arbitrary k, the value $d(k,n)$ is unknown, but the known examples permit us to state the following conjecture.

Conjecture 2. $d(k,n) \leq n^2 + 1 + 2n + k$.

The investigation was partially supported by the RFBR Grant no. 093-01-00225 and the International Science Foundation (ISF), Project NCE000.

The paper was written at Ruhr University (Germany, Bochum) and the University of Western Ontario (Canada). The author thanks Professors A. Hackleberry, A. Bouvet, and R. Dvilevich for hospitality and fruitful discussions.

REFERENCES

1. Chern, S. S. and Moser, J. K., 'Real hypersurfaces in complex manifold', *Acta Math.* vol. 133 no. 3-4 (1974), 219–271.
2. Vitushkin, A. G., 'Holomorphic mappings and geometry of surfaces' (1985), 125–166, Current problems in mathematics. Fundamental directions, vol. 7, Moscow.
3. Kohn, J.J., 'Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two', *J. Diff. Geom.* vol. 6 (1972), 542–553.
4. Beloshapka, V. K., 'On holomorphic transformations of a quadric', *Mat. sb.* vol. 182 no. 2 (1991), 203–219.
5. Shevchenko, S. N., 'Quadrics of codimension two and their automorphisms', *Izv. Ross. Akad. nauk, Ser. matem.* vol. 58 no. 4 (1994), 149–172.

Translated by A. I. SHTERN